Adapted from notes by
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ECE 5317-6351
Microwave Engineering

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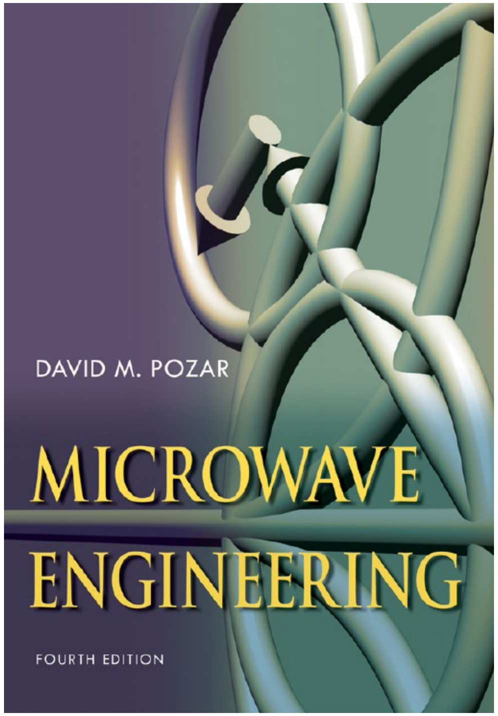
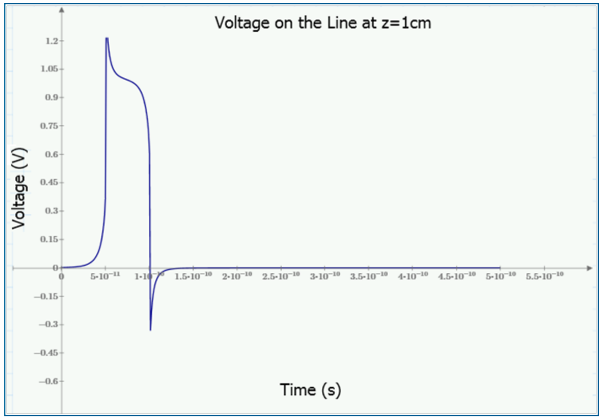
Dept. of ECE



Notes 13
Waveguiding Structures Part 8: Dispersion and Wave Velocities



Dispersion

Frequency dispersion: The phase velocity is not constant with frequency.

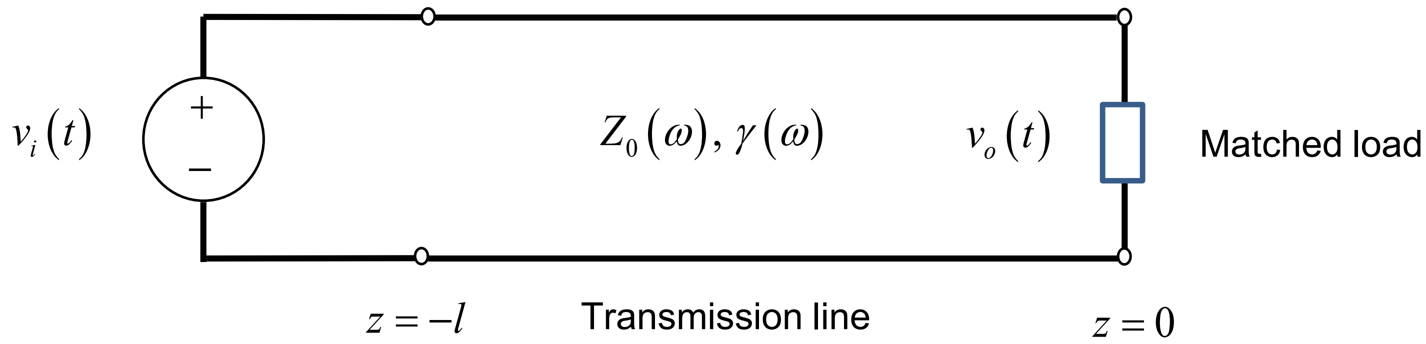
=> Different frequency components in the signal spectrum travel at different velocities.

In waveguiding structures, signal distortion is due to:

- Frequency dispersion
- Frequency-dependent attenuation
- Propagation of multiple modes that have different phase velocities

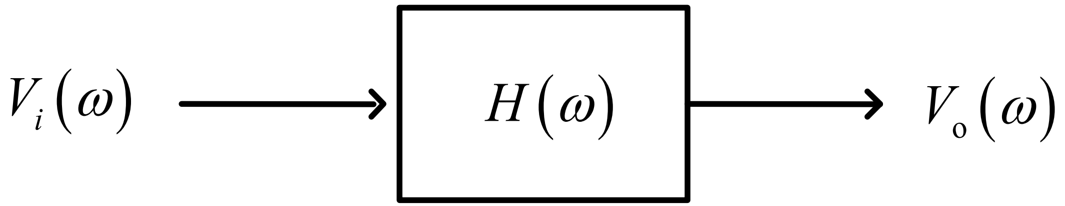
Voltage Signal on Transmission Line

Consider a signal applied to the input of a transmission line:



The transmission line as a transfer function in the phasor domain:

$$V_o(\omega) = H(\omega) V_i(\omega)$$



$$H(\omega) = e^{-\gamma(\omega)l} = A(\omega) e^{j\phi(\omega)} = e^{-\alpha(\omega)l} e^{-j\beta(\omega)l}$$

$$A(\omega) = e^{-\alpha(\omega)l}$$

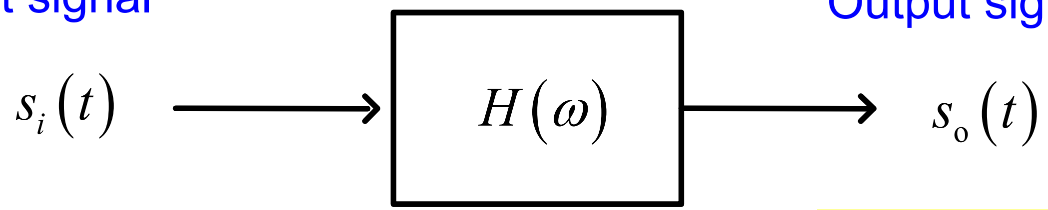
$$\phi(\omega) = -\beta(\omega)l$$

Signal Propagation

General System

Input signal

Output signal



$$H(\omega) \equiv \frac{S_o(\omega)}{S_i(\omega)}$$

$$\tilde{s}_i(\omega) = \int_{-\infty}^{\infty} s_i(t)\, e^{-j\omega t}\, dt$$

$$s_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega$$

Fourier transform pair

Note:
The signal will normally be assumed to represent the voltage.

Property of real-valued signal:

$$\tilde{s}_i(-\omega) = \tilde{s}_i^{*}(\omega)$$

(Please see the proof on the next slide.)

Signal Propagation (cont.)

Proof of Property:

$$\tilde{s}_i(-\omega) = \tilde{s}_i^*(\omega)$$

$$\tilde{s}_i(\omega) = \int_{-\infty}^{\infty} s_i(t) e^{-j\omega t} dt$$

$$s_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(\omega) e^{j\omega t} d\omega$$

Real signal : $s_i(t) = s_i^*(t)$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{s}_i(\omega)\right)^* e^{-j\omega t} d\omega$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(-\omega') e^{-j\omega' t} d\omega' = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{s}_i(\omega)\right)^* e^{-j\omega t} d\omega \qquad \left(\omega = -\omega'\right)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(-\omega) e^{-j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{s}_i(\omega)\right)^* e^{-j\omega t} d\omega \qquad \left(\omega' \to \omega\right)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(-\omega) e^{j\omega t'} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{s}_i(\omega)\right)^* e^{j\omega t'} d\omega \qquad \left(t = -t'\right)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{s}_i(-\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{s}_i(\omega)\right)^* e^{j\omega t} d\omega \qquad \left(t' \to t\right)$$

$$\Rightarrow F^{-1}\left\{\tilde{s}_i(-\omega)\right\} = F^{-1}\left\{\left(\tilde{S}_i(\omega)\right)^*\right\}$$

$$\Rightarrow \tilde{s}_i(-\omega) = \left(\tilde{s}_i(\omega)\right)^*$$

Signal Propagation (cont.)

We can then show:

$$s_i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad \rightarrow \quad s_i(t) = \frac{1}{\pi}\mathrm{Re}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega$$

(Please see the derivation on the next slide.)

The form on the right is convenient, since it only involves positive values of ω.
(In this case, ω has the nice interpretation of being radian frequency: $\omega = 2\pi f$.)

Signal Propagation (cont.)

Derivation:

$$s_i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad \text{(split integral into two parts)}$$

$$= \frac{1}{2\pi}(-1)\int_{\infty}^{0} \tilde{s}_i(-\omega')\, e^{-j\omega' t}\, d\omega' + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad (\omega = -\omega' \text{ in first integral})$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(-\omega')\, e^{-j\omega' t}\, d\omega' + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad \text{(reverse limits in first integral)}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(-\omega)\, e^{-j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad (\omega' \to \omega \text{ in first integral})$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \left(\tilde{s}_i(\omega)\right)^{*} e^{-j\omega t}\, d\omega' + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{s}_i(\omega)\, e^{j\omega t}\, d\omega \quad \text{(property of real - valued signal)}$$

$$= \frac{1}{2\pi}\left[\int_{0}^{\infty} \left(\tilde{s}_i(\omega)\, e^{j\omega t}\right)^{*} + \left(\tilde{s}_i(\omega)\, e^{j\omega t}\right)\right] d\omega \quad \text{(conjugate property)}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\left\{\tilde{s}_i(\omega)\, e^{j\omega t}\right\} d\omega \quad \text{(add the two integrands together)}$$

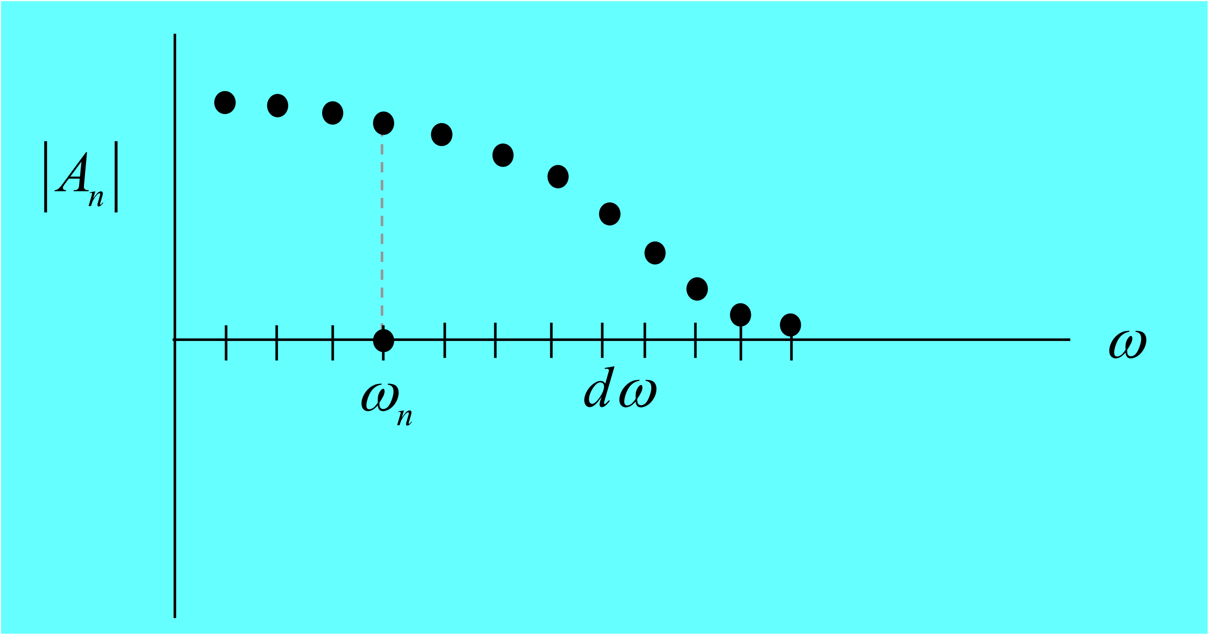
Signal Propagation (cont.)

Hence, we have

$$s_i(t) = \int_0^{\infty} \mathrm{Re}\left\{ \left(\frac{1}{\pi} \tilde{s}_i(\omega)\, d\omega \right) e^{j\omega t} \right\}$$

Interpreted as a phasor.

$$A_n = \text{complex amplitude of phasor } \#n = \frac{1}{\pi} \tilde{s}_i(\omega)\, d\omega$$



Signal Propagation (cont.)

$$s_i(t) = \int_0^\infty \mathrm{Re}\left\{ \left(\frac{1}{\pi} \tilde{s}_i(\omega)\, d\omega \right) e^{j\omega t} \right\}$$

Using the transfer function, we have:

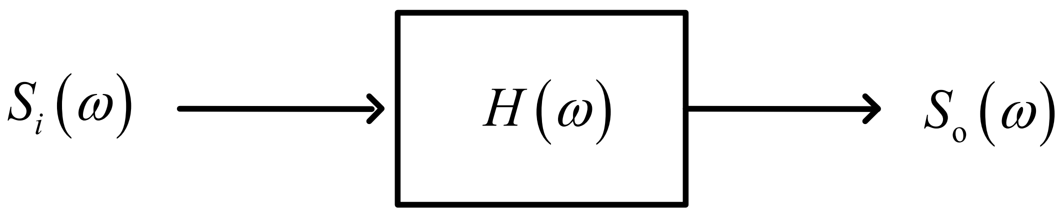
$$s_o(t) = \int_0^\infty \mathrm{Re}\left\{ H(\omega) \left(\frac{1}{\pi} \tilde{s}_i(\omega)\, d\omega \right) e^{j\omega t} \right\}$$

$$= \frac{1}{\pi} \mathrm{Re} \int_0^\infty H(\omega) \tilde{s}_i(\omega) e^{j\omega t}\, d\omega$$

Signal Propagation (cont.)

Summary

$$s_{o}(t)=\frac{1}{\pi}\operatorname{Re}\int_{0}^{\infty}H(\omega)\tilde{s}_{i}(\omega)e^{j\omega t}\,d\omega$$



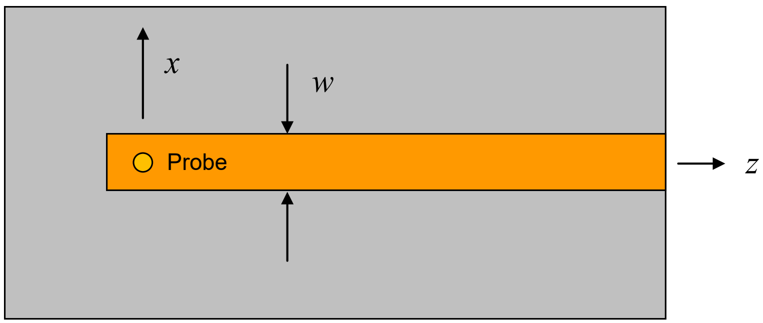
Phasor domain

Waveguiding system:

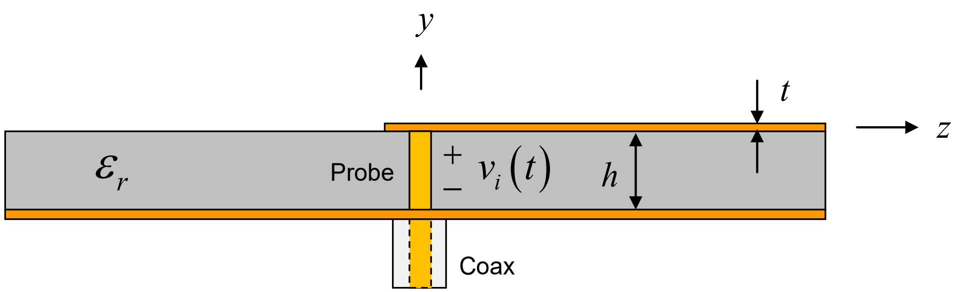
$$H(\omega)=A(\omega)e^{j\phi(\omega)}=e^{-\alpha(\omega)l}e^{-j\beta(\omega)l}$$

Signal Propagation (cont.)

Propagation on a microstrip line

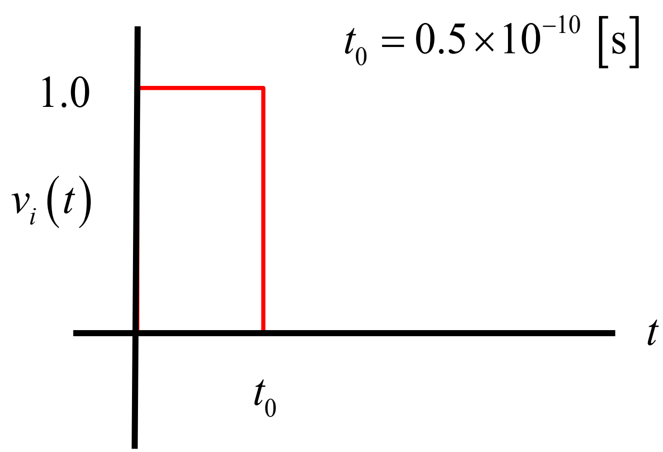


TOP VIEW



SIDE VIEW

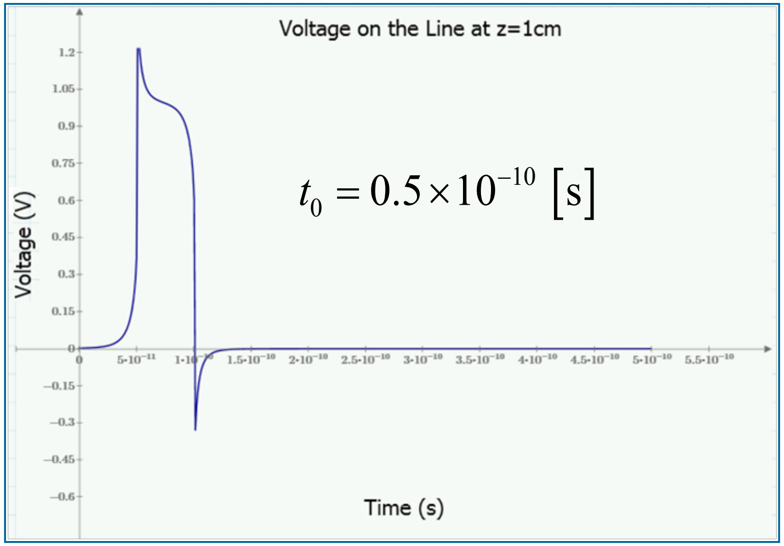
$$\varepsilon_r = 2.33$$
$$\tan\delta = 0.001$$
$$h = 0.787\ [\text{mm}]\ (31\,\text{mils})$$
$$w = 2.35\ [\text{mm}]$$
$$t = 0.0175\,[\text{mm}]\ (\text{"half oz" copper cladding})$$
$$\sigma = 3.0\times 10^{7}\ [\text{S/m}]$$

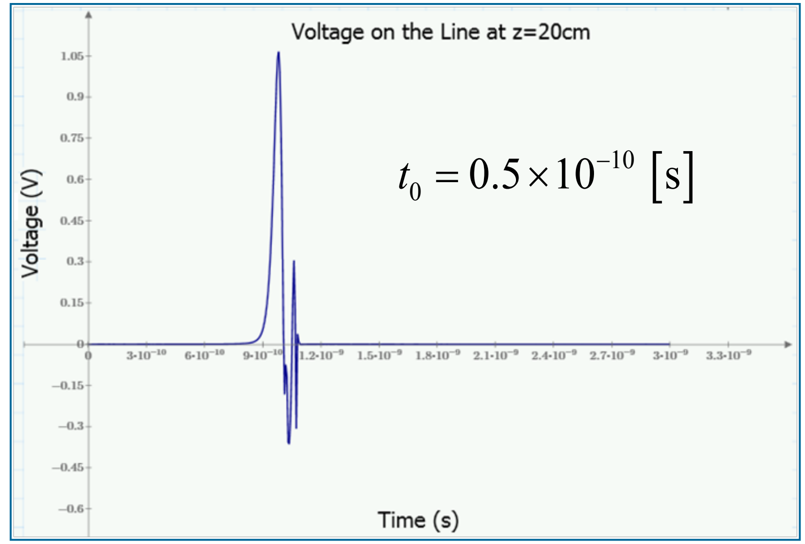


CAD formulas are used to get $\alpha(\omega)$, $\beta(\omega)$.

Signal Propagation (cont.)

Propagation on a microstrip line





$\varepsilon_r = 2.33$

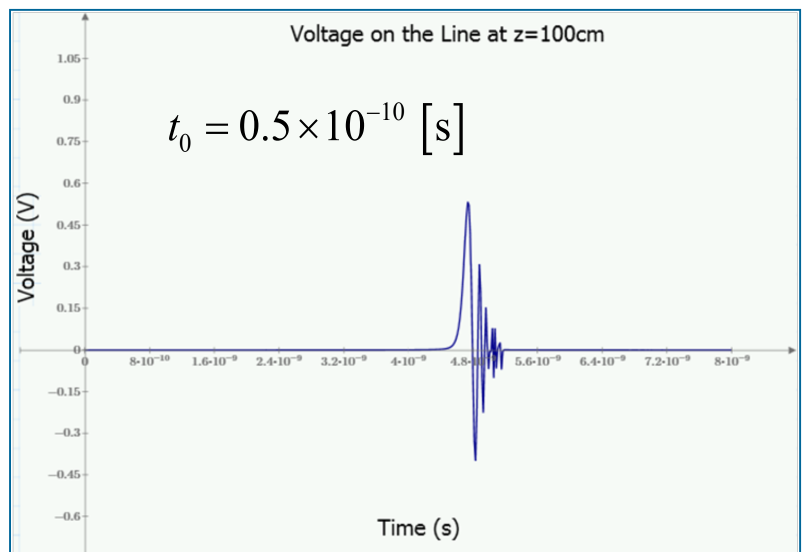
$\tan\delta = 0.001$

$h = 0.787\ [\text{mm}]\ (31\,\text{mils})$

$w = 2.35\ [\text{mm}]$

$t = 0.0175\,[\text{mm}]$ ("half oz" copper cladding)

$\sigma = 3.0\times10^{7}\,[\text{S/m}]$



Dispersionless System

Dispersionless System with Constant Attenuation

$$H(\omega) = A(\omega)e^{-j\beta(\omega)l}$$

$$A(\omega) \approx \text{constant} \equiv A_0 = e^{-\alpha(\omega_0)l}$$

$$\Rightarrow s_o(t) = \frac{1}{\pi}\int_0^\infty \text{Re}\left\{A_0 e^{-j\beta(\omega)l}\tilde{s}_i(\omega)e^{j\omega t}\right\}d\omega$$

$$= A_0 \frac{1}{\pi}\text{Re}\int_0^\infty \tilde{s}_i(\omega)e^{j\omega\left(t-\frac{l}{v_{p0}}\right)}d\omega$$

$$v_p \equiv \frac{\omega}{\beta} = \text{constant} = v_{p0}$$

No dispersion:
The phase velocity is constant (not a function of frequency).

$$s_o(t) = A_0 s_i\left(t - \frac{l}{v_{p0}}\right)$$

The output is a delayed and scaled version of the input.

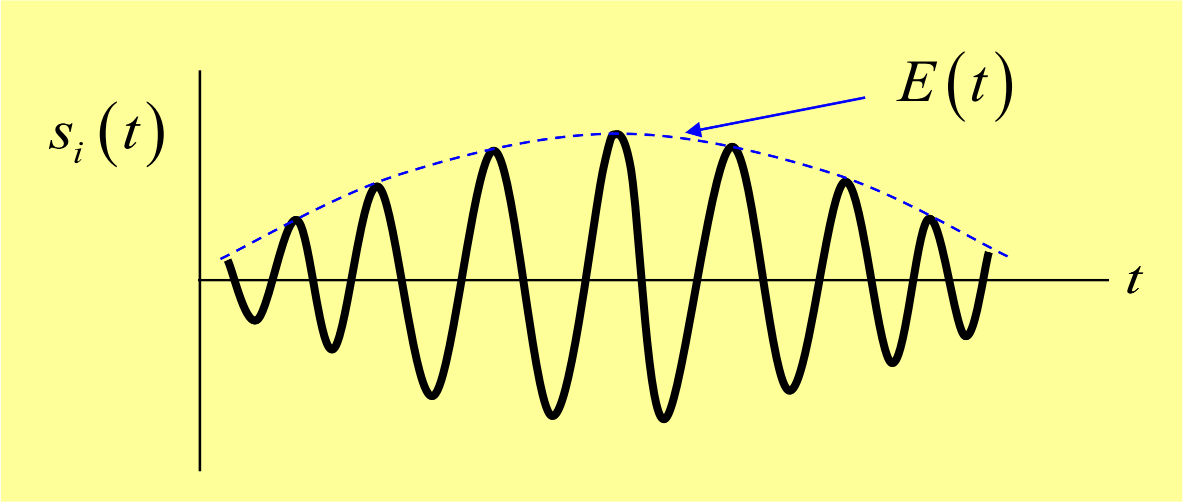
The output signal has no distortion.

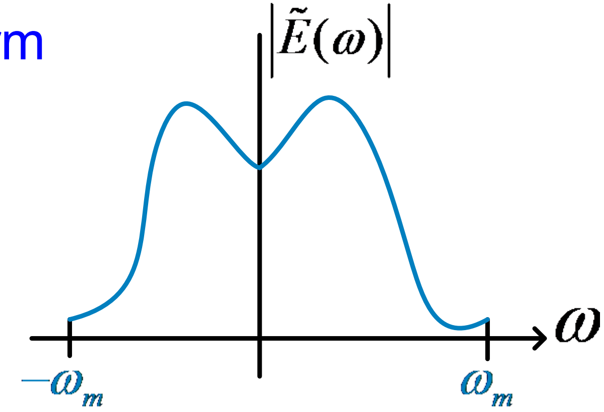
Narrow-Band Signal

Low-Loss System with Dispersion and Narrow-Band Signal

Now consider a narrow-band input signal of the form

$$s_i(t) = E(t)\cos(\omega_0 t)$$





Narrow band

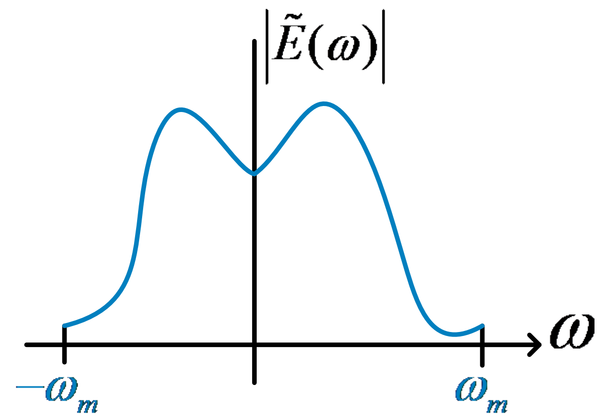
$$\Rightarrow \omega_m << \omega_0$$

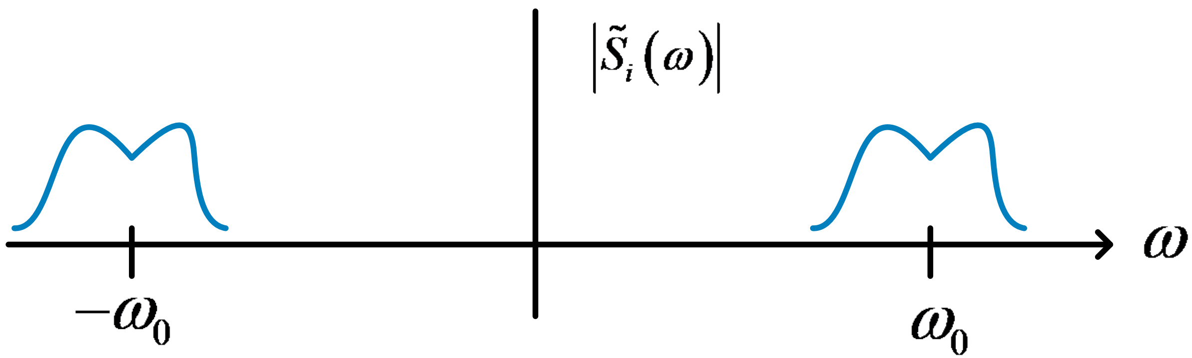
Physically, the envelope is slowing varying compared with the carrier.

Narrow-Band Signal (cont.)

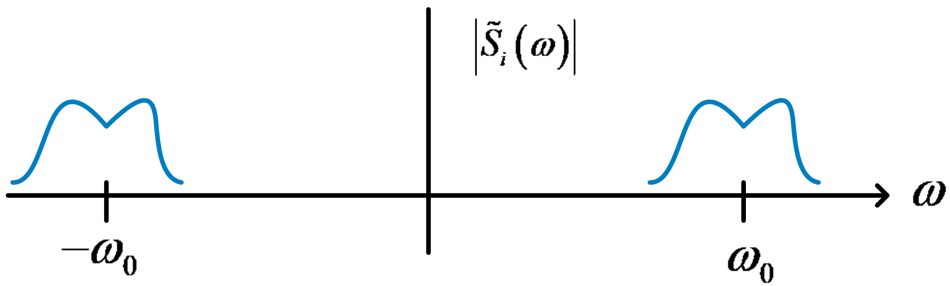
$$s_i(t) = E(t)\cos(\omega_0 t)$$

$$\tilde{s}_i(\omega) = F\left\{E(t)\cos(\omega_0 t)\right\}$$
$$= \frac{1}{2}F\left\{E(t)e^{j\omega_0 t}\right\} + \frac{1}{2}F\left\{E(t)e^{-j\omega_0 t}\right\}$$
$$= \frac{1}{2}\left[\tilde{E}(\omega-\omega_0) + \tilde{E}(\omega+\omega_0)\right]$$





Narrow-Band Signal (cont.)



$$\tilde{s}_i(\omega) = \frac{1}{2}\left[\tilde{E}(\omega-\omega_0) + \tilde{E}(\omega+\omega_0)\right]$$

Hence, we have:

$$s_o(t) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} A(\omega)\,\tilde{s}_i(\omega)e^{-j\beta(\omega)l}e^{j\omega t}d\omega \qquad \left(A(\omega) = e^{-\alpha(\omega)l}\right)$$

$$\approx \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} A(\omega)\frac{1}{2}\tilde{E}(\omega-\omega_0)e^{j(\omega t-\beta(\omega)l)}d\omega \qquad \left(\text{Ignore } \tilde{E}(\omega+\omega_0)\right)$$

$$= \frac{1}{2\pi}\mathrm{Re}\int_0^{\infty} A(\omega)\,\tilde{E}(\omega-\omega_0)e^{j(\omega t-\beta(\omega)l)}d\omega$$

Narrow-Band Signal (cont.)

Since the signal is narrow band, using a Taylor series expansion about ω_0 results in:

$$\beta(\omega) \approx \underbrace{\beta(\omega_0)}_{\beta_0} + \underbrace{\left.\frac{d\beta}{d\omega}\right|_{\omega_0}}_{\beta_0'} (\omega - \omega_0) + \ldots \approx \beta_0 + \beta_0'(\omega - \omega_0)$$

$$A(\omega) \approx \underbrace{A(\omega_0)}_{A_0} + \underbrace{\left.\frac{dA}{d\omega}\right|_{\omega_0} (\omega - \omega_0)}_{\text{neglect}} + \ldots \approx A_0$$

Low loss assumption

$$A(\omega) = e^{-\alpha(\omega)z}$$

$$\frac{dA(\omega)}{d\omega} = -z \frac{d\alpha(\omega)}{d\omega} e^{-\alpha(\omega)z}$$

Small

Narrow-Band Signal (cont.)

Thus,

$$s_o(t) \approx \frac{1}{2\pi}\,\mathrm{Re}\int_0^{\infty} A(\omega)\,\tilde{E}(\omega-\omega_0)\,e^{j(\omega t-\beta(\omega)l)}\,d\omega$$

$$\approx \frac{1}{2\pi}\,\mathrm{Re}\int_0^{\infty} A_0\tilde{E}(\omega-\omega_0)e^{j\omega t}e^{-j\beta_0 l}e^{-j\beta_0'(\omega-\omega_0)l}\,d\omega \qquad \text{(use Taylor approximation)}$$

$$= \frac{A_0}{2\pi}\,\mathrm{Re}\left[e^{-j\beta_0 l}e^{j\omega_0 t}\int_0^{\infty}\tilde{E}(\omega-\omega_0)e^{-j\beta_0'(\omega-\omega_0)l}e^{j(\omega-\omega_0)t}\,d\omega\right] \qquad \left(\text{multiply and divide by } e^{j\omega_0 t}\right)$$

$$= \frac{A_0}{2\pi}\,\mathrm{Re}\left[e^{j(\omega_0 t-\beta_0 l)}\int_{-\omega_0}^{\infty}\tilde{E}(\omega_s)e^{-j\beta_0'\omega_s l}e^{j\omega_s t}\,d\omega_s\right] \qquad \left(\omega_s \equiv \omega-\omega_0\right)$$

$$\approx \frac{A_0}{2\pi}\,\mathrm{Re}\left[e^{j(\omega_0 t-\beta_0 l)}\int_{-\infty}^{\infty}\tilde{E}(\omega_s)e^{-j\beta_0'\omega_s l}e^{j\omega_s t}\,d\omega_s\right] \qquad \left(\text{extend lower limit to } -\infty\right)$$

The spectrum of $E(t)$ is concentrated near $\omega = 0$.

Narrow-Band Signal (cont.)

$$s_o(t) \approx \frac{A_0}{2\pi} \mathrm{Re}\left[e^{j(\omega_0 t - \beta_0 l)} \int_{-\infty}^{\infty} \tilde{E}(\omega_s) e^{j\omega_s (t - \beta_0' l)} \, d\omega_s \right]$$

$$= A_0 \,\mathrm{Re}\left[e^{j(\omega_0 t - \beta_0 l)} E(t - \beta_0' l) \right]$$

$$= A_0 \cos\left(\omega_0 \left(t - \frac{\beta_0}{\omega_0} l \right) \right) E(t - \beta_0' l)$$

Define:

$v_p \equiv \dfrac{\omega_0}{\beta_0}$ phase velocity @ ω_0

Define:

$v_g \equiv \dfrac{1}{\beta_0'} = \left(\dfrac{d\omega}{d\beta} \right)\Bigg|_{\omega_0}$ group velocity @ ω_0

Narrow-Band Signal (cont.)

Summary

$$s_o(t) \approx A_0 E\left(t - \frac{l}{v_g}\right)\cos\left(\omega_0\left(t - \frac{l}{v_p}\right)\right)$$

Envelope travels with group velocity

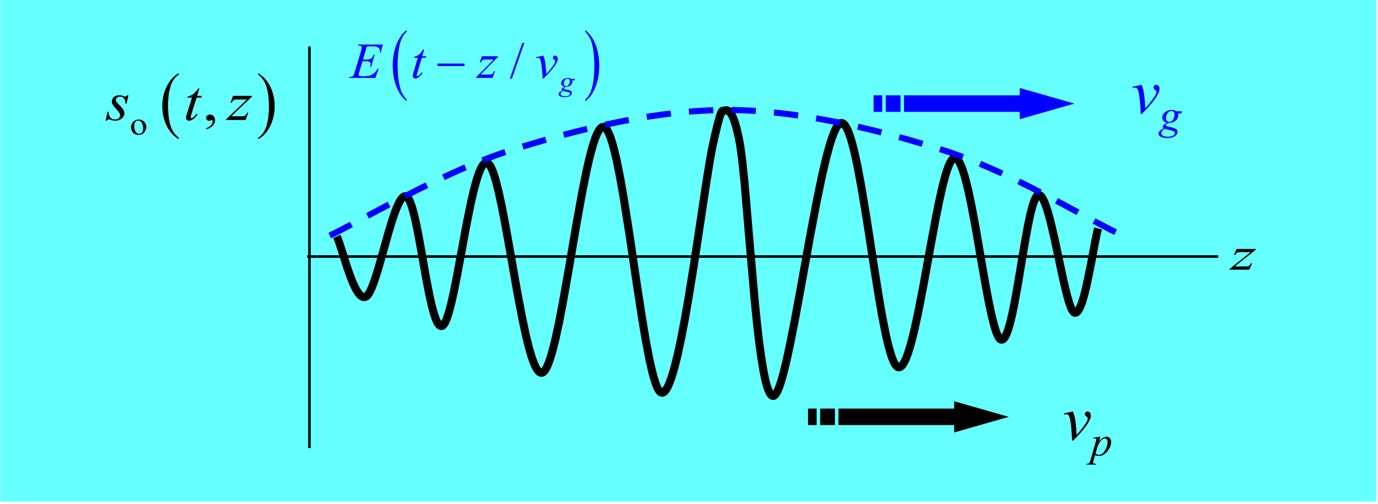
Carrier travels with phase velocity

$$v_p \equiv \frac{\omega_0}{\beta_0}$$

$$v_g \equiv \left.\left(\frac{d\omega}{d\beta}\right)\right|_{\omega_0}$$

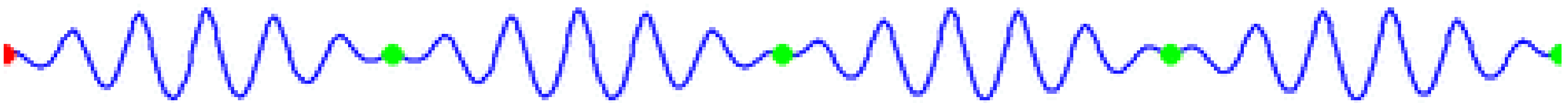
Narrow-Band Signal (cont.)

$$s_o(t) \approx A_0 E\left(t - \frac{l}{v_g}\right)\cos\left(\omega_0\left(t - \frac{l}{v_p}\right)\right)$$



Narrow-Band Signal (cont.)

Example from Wikipedia (view in full-screen mode with pptx)



Red dot: phase velocity

Green dot: group velocity

Phase velocity > group velocity

http://en.wikipedia.org/wiki/Group_velocity

Narrow-Band Signal (cont.)

Note on dispersion

Assume:

No dispersion

$$\Rightarrow \quad v_p = v_g$$

In this case the envelope and carrier are delayed the same.

Example: lossless transmission line

Proof

Assume :

$$v_p = \frac{\omega}{\beta} = \text{constant} = c_1$$

$$\Rightarrow \omega = c_1 \beta$$

$$\Rightarrow \frac{d\omega}{d\beta} = c_1$$

$$\Rightarrow v_g = c_1$$

Example: TE_{10} Mode of Rectangular Waveguide

Recall: $$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2}$$

Phase velocity: $$v_p = \frac{\omega}{\beta}$$

Group velocity: $$v_g = \frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1}$$

After simple calculation:

$$v_p = \frac{\omega}{\sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2}}$$

$$v_g = \frac{1}{\omega\mu\varepsilon}\sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2}$$

Observation:

$$v_p v_g = \frac{1}{\mu\varepsilon} = c_d^2$$

(This final result is valid for any mode of a lossless waveguide.)

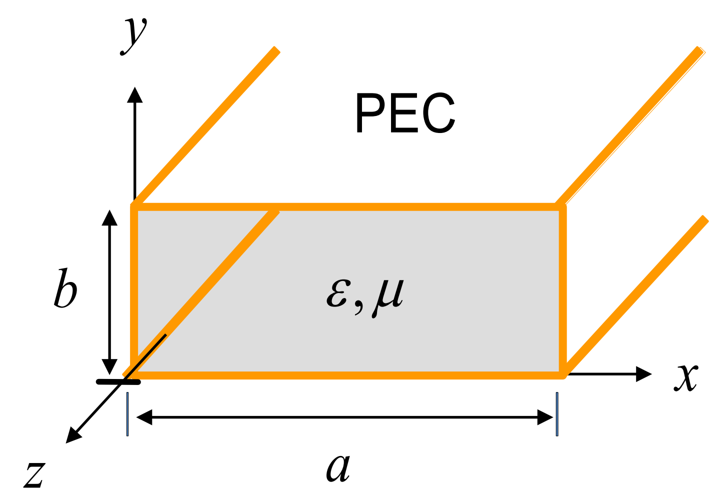
Example (cont.)

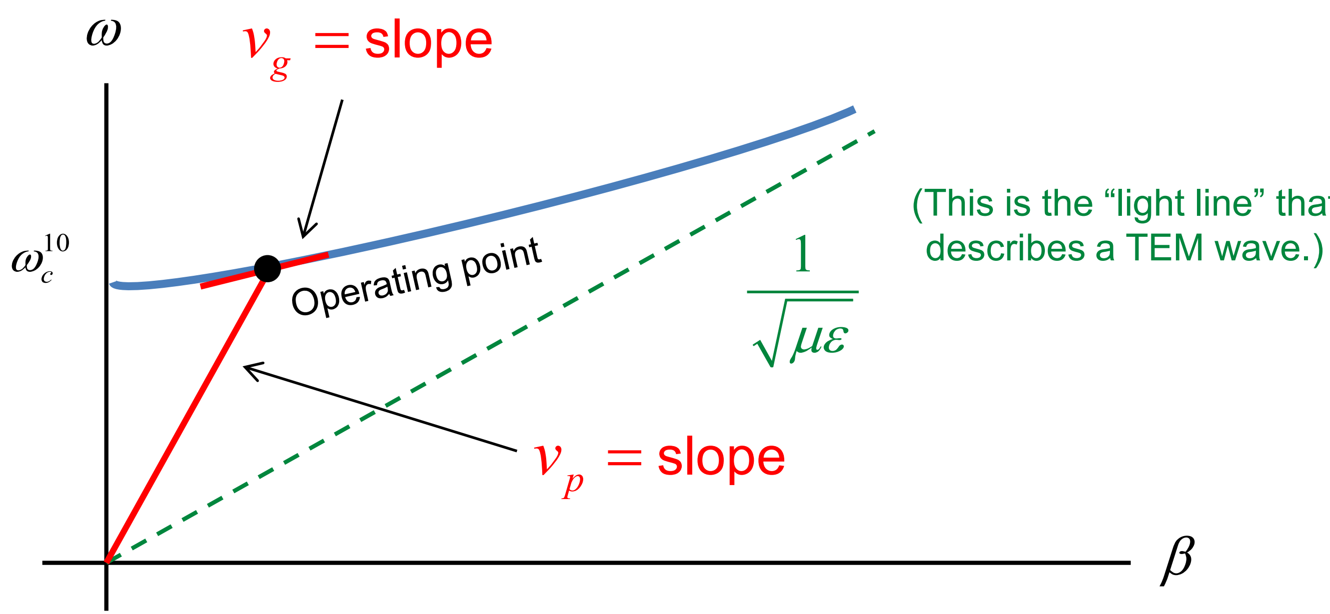
Lossless Case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$f > f_c$

$$v_p = \frac{\omega}{\beta}$$

$$v_g = \frac{d\omega}{d\beta}$$





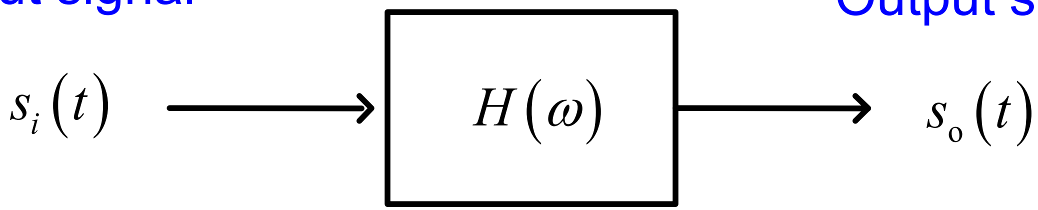
$v_p > c_d$

$v_g < c_d$

Filter Response

Input signal

Output signal



$$H(\omega) \equiv \frac{S_o(\omega)}{S_i(\omega)}$$ (phasor domain)

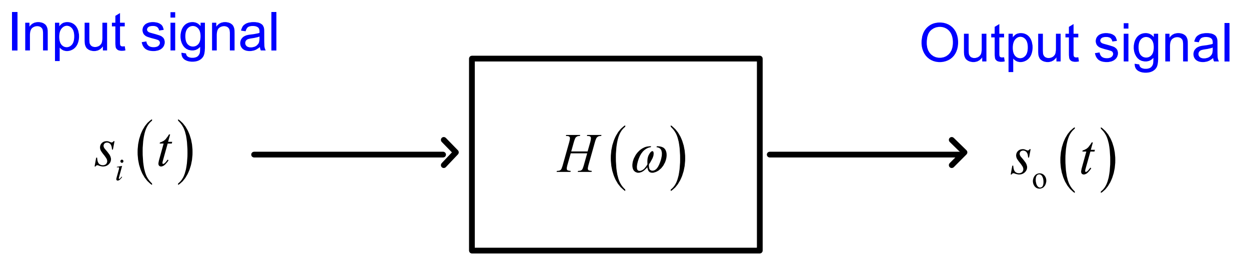
What we have done also applies to a filter, but here we use the transfer function phase directly, and do not introduce a phase constant β.

$$H(\omega) = A(\omega)e^{j\phi(\omega)}$$

From the previous results, we have

$$s_o(t) = \frac{1}{\pi}\int_0^\infty \text{Re}\left\{A(\omega)e^{j\phi(\omega)}\tilde{s}_i(\omega)e^{j\omega t}\right\}d\omega$$

Filter Response (cont.)



$$H(\omega) = A(\omega)e^{j\phi(\omega)}$$

Let $\beta l \rightarrow -\phi$

$$s_o(t) \approx A_0\, E(t - \beta_0' l)\cos\left(\omega_0\left(t - \frac{\beta_0}{\omega_0} l\right)\right)$$

Assume we have our modulated input signal:

$$s_i(t) = E(t)\cos(\omega_0 t)$$

The output is:

$$s_o(t) \approx A_0 E(t + \phi_0')\cos\left(\omega_0\left(t + \frac{\phi_0}{\omega_0}\right)\right)$$

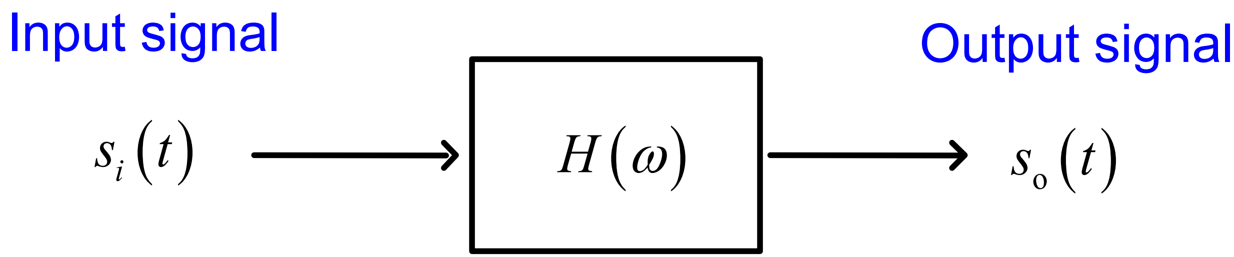
where

$$A_0 = A(\omega_0)$$

$$\phi_0 = \phi(\omega_0)$$

$$\phi_0' = \left.\frac{d\phi}{d\omega}\right|_{\omega_0}$$

Filter Response (cont.)



$$s_o(t) \approx A_0 E(t + \phi_0') \cos\left(\omega_0 \left(t + \frac{\phi_0}{\omega_0}\right)\right)$$

This motivates the following definitions:

Phase delay:

$$\tau_p \equiv -\frac{\phi_0}{\omega_0}$$

Group delay:

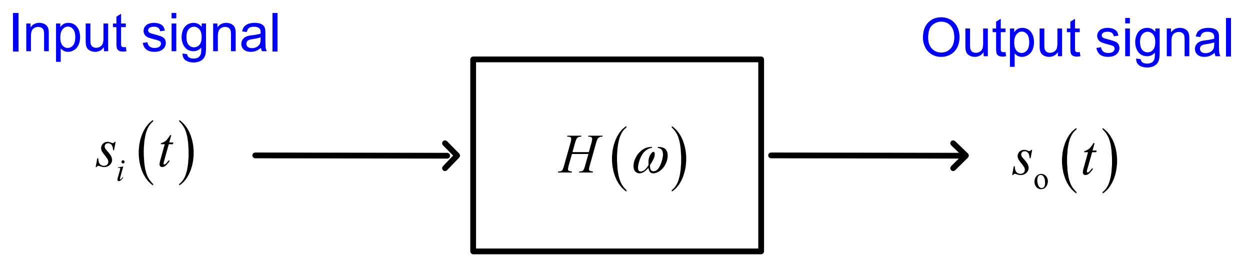
$$\tau_g \equiv -\left.\frac{d\phi}{d\omega}\right|_{\omega_0}$$

(The units are seconds.)

$$\Rightarrow \quad s_o(t) \approx A_0 E(t - \tau_g) \cos\left(\omega_0 \left(t - \tau_p\right)\right)$$

Filter Response (cont.)

Summary



$$H(\omega) = A(\omega)e^{j\phi(\omega)}$$

$$s_i(t) = E(t)\cos(\omega_0 t)$$

$$s_o(t) \approx A_0 E(t-\tau_g)\cos\left(\omega_0\left(t-\tau_p\right)\right)$$

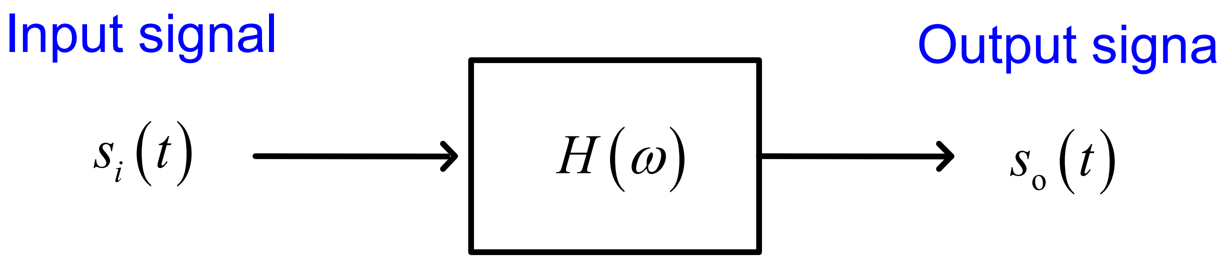
Phase delay:

$$\tau_p \equiv -\frac{\phi_0}{\omega_0}$$

Group delay:

$$\tau_g \equiv -\left.\frac{d\phi}{d\omega}\right|_{\omega_0}$$

Linear-Phase Filter Response



$$H(\omega) = A(\omega)e^{j\phi(\omega)}$$

Ideal linear phase filter: $\phi(\omega) = -\tau\,\omega$ $\quad A(\omega) \approx A_0$

The attenuation of the ideal filter is constant, at least over the bandwidth of the filter.

Recall: $$s_o(t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left\{A(\omega)e^{j\phi(\omega)}\tilde{s}_i(\omega)e^{j\omega t}\right\}d\omega$$

Hence
$$s_o(t) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left\{\left(A_0 e^{-j\tau\omega}\right)\tilde{s}_i(\omega)e^{j\omega t}\right\}d\omega$$

$$\phi(\omega) = -\tau\,\omega$$

$$\Rightarrow -\frac{\phi}{\omega} = -\frac{d\phi}{d\omega} = \tau$$

$\Rightarrow \tau_p = \tau_g = \tau =$ constant

The envelope and carrier are delayed the same.

Linear-Phase Filter Response (cont.)

We then have:

$$s_o(t) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\left(A_0 e^{-j\tau\omega}\right)\tilde{s}_i(\omega)e^{j\omega t}\right\}d\omega$$

$$= \frac{1}{\pi}A_0\,\mathrm{Re}\left\{\int_0^\infty \tilde{s}_i(\omega)e^{j\omega(t-\tau)}d\omega\right\}$$

$$= A_0 s_i\left(t-\tau\right)$$

so

$$s_o(t) = A_0 s_i\left(t-\tau\right)$$

An ideal linear-phase filter does not distort the signal.

It may be desirable to have a filter maintain a linear phase, at least over the bandwidth of the filter. This will tend to minimize signal distortion.