Adapted from notes by
Prof. Jeffery T. Williams

ECE 5317-6351
Microwave Engineering

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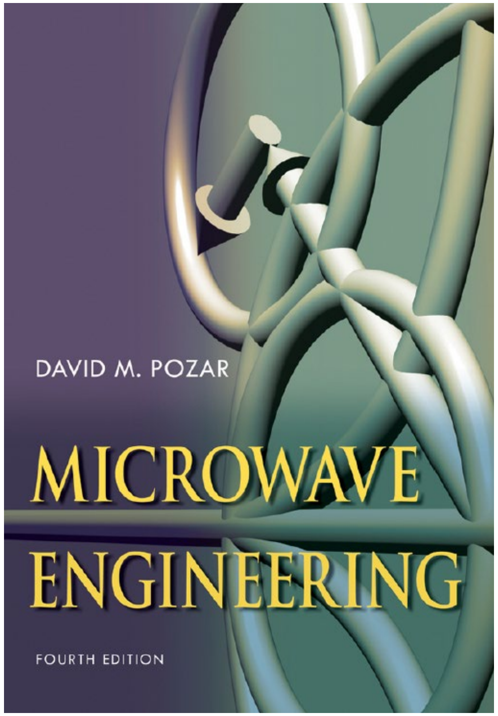
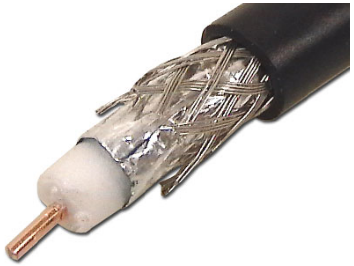
Prof. David R. Jackson
Dept. of ECE



Notes 10

Waveguiding Structures
Part 5: Coaxial Cable



TEM Solution Process

A) Solve Laplace's equation subject to appropriate B.C.s.:

$$\nabla^2 \Phi(x, y) = 0$$

B) Find the transverse electric field: $\underline{e}_t(x, y) = -\nabla \Phi(x, y)$

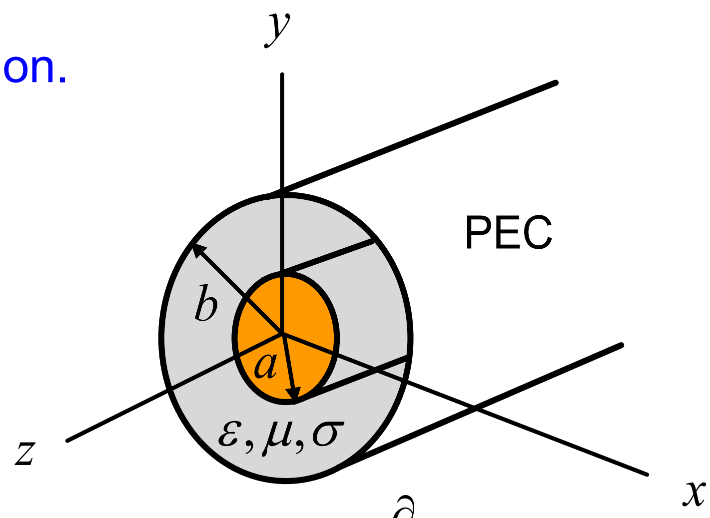
C) Find the total electric field: $\underline{E}(x, y, z) = \underline{e}_t(x, y)\, e^{\mp j k_z z}, \;\; k_z = k$

D) Find the magnetic field: $\underline{H} = \dfrac{1}{\eta}\left(\pm \hat{z} \times \underline{E}\right); \quad \pm z$ propagating

Note: The only frequency dependence is in the wavenumber $k_z = k$.

Coaxial Line: TEM Mode

Assume wave going in $+z$ direction.



To find the TEM mode fields, we need to solve:

$$\nabla^2 \Phi(\rho, \phi) = 0 \; ; \qquad \Phi(a) = V_0$$
$$\Phi(b) = 0$$

$$\frac{\partial}{\partial \phi} = 0$$

$$\Rightarrow \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) = 0$$

$$\Rightarrow \Phi(\rho) = C \ln \rho + D$$

or $\Phi(\rho) = C \ln\left(\frac{\rho}{\rho_0}\right)$

$@ \rho = a$

$$V_0 = C \ln\left(\frac{a}{b}\right)$$

$$\Rightarrow C = \frac{V_0}{\ln\left(\frac{a}{b}\right)}$$

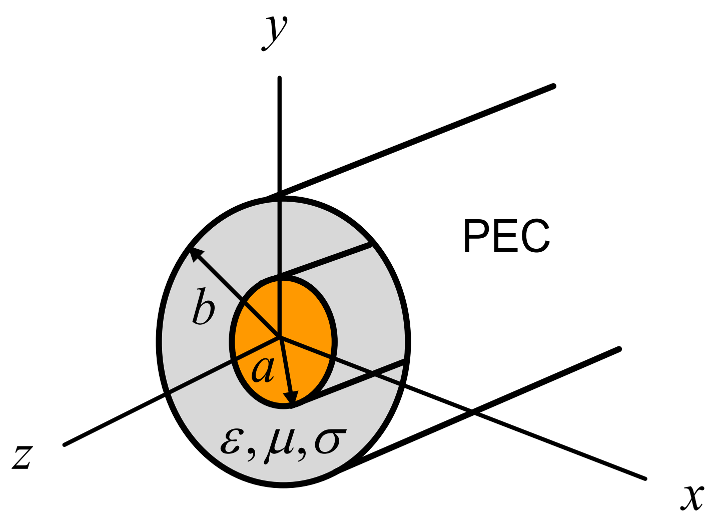
$$\begin{aligned}
\varepsilon_c &= \varepsilon - j\frac{\sigma}{\omega} \\
&= \varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega} \\
&= \varepsilon_c' - j\varepsilon_c'' \\
&= \varepsilon_c'\left(1 - j\frac{\varepsilon_c''}{\varepsilon_c'}\right) \\
&= \varepsilon_c'(1 - j\tan\delta_d) \\
&= \varepsilon_0\varepsilon_r(1 - j\tan\delta_d)
\end{aligned}$$

Here ρ_0 = zero volt potential reference location ($\rho_0 = b$).

Coaxial Line: TEM Mode (cont.)

Hence

$$\Phi(\rho) = \frac{V_0}{\ln\left(\frac{b}{a}\right)} \ln\left(\frac{b}{\rho}\right)$$



Thus,

$$\underline{E}(x,y,z) = \underline{e}_t(x,y)\, e^{-jk_z z}$$

$$= \left(-\nabla_t \Phi(x,y)\right) e^{-jk_z z} = \left(-\underline{\hat{\rho}} \frac{\partial \Phi}{\partial \rho}\right) e^{-jk_z z}$$

TEM: $k_z = k = \omega\sqrt{\mu\varepsilon_c} = k' - jk''$

$$\underline{E}(x,y,z) = \underline{\hat{\rho}} \frac{V_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho} e^{-jk_z z}$$

$$\underline{H} = \frac{1}{\eta}\left(\hat{z} \times \underline{E}\right) \;\Rightarrow\; \underline{H} = \underline{\hat{\phi}} \frac{V_0}{\eta \ln\left(\frac{b}{a}\right)} \frac{1}{\rho} e^{-jk_z z}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

Coaxial Line: TEM Mode (cont.)

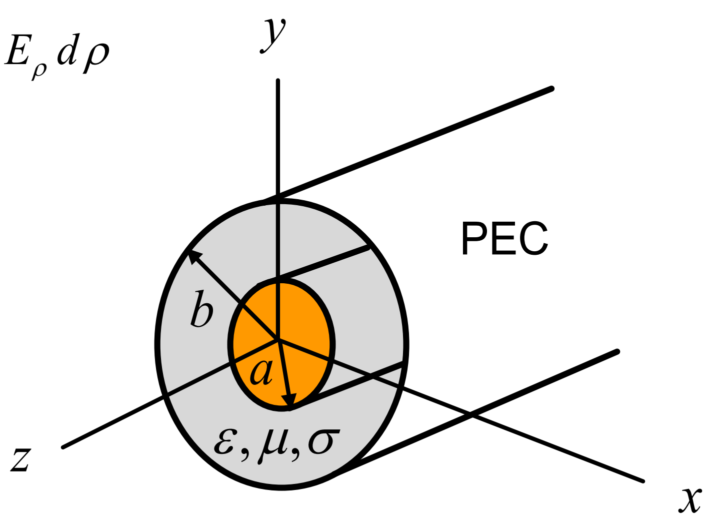
$$V(z) = V_{AB}(z) = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r} = \int_{\underline{A}}^{\underline{B}} \left(\underline{\hat{\rho}} E_\rho\right) \cdot \left(\underline{\hat{\rho}} d\rho + \underline{\hat{\phi}}\, \rho d\phi + \underline{\hat{z}} dz\right) = \int_a^b E_\rho \, d\rho$$

$$= \int_a^b \frac{V_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho} e^{-jk_z z} \, d\rho$$

$\underline{A}$ = point on inner conductor

$\underline{B}$ = point on outer conductor

$$V(z) = V^+(z) = V_0 \, e^{-jk_z z}$$



$$I(z) = \int_0^{2\pi} J_{sz}^a \, a \, d\phi = \int_0^{2\pi} H_\phi \Big|_{\rho=a} a \, d\phi$$

$$= \int_0^{2\pi} \frac{V_0}{\eta \ln\left(\frac{b}{a}\right)} \frac{1}{a} e^{-jk_z z} \, a \, d\phi$$

$$I(z) = I^+(z) = \frac{2\pi V_0}{\eta \ln\left(\frac{b}{a}\right)} e^{-jk_z z}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

$$Z_0 = \frac{V^+(z)}{I^+(z)}$$

Hence

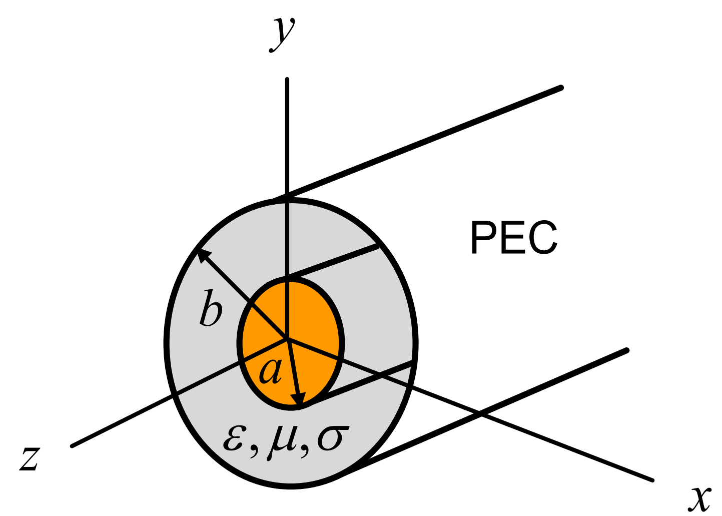
$$Z_0 = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right)$$

Note:
This formula does not account for conductor loss.

Coaxial Line: TEM Mode (cont.)

Attenuation:

$$\alpha = \alpha_d + \alpha_c$$



Geometry for dielectric attenuation

Dielectric attenuation:

$$\text{TEM}: \ \alpha_d = k''$$

$$\alpha_d \approx \frac{k_0 \sqrt{\varepsilon_r}}{2} \tan \delta_d$$

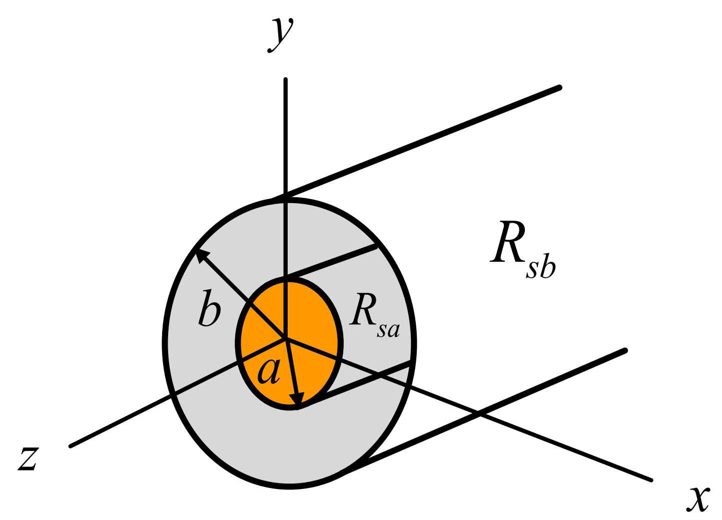
$$\text{TEM}: k_z = \beta - j\alpha_d = k = \omega \sqrt{\mu \varepsilon_c} = k' - jk''$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon_0 \varepsilon_r \left(1 - j \tan \delta_d\right)$$

Coaxial Line: TEM Mode (cont.)

Attenuation:

$$\alpha = \alpha_d + \alpha_c$$



$$\varepsilon_c = \varepsilon = \varepsilon'$$

Geometry for conductor attenuation

(We ignore dielectric loss here.)

Conductor attenuation:

$$\alpha_c = \frac{P_l(0)}{2P_0}$$

$$P_0 \approx \frac{1}{2} Z_0 |I_0|^2$$

(We assume Z_0 is real here.)

Coaxial Line: TEM Mode (cont.)

Conductor attenuation:

$$P_l(0) = \frac{1}{2} \int_{C_1+C_2} R_s \left| \underline{J}_s \right|^2 \bigg|_{z=0} d\ell$$

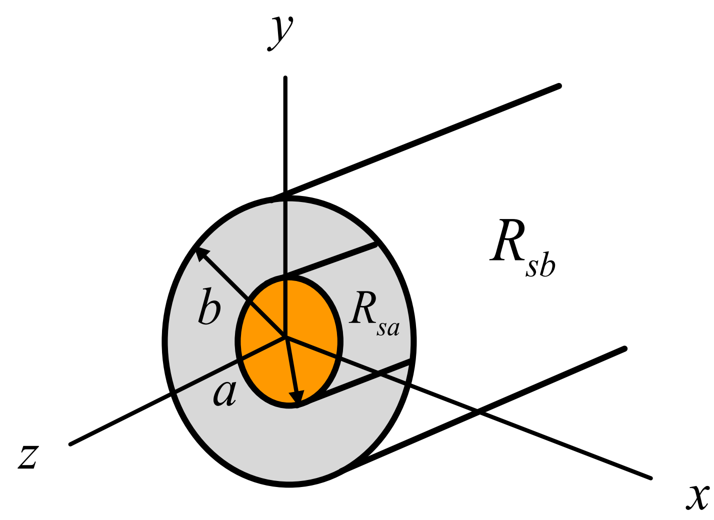
$$= \frac{R_{sa}}{2} \int_0^{2\pi} \left| J_{sz} \right|^2 a d\phi + \frac{R_{sb}}{2} \int_0^{2\pi} \left| J_{sz} \right|^2 b d\phi$$

$$= \frac{R_{sa}}{2} \int_0^{2\pi} \left| \frac{I_0}{2\pi a} \right|^2 a d\phi + \frac{R_{sb}}{2} \int_0^{2\pi} \left| \frac{-I_0}{2\pi b} \right|^2 b d\phi$$

$$= \left| I_0 \right|^2 \frac{R_{sa}}{2} \int_0^{2\pi} \left(\frac{1}{2\pi a} \right)^2 a d\phi + \left| I_0 \right|^2 \frac{R_{sb}}{2} \int_0^{2\pi} \left(\frac{1}{2\pi b} \right)^2 b d\phi$$

$$= \left| I_0 \right|^2 \frac{R_{sa}}{2} \left(\frac{1}{2\pi a} \right) + \left| I_0 \right|^2 \frac{R_{sb}}{2} \left(\frac{1}{2\pi b} \right)$$

$$= \left| I_0 \right|^2 \left(\frac{1}{4\pi} \right) \left(\frac{R_{sa}}{a} + \frac{R_{sb}}{b} \right)$$



$$\varepsilon_c = \varepsilon = \varepsilon'$$

Geometry for conductor attenuation

$$R_s = \sqrt{\frac{\omega \mu}{2\sigma}}$$

(Here σ denotes the conductivity of the metal.)

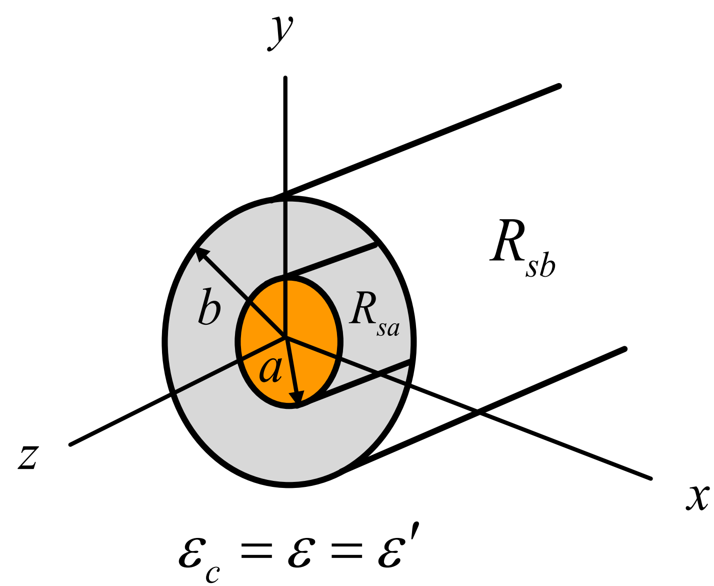
Coaxial Line: TEM Mode (cont.)

Conductor attenuation:

$$\alpha_c = \frac{P_l(0)}{2P_0}$$

$$P_l(0) = |I_0|^2 \left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a} + \frac{R_{sb}}{b}\right)$$

$$P_0 = \frac{1}{2} Z_0 |I_0|^2$$



Geometry for conductor attenuation

Hence we have

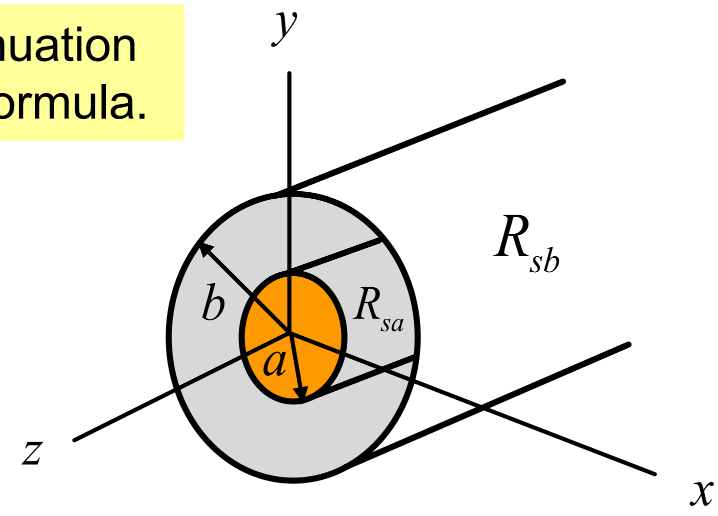
$$\alpha_c = \frac{|I_0|^2 \left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a} + \frac{R_{sb}}{b}\right)}{2\left(\frac{1}{2} Z_0 |I_0|^2\right)}$$

or

$$\alpha_c = \left(\frac{1}{Z_0}\right)\left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a} + \frac{R_{sb}}{b}\right)$$

Coaxial Line: TEM Mode (cont.)

Let's redo the calculation of conductor attenuation using the Wheeler incremental inductance formula.



$$\varepsilon_c = \varepsilon = \varepsilon'$$

Geometry for conductor attenuation

Wheeler's formula:

$$\alpha_c^{cond} = \left(\frac{R_s}{2 Z_0 \eta} \right) \frac{dZ_0}{d\ell} \qquad \eta = \frac{\eta_0}{\sqrt{\varepsilon_r}}$$

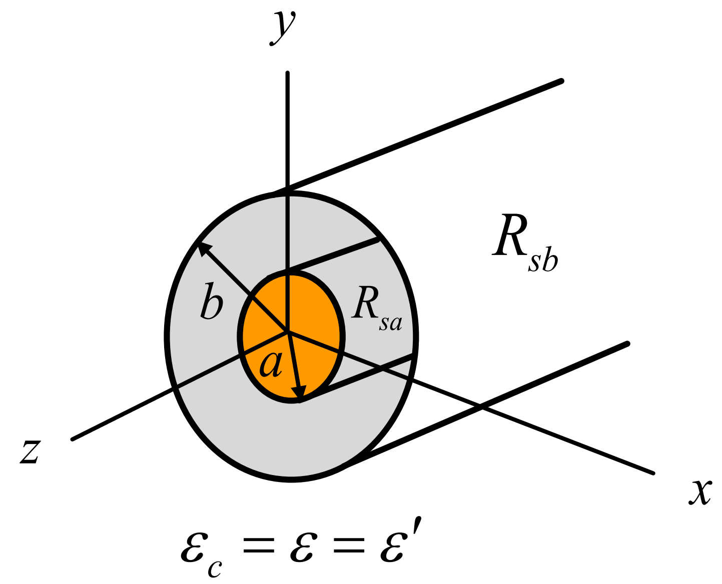
The formula is applied for <u>each</u> conductor and the conductor attenuation from each of the two conductors is then added.

In this formula, $d\ell$ (for a given conductor) is the distance by which the conducting boundary is <u>receded</u> away from the field region.

Coaxial Line: TEM Mode (cont.)

$$\alpha_c = \left(\frac{R_s}{2Z_0\eta}\right)\frac{dZ_0}{d\ell}$$

$$Z_0 \approx \frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right)$$



Geometry for conductor attenuation

$$\alpha_c^a = \left(\frac{R_{sa}}{2Z_0\eta}\right)\left(-\frac{dZ_0}{da}\right) \qquad (d\ell = -da)$$

$$\alpha_c^b = \left(\frac{R_{sb}}{2Z_0\eta}\right)\left(+\frac{dZ_0}{db}\right) \qquad (d\ell = db)$$

Hence

$$\alpha_c^a = \left(\frac{R_{sa}}{2Z_0\eta}\right)\left(-\frac{\eta}{2\pi}\left(-\frac{1}{a}\right)\right)$$

$$\alpha_c^b = \left(\frac{R_{sb}}{2Z_0\eta}\right)\left(\frac{\eta}{2\pi}\left(\frac{1}{b}\right)\right)$$

so

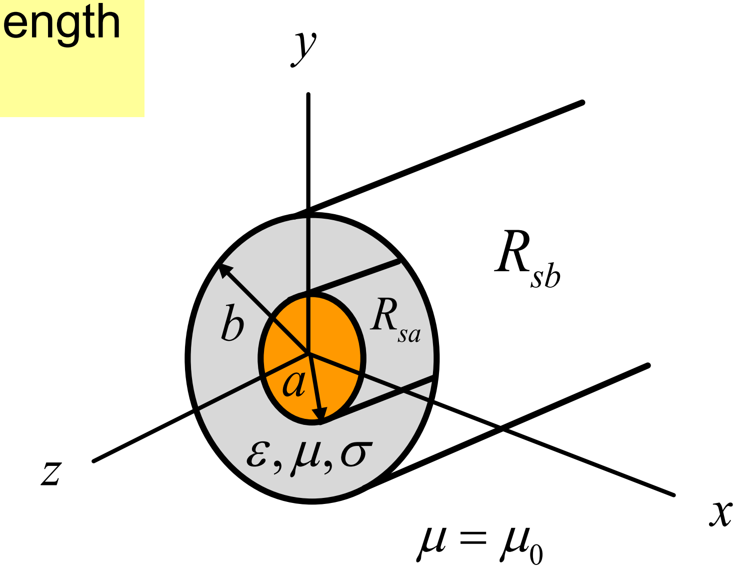
$$\alpha_c = \left(\frac{1}{2Z_0\eta}\right)\left(\frac{\eta}{2\pi}\right)\left(\frac{R_{sa}}{a}+\frac{R_{sb}}{b}\right)$$

or

$$\alpha_c = \left(\frac{1}{Z_0}\right)\left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a}+\frac{R_{sb}}{b}\right)$$

Coaxial Line: TEM Mode (cont.)

We can also calculate the fundamental per-unit-length parameters of the lossy coaxial line.



From previous calculations:

(From Notes 3)

$$L = Z_0^{lossless}\sqrt{\mu\varepsilon'}$$

$$C = \sqrt{\mu\varepsilon'}\,/\,Z_0^{lossless}$$

$$G = (\omega C)\tan\delta$$

(From Notes 7)

$$R = \alpha_c\left(2Z_0^{lossless}\right)$$

where

$$Z_0^{lossless} = \frac{1}{2\pi}\frac{\eta_0}{\sqrt{\varepsilon_r}}\ln\left(\frac{b}{a}\right)$$

$$\varepsilon' = \varepsilon_0\varepsilon_r$$

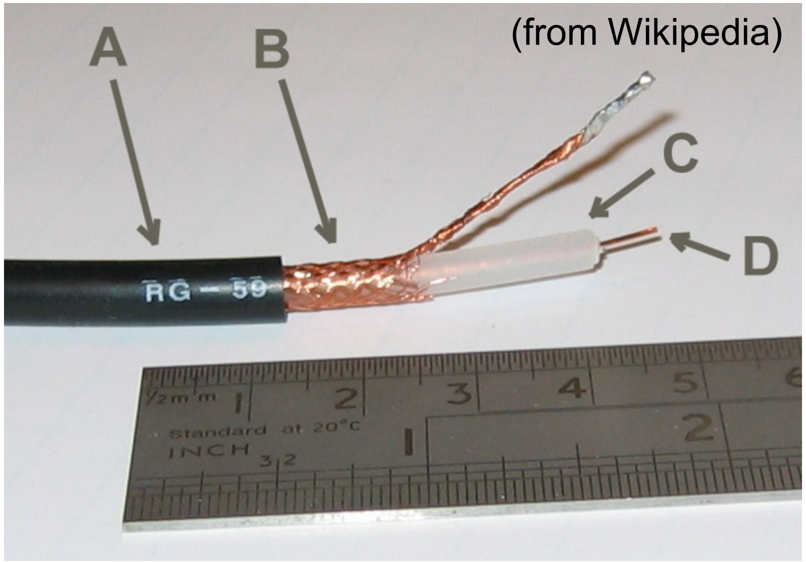
The "lossless" superscript means that we ignore all loss.

Attenuation for RG59 Coax

Approximate attenuation in dB/m

Frequency	RG59 Coax
1 [MHz]	0.01
10 [MHz]	0.03
100 [MHz]	0.11
1 [GHz]	0.40
5 [GHz]	1.0
10 [GHz]	1.5
20 [GHz]	2.3
50 [GHz]	OM*
100 [GHz]	OM*

*OM = overmoded



(from Wikipedia)

$$Z_0 = 75\,\Omega$$

$$a = 0.292\,\text{mm}$$

$$b = 1.85\,\text{mm}$$

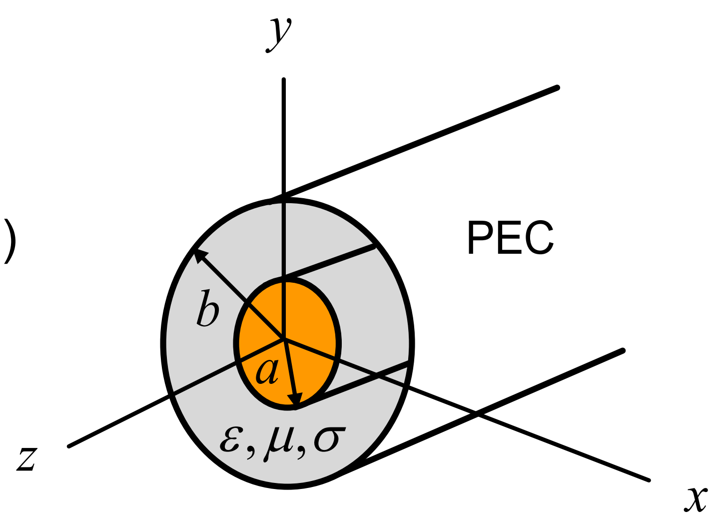
$$\varepsilon_r = 2.25$$

$f_c = 29.7$ GHz (TE_{11} waveguide mode)

Coaxial Line: Power Flow

Power flow at $z = 0$:

$$P_0 = \frac{1}{2}\frac{|V_0|^2}{Z_0}$$ (Z_0 is assumed to be real here.)



$$V_0 = V_{AB}(0) = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot d\underline{r}$$

$$= \int_{\underline{A}}^{\underline{B}} \left(\hat{\underline{\rho}} E_\rho\right) \cdot \left(\hat{\underline{\rho}} d\rho + \hat{\underline{\phi}} \rho d\phi + \hat{\underline{z}} dz\right)$$

$$= \int_a^b E_\rho \, d\rho$$

$$= \int_a^b E_{\rho a} \left(\frac{a}{\rho}\right) d\rho$$

$$= E_{\rho a} a \ln\left(\frac{b}{a}\right)$$

$$\frac{\partial}{\partial \phi} = 0$$

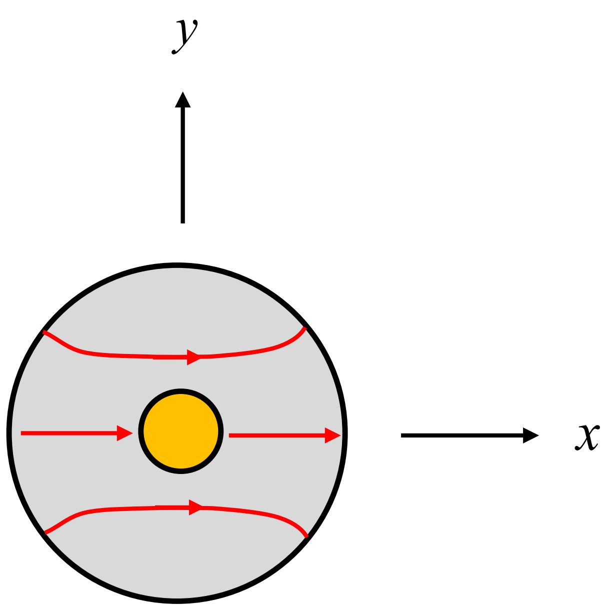
$$P_0 = \frac{a^2}{2Z_0} \ln^2\left(\frac{b}{a}\right) |E_{\rho a}|^2$$

Note: At dielectric breakdown $E_{\rho a} = E_c$

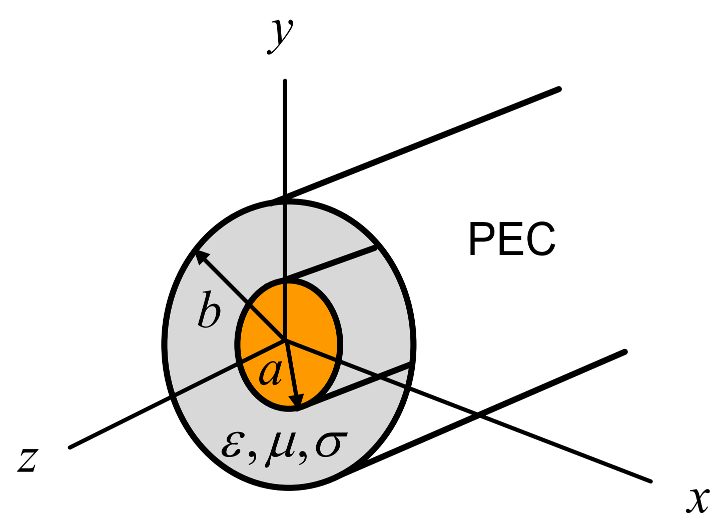
Coaxial Line: Higher-Order Modes

We look at the higher-order modes* of a coaxial line.

The lowest waveguide mode is the TE$_{11}$ mode.



Sketch of field lines for TE$_{11}$ mode



*Here the term "higher-order modes" means the waveguide modes that exist in addition to the desired TEM mode.

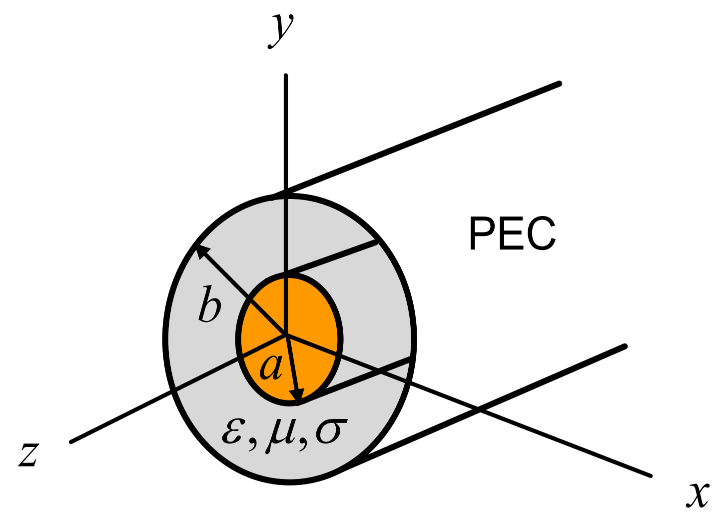
Coaxial Line: Higher-Order Modes (cont.)

TE$_z$:

$$\nabla^2 h_z(\rho,\phi) = -k_c^2 h_z(\rho,\phi)$$

eigenvalue problem

$$k_z^2 = k^2 - k_c^2$$



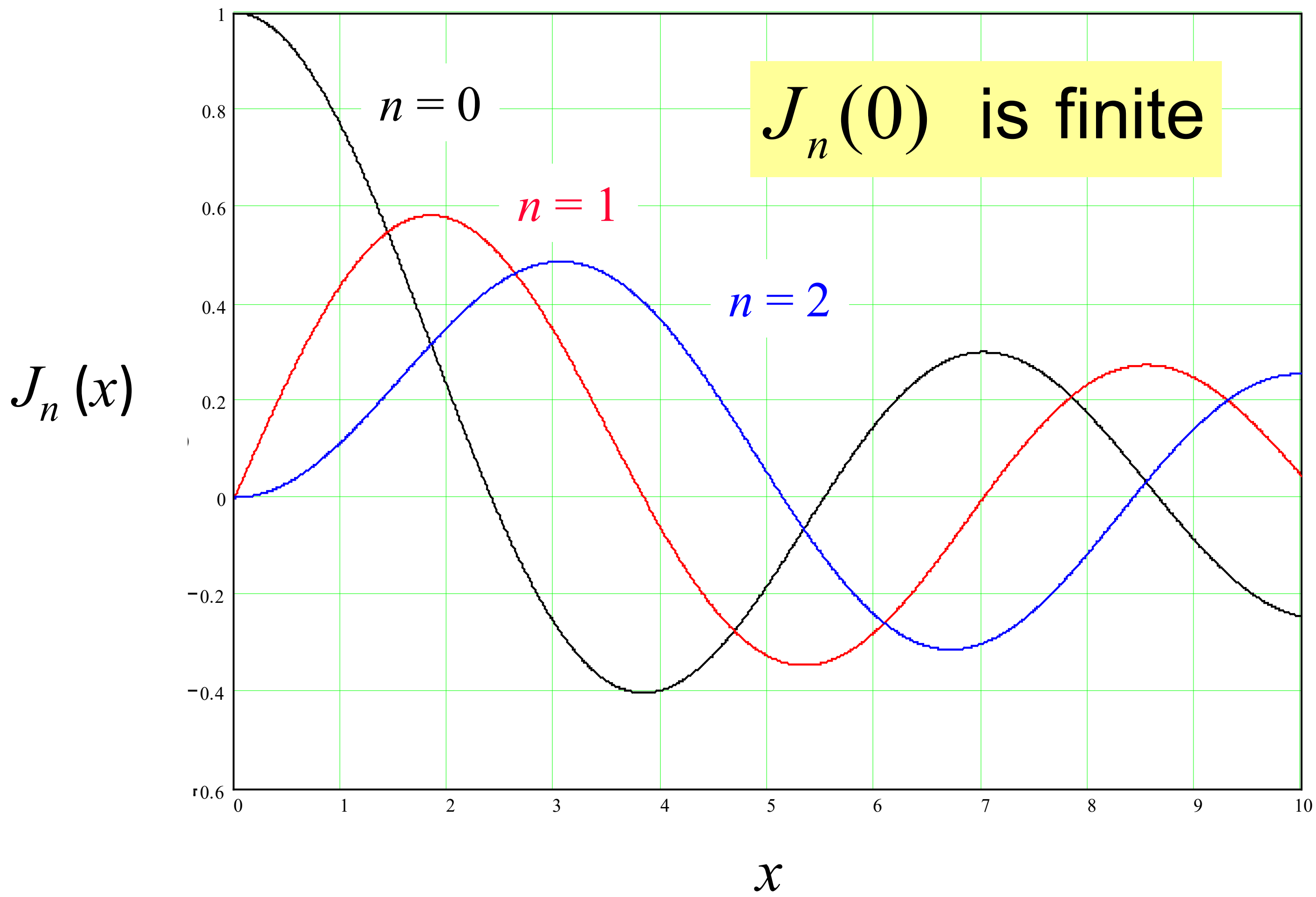
$$H_z(\rho,\phi,z) = h_z(\rho,\phi)\,e^{-jk_z z}$$

The solution in cylindrical coordinates is:

$$h_z(\rho,\phi) = \begin{Bmatrix} J_n(k_c\rho) \\ Y_n(k_c\rho) \end{Bmatrix} \begin{Bmatrix} \sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}$$

Note: The value n must be an integer to have unique fields.

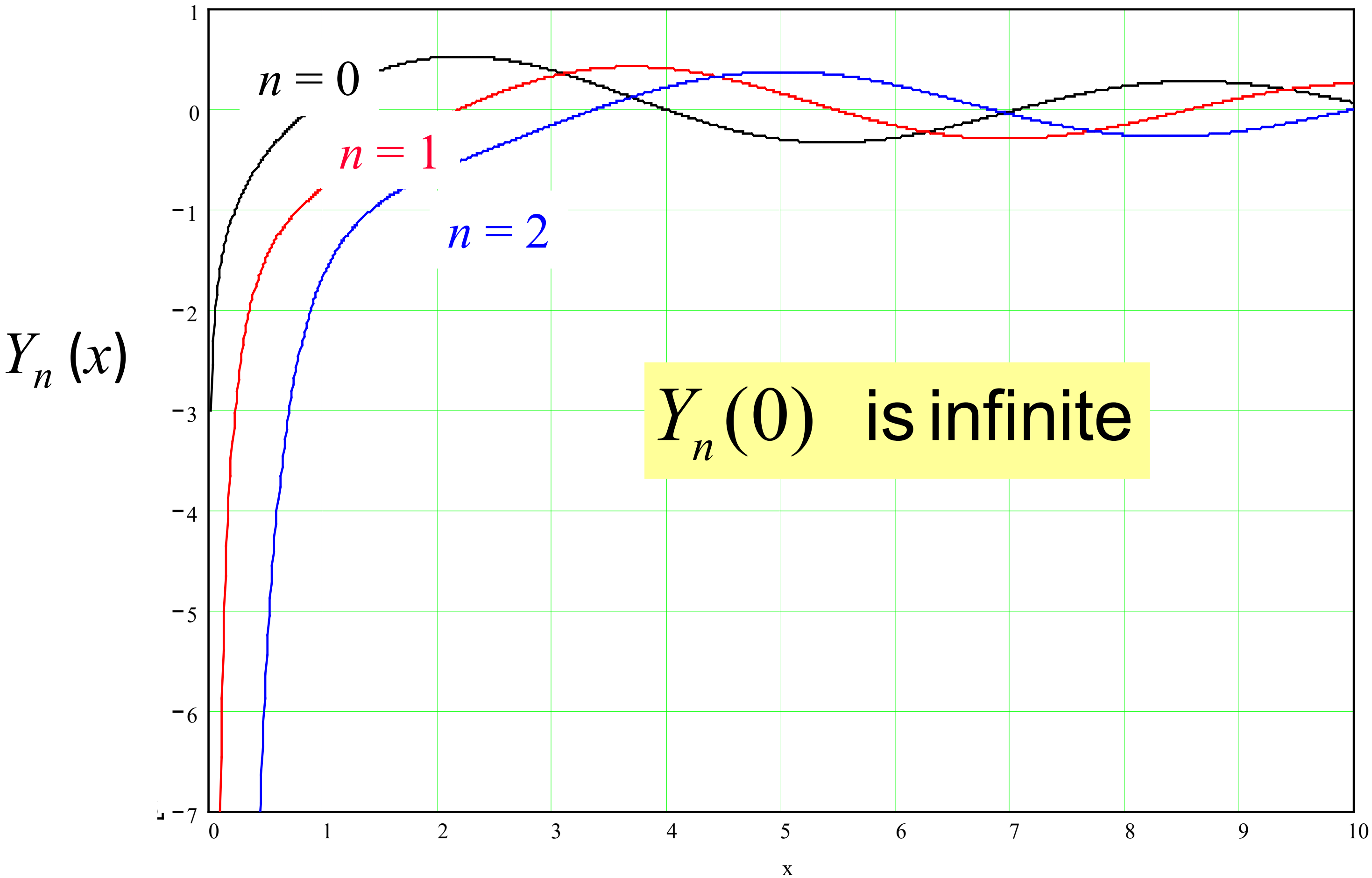
Plot of Bessel Functions



$$J_n(x) \sim x^n \left(\frac{1}{2^n n!} \right) \quad n = 0, 1, 2, \ldots, \quad x \to 0$$

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4} \right), \quad x \to \infty$$

Plot of Bessel Functions (cont.)



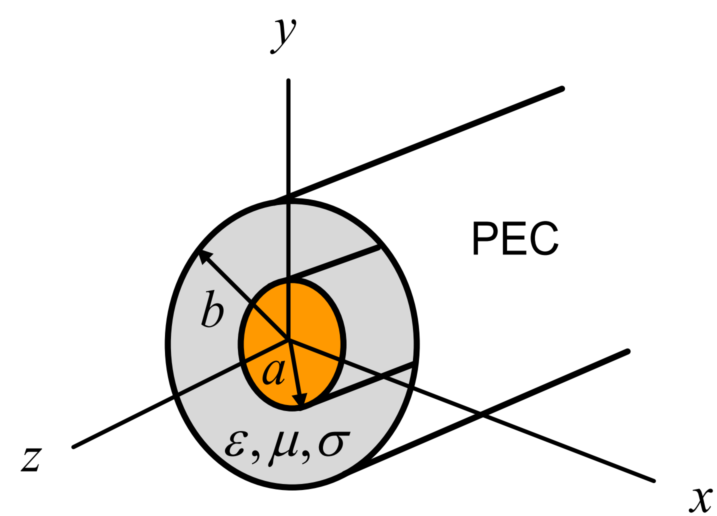
$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad x \to \infty$$

$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + \gamma\right], \; \gamma = 0.5772156, \; x \to 0$$

$$Y_n(x) \sim -\frac{1}{\pi}(n-1)!\left(\frac{2}{x}\right)^n, \; n = 1, 2, 3, \ldots\ldots, \; x \to 0$$

Coaxial Line: Higher-Order Modes (cont.)

We choose (somewhat arbitrarily) the cosine function for the angle variation.



Wave traveling in $+z$ direction:

$$h_z(\rho, \phi, z) = h_z(\rho, \phi) e^{-jk_z z}$$

$$h_z(\rho, \phi) = \cos(n\phi)\left(AJ_n(k_c\rho) + BY_n(k_c\rho)\right)$$

The cosine choice corresponds to having the transverse electric field E_ρ being an even function of ϕ, which is the field that would be excited by a probe located at $\phi = 0$.

Coaxial Line: Higher-Order Modes (cont.)

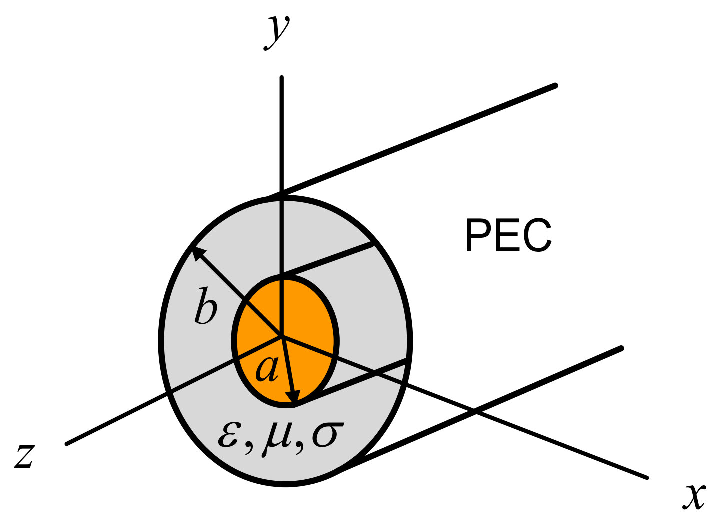
Boundary Conditions:

$$E_\phi\left(a,\phi\right)=0 \qquad E_\phi\left(b,\phi\right)=0$$

$$\left(H_\rho = 0\Big|_{\rho=a}\right)$$

$$E_\phi = \frac{1}{j\omega\varepsilon_c}\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)$$

$$\Rightarrow \frac{\partial H_z}{\partial \rho} = 0\Bigg|_{\rho=a,b}$$



Hence

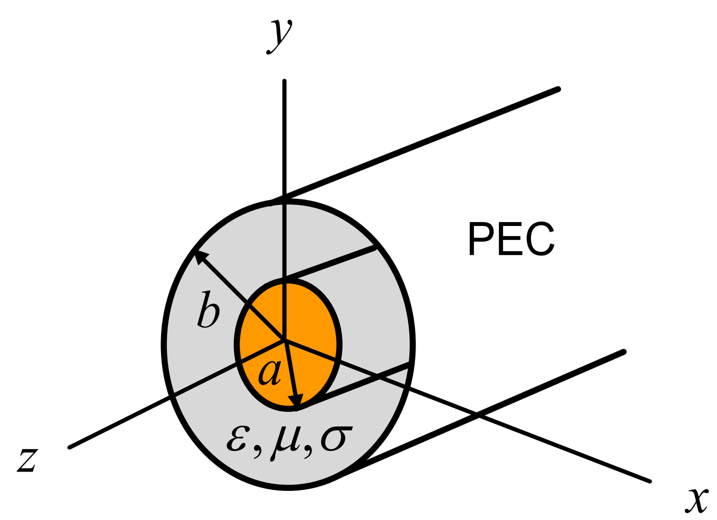
$$k_c\left(AJ_n'(k_c a) + BY_n'(k_c a)\right) = 0$$

$$k_c\left(AJ_n'(k_c b) + BY_n'(k_c b)\right) = 0$$

Note:
The prime denotes derivative with respect to the argument.

Coaxial Line: Higher-Order Modes (cont.)

$$AJ_n'(k_c a) + BY_n'(k_c a) = 0$$
$$AJ_n'(k_c b) + BY_n'(k_c b) = 0$$



In order for this homogenous system of equations for the unknowns A and B to have a non-trivial solution, we require the determinant to be zero.

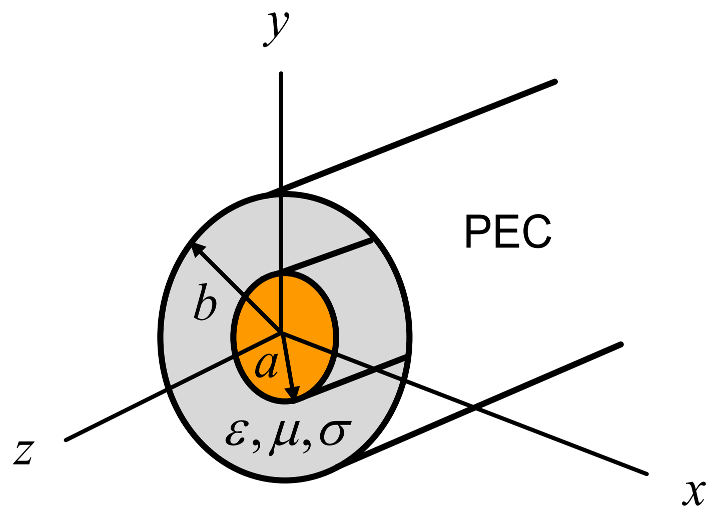
$$\mathrm{Det}\left(k_c\right) = \begin{vmatrix} J_n'(k_c a) & Y_n'(k_c a) \\ J_n'(k_c b) & Y_n'(k_c b) \end{vmatrix} = 0$$

Hence

$$J_n'(k_c a) Y_n'(k_c b) - J_n'(k_c b) Y_n'(k_c a) = 0$$

Coaxial Line: Higher-Order Modes (cont.)

$$J_n'(k_c a)Y_n'(k_c b) - J_n'(k_c b)Y_n'(k_c a) = 0$$



Denote

$$x = k_c a$$

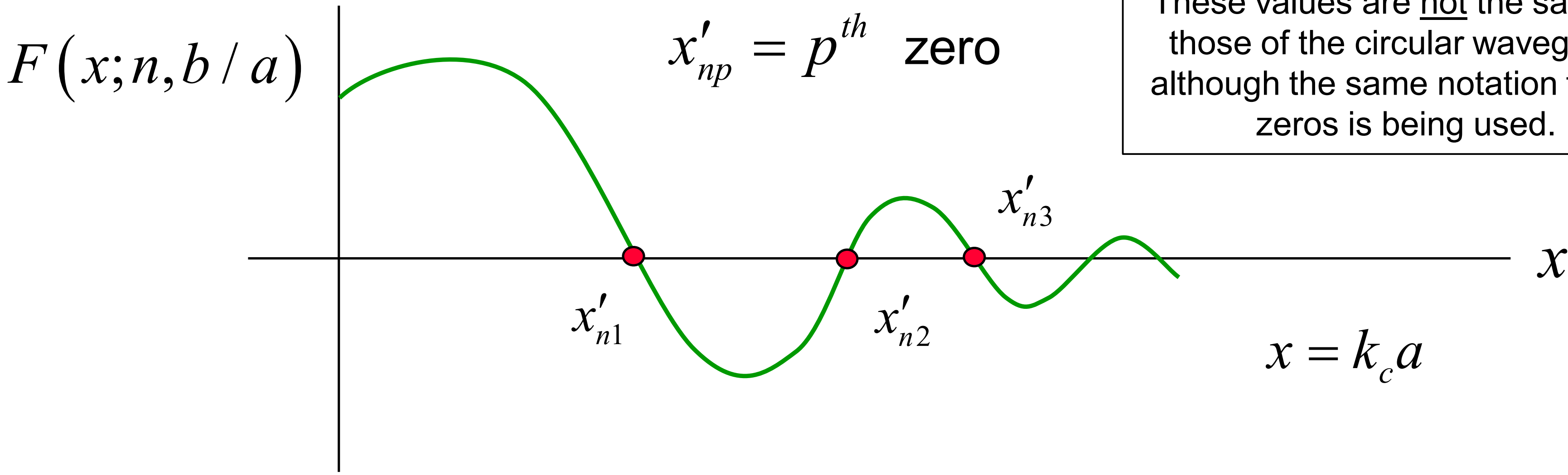
Then we have:

$$F(x;n,b/a) = J_n'(x)Y_n'\left(x\left(b/a\right)\right) - J_n'\left(x\left(b/a\right)\right)Y_n'(x) = 0$$

For a given choice of n and a given value of b/a, we can solve the above equation for x to find the zeros.

Coaxial Line: Higher-Order Modes (cont.)

A graph of the determinant reveals the zeros of the determinant.



Note:
These values are not the same as those of the circular waveguide, although the same notation for the zeros is being used.

$$x = x'_{np} \quad \Rightarrow \quad k_c a = x'_{np}$$

TE$_{11}$ mode: $k_c a = x'_{11}$

Coaxial Line: Higher-Order Modes (cont.)

Approximate solution:

$$k_c a = \frac{2}{1 + b/a}$$

The TE_{11} mode is the dominant higher-order mode of the coax (i.e., the waveguide mode with the lowest cutoff frequency).

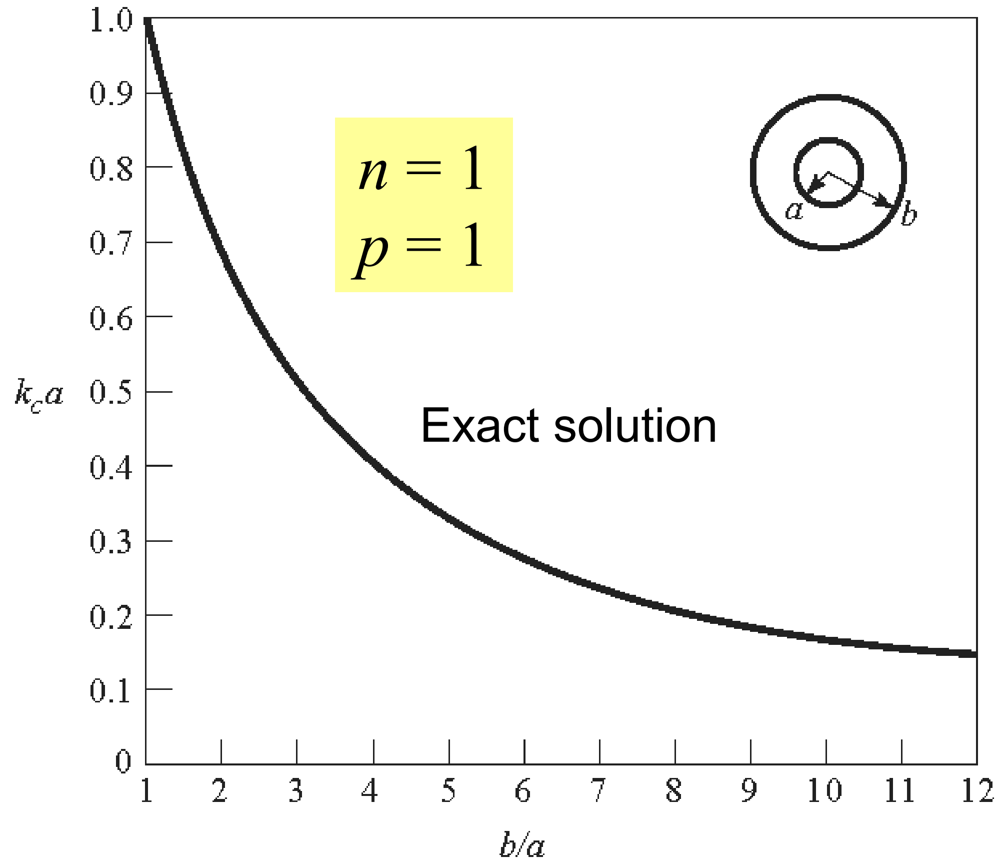


Figure 3.16 from the Pozar book

Cutoff Frequency of TE_{11} Mode

Lossless case:

$$\varepsilon_c = \varepsilon = \varepsilon'$$

$$k_z = \sqrt{k^2 - k_c^2} \quad (k \text{ is real here})$$

$$k\big|_{f=f_c} = k_c$$

$$\Rightarrow 2\pi f_c \sqrt{\mu\varepsilon} = k_c$$

Use formula on previous slide

$$f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi a}\frac{1}{\sqrt{\mu\varepsilon}}k_c a = \left(\frac{1}{2\pi a}\right)\frac{c}{\sqrt{\varepsilon_r}}k_c a \qquad c = 2.99792458 \times 10^8 \text{ [m/s]}$$

TE_{11} mode of coax:

$$f_c \approx \left(\frac{1}{2\pi a}\right)\frac{c}{\sqrt{\varepsilon_r}}\left(\frac{2}{1 + b/a}\right)$$

Coaxial Line: Lossless Case (cont.)

$$f_c \approx \frac{c}{a\sqrt{\varepsilon_r}}\left(\frac{1}{\pi}\right)\left(\frac{1}{1+b/a}\right)$$

At the <u>cutoff frequency</u>, the wavelength (in the dielectric) is then:

$$\lambda_d = \frac{c_d}{f} = \frac{c}{f_c\sqrt{\varepsilon_r}}$$

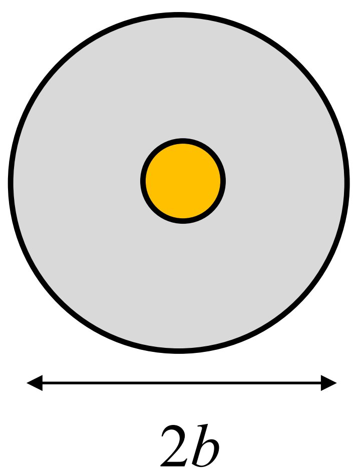
$$\approx \pi a\left(1+b/a\right)$$

$$= \pi\left(a+b\right)$$

$$\approx \pi b$$

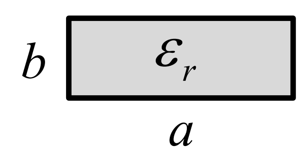
so

$$2b \approx \frac{\lambda_d}{\pi/2}$$



Compare with the cutoff frequency condition of the TE_{10} mode of RWG:

$$2a = \lambda_d$$



Example

Example 3.3, p. 133 of the Pozar book:

RG 142 coax:

$$a = 0.035 \text{ inches} = 8.89 \times 10^{-4} \text{ [m]}$$

$$b = 0.116 \text{ inches} = 29.46 \times 10^{-4} \text{ [m]}$$

$$\varepsilon_r = 2.2$$

$$\Rightarrow b / a = 3.31$$

$$f_c \approx \frac{c}{a\sqrt{\varepsilon_r}} \left(\frac{1}{\pi}\right) \left(\frac{1}{1 + b / a}\right)$$

$$f_c \approx 16.8 \text{ [GHz]}$$