Adapted from notes by
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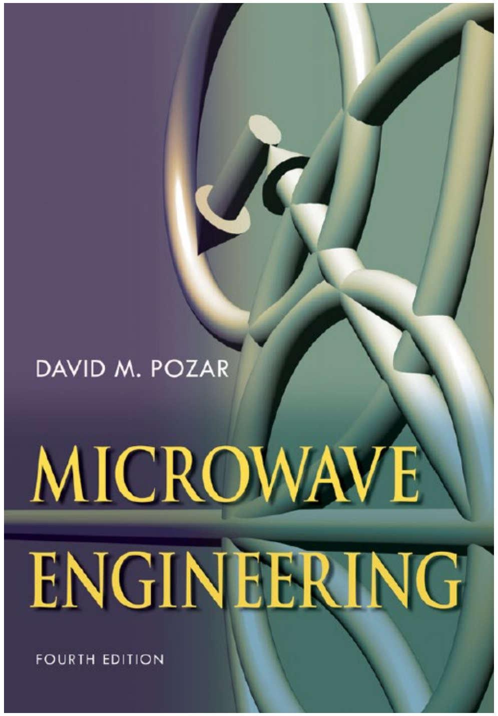

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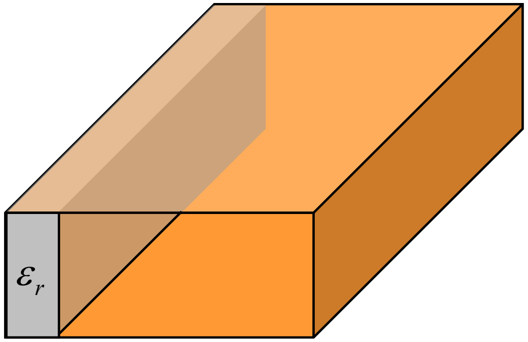
Transverse Resonance Method



Transverse Resonance Method

This is a general method that can be used to help us calculate various important quantities:

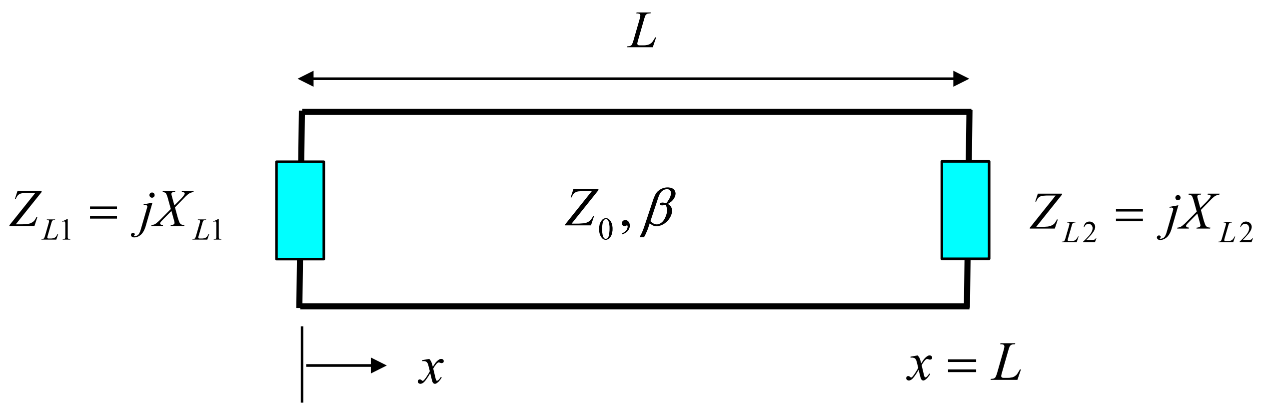
- Wavenumbers for complicated waveguiding structures (dielectric-loaded waveguides, surface waves, etc.)
- Resonance frequencies of resonant cavities (resonators)

The transverse resonance method involves establishing a reference plane and enforcing the KVL and KCL.

This leads to a "Transverse Resonance Equation (TRE)."

Transverse Resonance Method

To illustrate the method, consider a lossless resonator formed by a lossless transmission line with reactive loads at the ends.



Transmission line resonator

We wish to find the resonance frequencies of this transmission-line resonator.

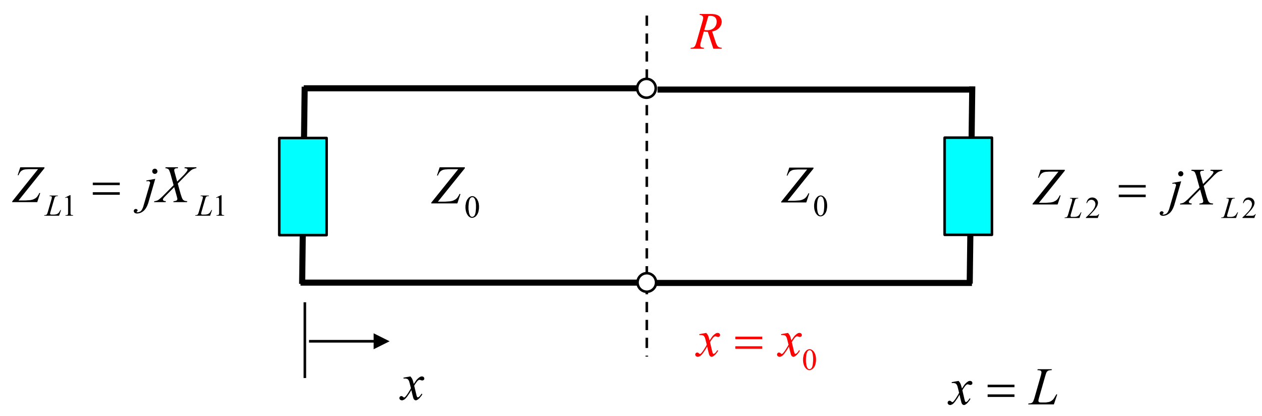
(Here we develop the method. We will do the actual algebra for this structure a little later as an example.)

A resonator can have nonzero fields at a resonance frequency, when there is no source.

Note: The transmission line in the model might be a TEN model for a waveguide type of problem.

Transverse Resonance Method (cont.)

We start by selecting an (arbitrary) reference plane R.

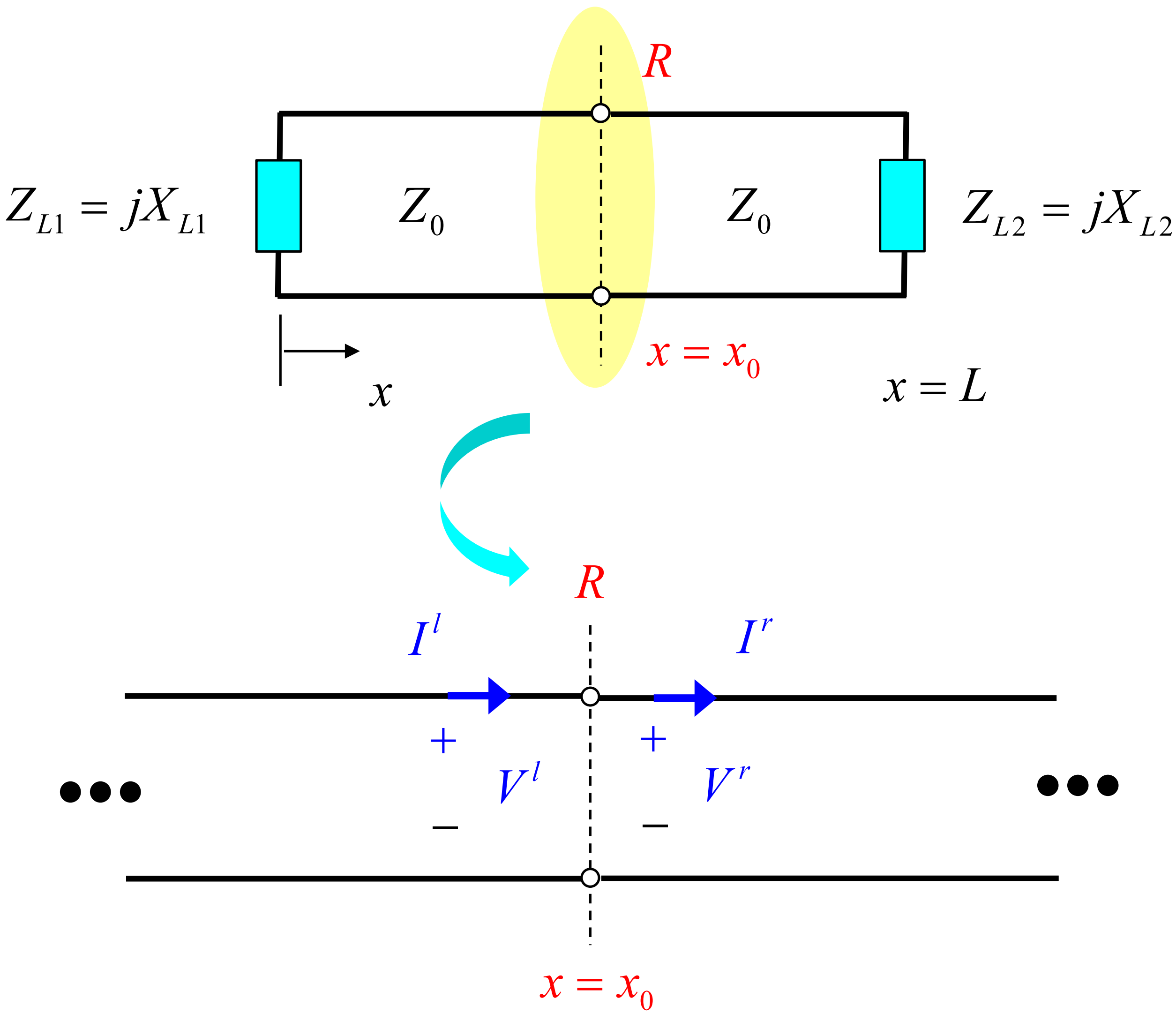


R = reference plane at arbitrary $x = x_0$

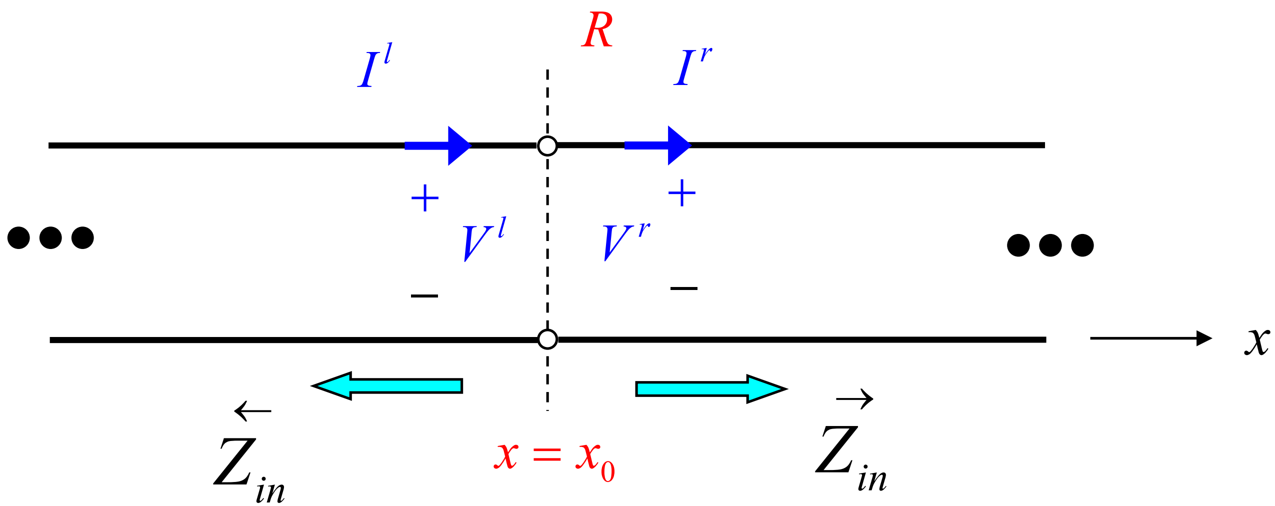
Note:
Although the location of the reference plane is arbitrary, a good choice will often simplify the derivation of the TRE and the complexity of the final TRE.

Transverse Resonance Method (cont.)

Examine the voltages and currents at the reference plane:



Transverse Resonance Method (cont.)



Define impedances:

$$\overrightarrow{Z}_{in} = \frac{V^r}{I^r}$$

$$\overleftarrow{Z}_{in} = \frac{V^l}{-I^l}$$

Boundary conditions:

$$V^r = V^l$$

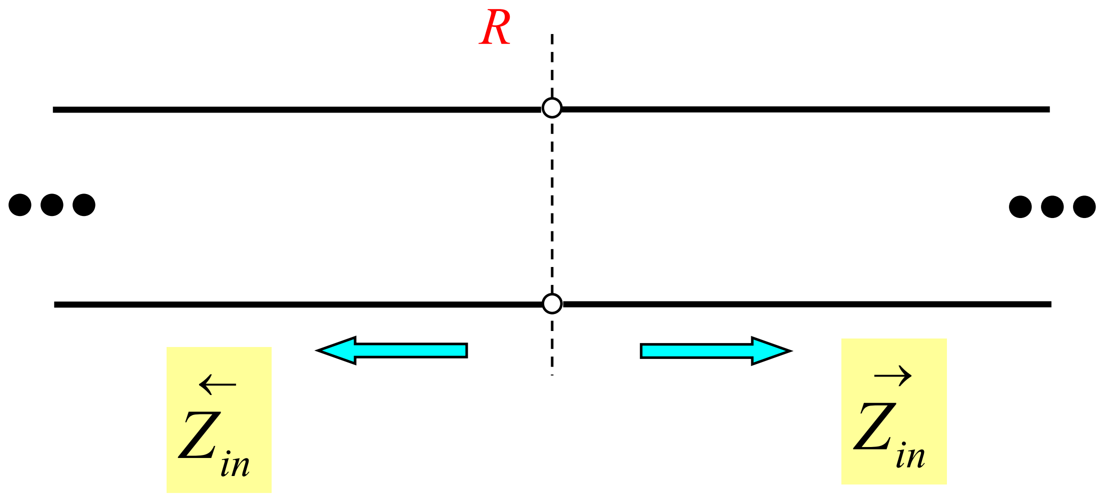
$$I^r = I^l$$

Hence:

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

TRE

Summary



TRE:

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

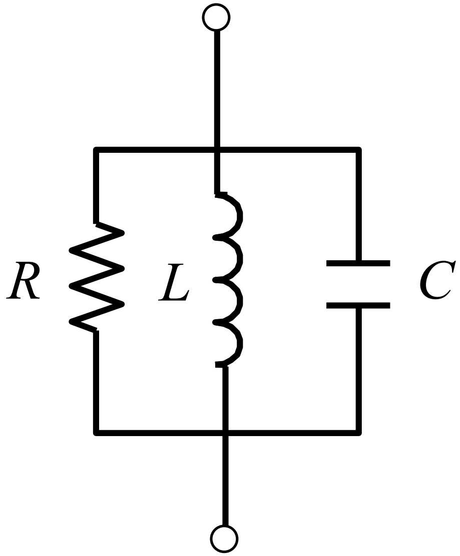
or

$$\overleftarrow{Y}_{in} = -\overrightarrow{Y}_{in}$$

RLC Resonator

Example:

Derive the resonance frequency of a parallel RLC resonator.



Note:
A complex resonance frequency corresponds to the fields being damped (decaying) with time.

$$\omega_0 = \omega_0' + j\omega_0''$$

$$\omega_0' = 2\pi f_0'$$

$$\omega = \omega_0 \;\; (\text{complex resonance frequency})$$

At the resonance frequency, voltages and currents exist with no sources.

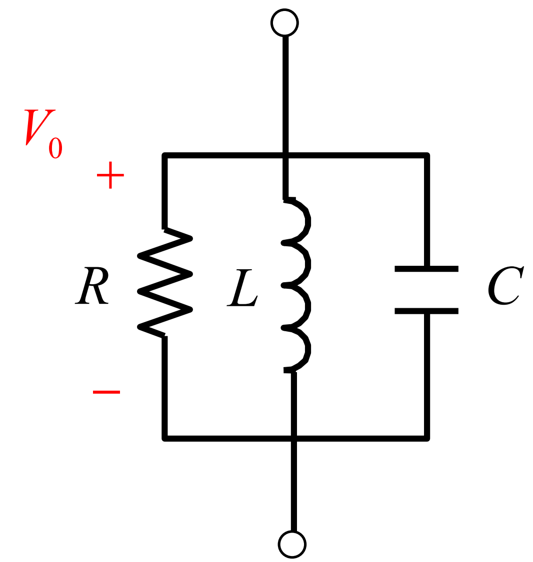
RLC Resonator (cont.)

In the time domain we have:

$$v(t) = \mathrm{Re}\left(V_0 e^{j\omega_0 t}\right)$$

$$= \mathrm{Re}\left(|V_0| e^{j\phi}\, e^{j\omega_0' t} e^{-\omega_0'' t}\right)$$

$$= |V_0| e^{-\omega_0'' t} \cos\left(\omega_0' t + \phi\right)$$



$$\omega_0 = \omega_0' + j\omega_0''$$

$$v(t) = |V_0| e^{-\omega_0'' t} \cos\left(\omega_0' t + \phi\right)$$

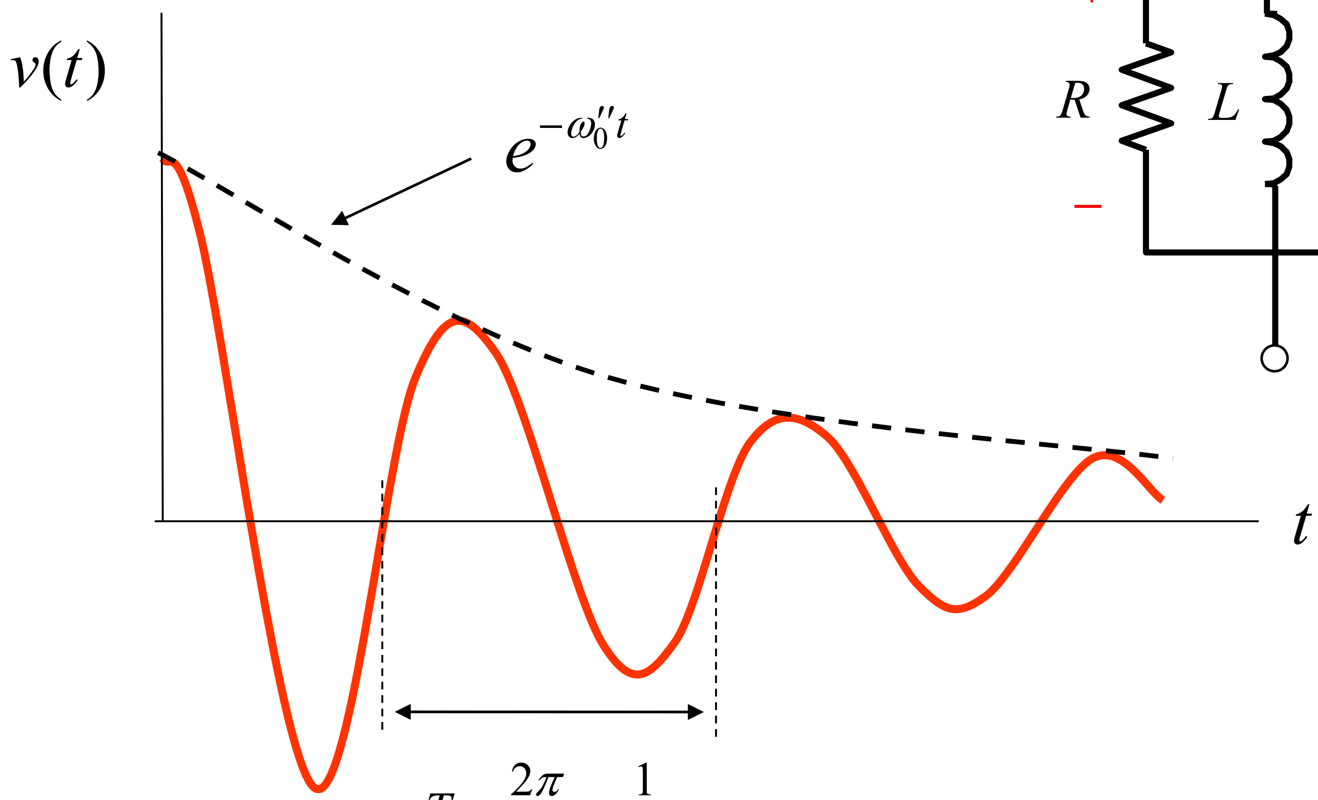
In the phasor domain:

$$V_0 = |V_0| e^{j\phi}$$

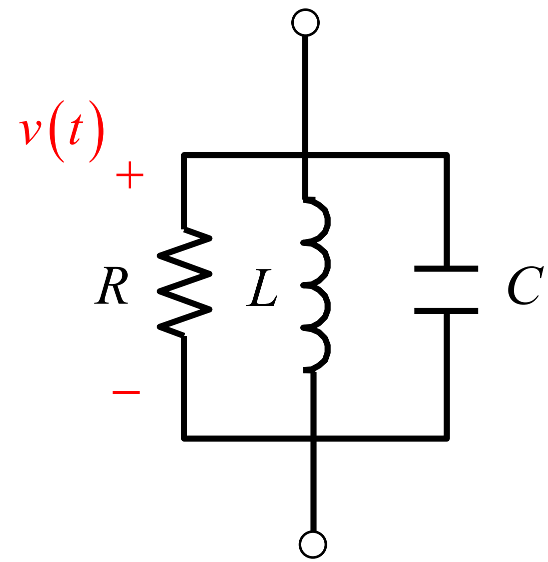
Note: The phasor domain concept applies to complex frequencies as well as real frequencies.

RLC Resonator (cont.)

$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$



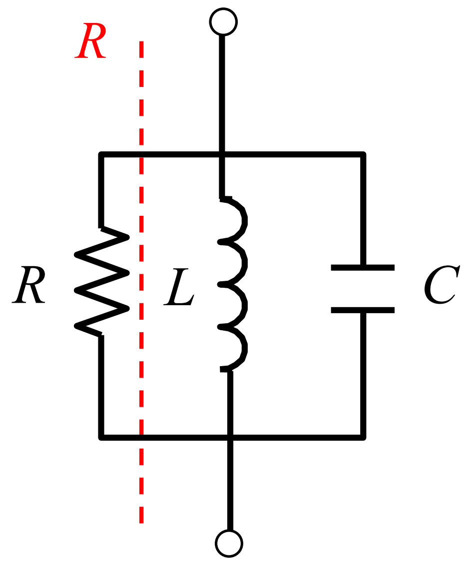
$$T = \frac{2\pi}{\omega_0'} = \frac{1}{f_0'}$$



RLC Resonator (cont.)

Apply the Transverse Resonance Equation (TRE):

❖ A reference plane is first chosen (arbitrary).

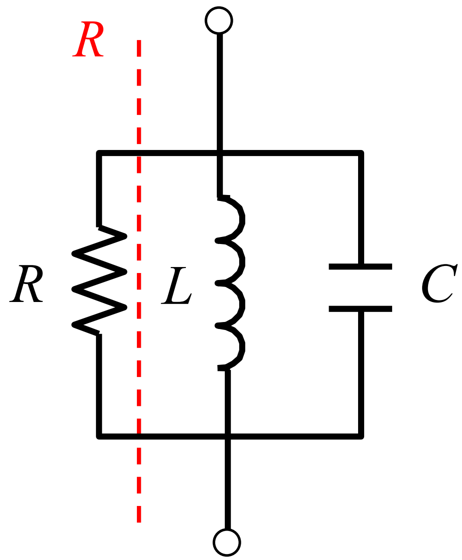


RLC Resonator (cont.)

The TRE is obtained.

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

$$\overleftarrow{Y}_{in} = -\overrightarrow{Y}_{in}$$



$$G = \frac{1}{R}$$

$$\Rightarrow G = -\left[\frac{1}{j\omega_0 L} + j\omega_0 C\right]$$

RLC Resonator (cont.)

$$G = -\left[\frac{1}{j\omega_0 L} + j\omega_0 C\right]$$

⇓

$$-j\omega_0 LG = 1 - \omega_0^2 LC$$

⇓

$$\omega_0^2 \left(LC\right) + \omega_0 \left(-jLG\right) - 1 = 0$$

⇓ Solve the quadratic equation.

$$\omega_0 = \frac{jLG \pm \sqrt{-L^2G^2 + 4LC}}{2LC}$$

RLC Resonator (cont.)

$$\omega_0 = \frac{jLG \pm \sqrt{4LC - L^2G^2}}{2LC}$$

Factor out $4LC$ from the square root.

$$\omega_0 = j\left(\frac{G}{2C}\right) \pm \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{LG^2}{4C}}$$

For the lossless limit, $G \to 0$:

$$\omega_0 \to +\frac{1}{\sqrt{LC}}$$

(must be a positive real number)

Hence, the plus sign is the correct choice.

RLC Resonator (cont.)

Hence, we have

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{LG^2}{4C}} + j\left(\frac{G}{2C}\right)$$

Complex resonance frequency

We can write this as

$$\omega_0 = \omega_0' + j\omega_0''$$

where

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{LG^2}{4C}}$$

$$\omega_0'' = \left(\frac{G}{2C}\right)$$

RLC Resonator (cont.)

Ratio of imaginary and real parts of complex frequency:

$$\omega_0' = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{LG^2}{4C}} \approx \frac{1}{\sqrt{LC}}$$

$$\omega_0'' = \left(\frac{G}{2C}\right)$$

so

$$\frac{\omega_0'}{\omega_0''} = \left(\frac{2C}{G}\right)\frac{1}{\sqrt{LC}} = 2R\sqrt{\frac{C}{L}} = \frac{2R}{L}\sqrt{LC} = 2\left(\frac{R}{\omega_0' L}\right)$$

or

$$\frac{\omega_0'}{\omega_0''} = 2Q$$ where $$Q \equiv \frac{R}{\omega_0' L}$$

Quality factor of RLC resonator

Q of a General Resonator

The quality factor (Q) for a general resonator is defined as:

$$Q \equiv \omega_0' \frac{U}{P_d^{ave}}$$

$$U = U_E + U_H = \text{energy stored}$$

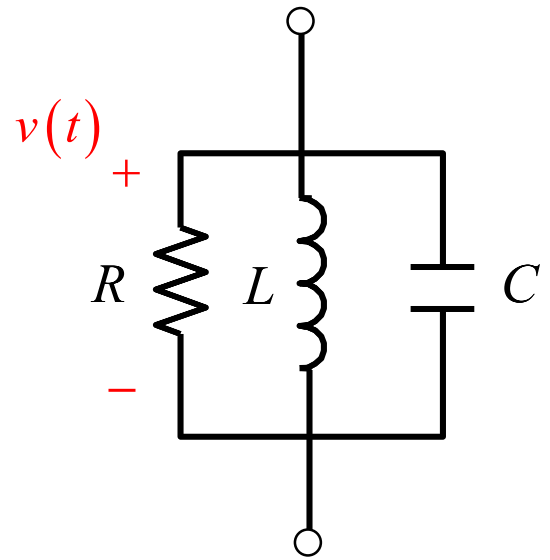
$$P_d^{ave} = \text{average power dissipated}$$

Note:
ω_0' is often denoted simply as ω_0 in this equation.

Q for RLC Resonator

For the RLC resonator we have:

$$Q \approx \frac{1}{\sqrt{LC}} \frac{U}{P_d^{ave}}$$



$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

$$P_d^{ave} = \langle G v^2(t) \rangle = G \langle v^2(t) \rangle$$

$$\langle v(t) v(t) \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)^2 \, dt$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} |V_0|^2 e^{-2\omega_0'' t} \cos^2(\omega_0' t + \phi) \, dt$$

$$\approx \frac{1}{T} |V_0|^2 e^{-2\omega_0'' t_0} \int_{t_0}^{t_0+T} \cos^2(\omega_0' t + \phi) \, dt$$

$$= \frac{1}{T} |V_0|^2 e^{-2\omega_0'' t_0} \left(\frac{T}{2} \right)$$

$$= \frac{1}{2} |V_0|^2 e^{-2\omega_0'' t_0}$$

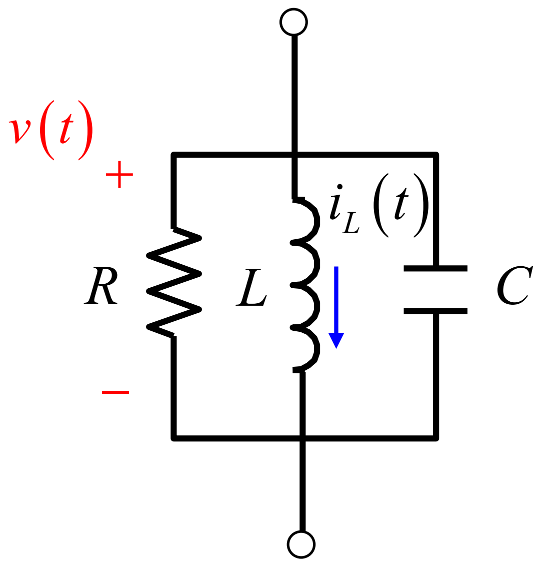
Hence

$$P_d^{ave} = \frac{1}{2} G |V_0|^2 e^{-2\omega_0'' t_0}$$

Q for RLC Resonator (cont.)

$$Q \approx \frac{1}{\sqrt{LC}}\frac{U}{P_d^{ave}}$$

Similarly:



$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

$$U = \left(\frac{1}{2}C|v(t)|^2 + \frac{1}{2}L|i_L(t)|^2\right)$$

$$\approx \left(\frac{1}{2}C|V_0|^2 \cos^2(\omega_0' t) + \frac{1}{2}L\left|\frac{V_0}{\omega_0 L}\right|^2 \sin^2(\omega_0' t)\right) e^{-2\omega_0'' t_0}$$

$$= \frac{|V_0|^2}{2}\left(C\cos^2(\omega_0' t) + L\left|\frac{1}{\omega_0 L}\right|^2 \sin^2(\omega_0' t)\right) e^{-2\omega_0'' t_0}$$

$$= \frac{|V_0|^2}{2}\left(C\cos^2(\omega_0' t) + C\sin^2(\omega_0' t)\right) e^{-2\omega_0'' t_0}$$

$$= \frac{|V_0|^2}{2}(C) e^{-2\omega_0'' t_0}$$

$$= \frac{C}{2}|V_0|^2 \; e^{-2\omega_0'' t_0}$$

$$V_0 = |V_0| e^{j\phi} \qquad I_L = \frac{V_0}{j\omega L} = \frac{|V_0| e^{j\phi}}{j\omega L}$$

$$i_L(t) = \left|\frac{V_0}{\omega L}\right| e^{-\omega_0'' t} \cos(\omega_0' t + \phi - \pi/2)$$

$$i_L(t) = \left|\frac{V_0}{\omega L}\right| e^{-\omega_0'' t} \sin(\omega_0' t + \phi)$$

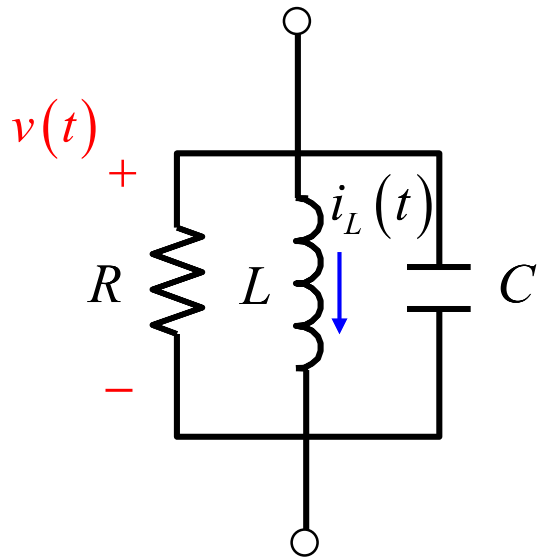
Q for RLC Resonator (cont.)

$$Q \approx \frac{1}{\sqrt{LC}} \frac{U}{P_d^{ave}}$$

We have:

$$P_d^{ave} = \frac{1}{2} G |V_0|^2 e^{-2\omega_0'' t_0}$$

$$U = \frac{1}{2} C |V_0|^2 \; e^{-2\omega_0'' t_0}$$



$$v(t) = |V_0| e^{-\omega_0'' t} \cos(\omega_0' t + \phi)$$

Hence:

$$Q \approx \frac{1}{\sqrt{LC}} \frac{C}{G} = R\sqrt{\frac{C}{L}} = R\sqrt{LC}\,\frac{1}{L} = \frac{R}{\omega_0' L}$$

so

$$Q \approx \frac{R}{\omega_0' L}$$

General Q Formulas

These formulas hold for any resonator:

$$Q \equiv \omega_0' \frac{U}{P_d^{ave}}$$

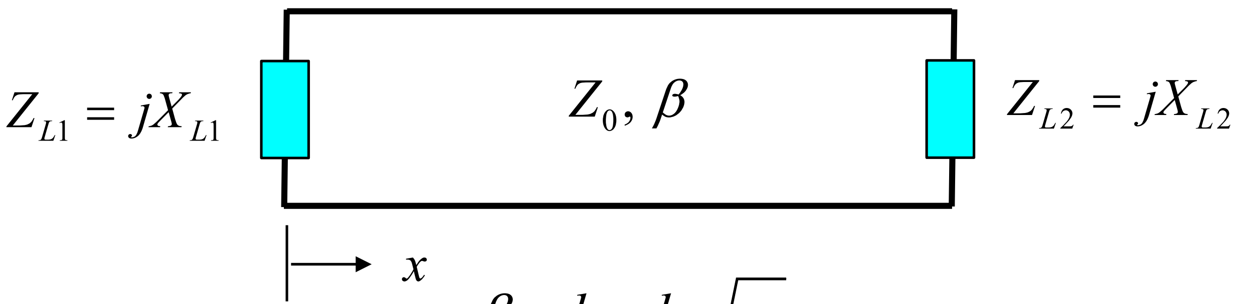
$$Q = \frac{1}{2}\left(\frac{\omega_0'}{\omega_0''}\right)$$

$$\omega_0 = \omega_0' + j\omega_0''$$

Transmission Line Resonator

Example:

Derive a transcendental equation for the resonance frequency of this lossless transmission-line resonator.



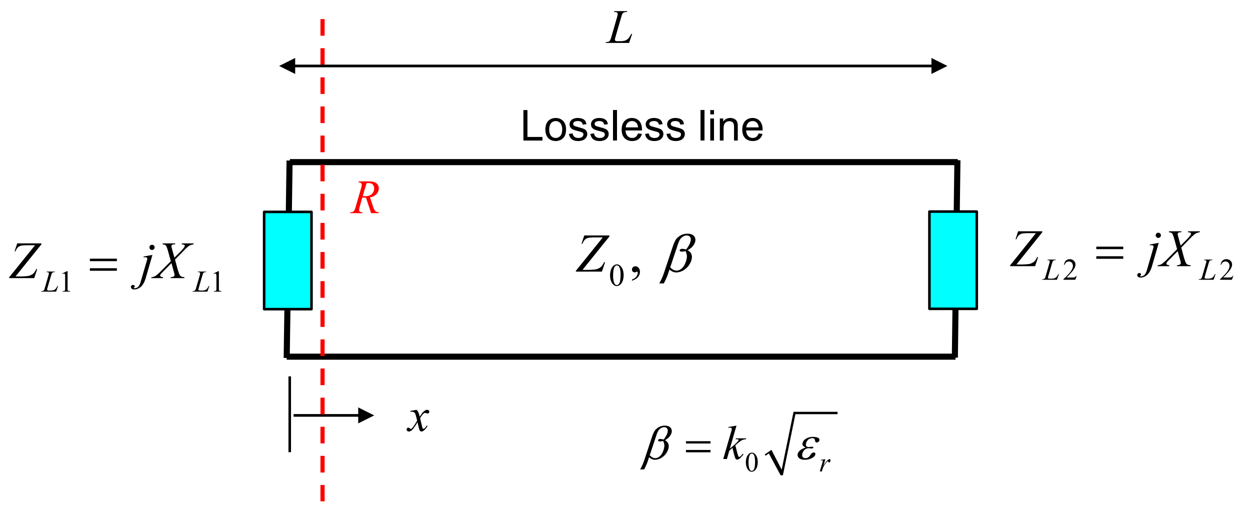
$$\beta = k = k_0 \sqrt{\varepsilon_r}$$

We choose a reference plane at $x = 0^+$.

The resonance frequencies will be real if the loads are lossless.

Note: The load reactances may be functions of frequency.

Transmission Line Resonator (cont.)



Apply TRE: $\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$

$$\Rightarrow \quad Z_{L1} = -\left[Z_0 \left(\frac{Z_{L2} + jZ_0 \tan(\beta L)}{Z_0 + jZ_{L2} \tan(\beta L)} \right) \right]$$

Lossless: $k_z = \beta$

Transmission Line Resonator (cont.)

$$Z_{L1} = -\left[Z_0 \left(\frac{Z_{L2} + jZ_0 \tan(\beta L)}{Z_0 + jZ_{L2} \tan(\beta L)} \right) \right]$$

↓

$$jX_{L1} = -\left[Z_0 \left(\frac{jX_{L2} + jZ_0 \tan(\beta L)}{Z_0 + j(jX_{L2}) \tan(\beta L)} \right) \right]$$

TEM: $\beta = k_1 = k_0 \sqrt{\varepsilon_r}$

↓

$$jX_{L1} = -\left[Z_0 \left(\frac{jX_{L2} + jZ_0 \tan\left(k_0 L \sqrt{\varepsilon_r}\right)}{Z_0 + j(jX_{L2}) \tan\left(k_0 L \sqrt{\varepsilon_r}\right)} \right) \right]$$

Note:

$$k_0 = 2\pi f_n^{res} \sqrt{\mu_0 \varepsilon_0}$$

↓

$$jX_{L1}\left(Z_0 + j(jX_{L2}) \tan\left(k_0 L \sqrt{\varepsilon_r}\right)\right) = -Z_0\left(jX_{L2} + jZ_0 \tan\left(k_0 L \sqrt{\varepsilon_r}\right)\right)$$

Transmission Line Resonator (cont.)

After simplifying, we have

$$\tan\left(k_0 L\sqrt{\varepsilon_r}\right) = \frac{Z_0\left(X_{L1} + X_{L2}\right)}{X_{L1}X_{L2} - Z_0^2}$$

Special cases:

$$X_{L1} = X_{L2} = 0 \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r} = n\pi, \quad n = 1, 2, \ldots$$

$$X_{L1} = X_{L2} = \infty \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r} = n\pi, \quad n = 1, 2, \ldots$$

$$X_{L1} = 0,\ X_{L2} \to \infty \quad \Rightarrow \quad k_0 L\sqrt{\varepsilon_r} = \left(2n-1\right)\pi / 2, \quad n = 1, 2, \ldots$$

Transmission Line Resonator (cont.)

For the resonance frequencies, we have

$$k_0 = 2\pi f_n^{res}\sqrt{\mu_0\varepsilon_0} = 2\pi\frac{f_n^{res}}{c}$$

$$c = 2.99792458\times 10^8\ [\mathrm{m/s}]$$

f_n^{res} = resonance frequnecy of nth mode

We then have

$$\tan\left(2\pi L\frac{f_n^{res}}{c}\sqrt{\varepsilon_r}\right) = \frac{Z_0\left(X_{L2}+X_{L1}\right)}{X_{L1}X_{L2}-Z_0^2}$$

↑

The RHS may also be a function of frequency.

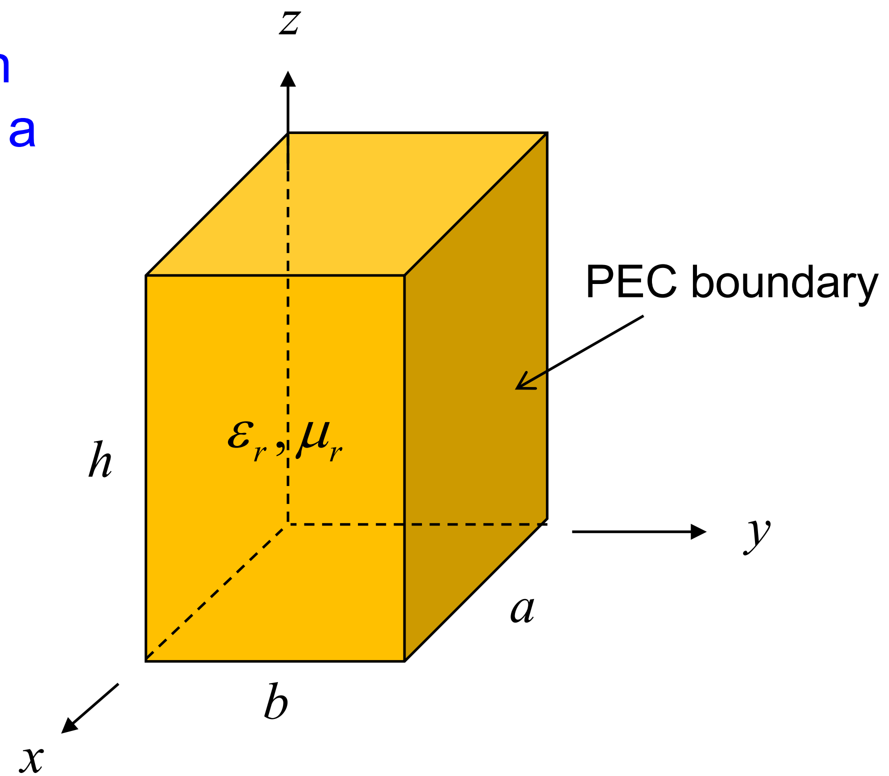
Rectangular Resonator

Example:

Derive a transcendental equation for the resonance frequencies of a lossless rectangular resonator.

Orient the structure so that

$$b < a < h$$



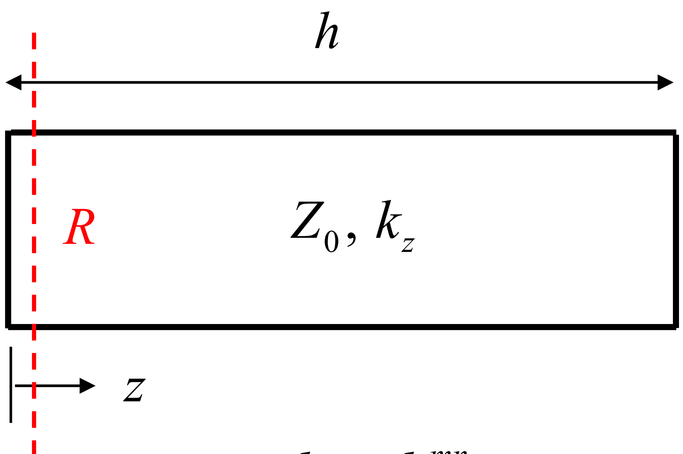
The structure is thought of as supporting rectangular waveguide modes bouncing back and forth in the z direction.

We have TM_{mnp} and TE_{mnp} modes.

The index p describes the variation in the z direction.

Rectangular Resonator (cont.)

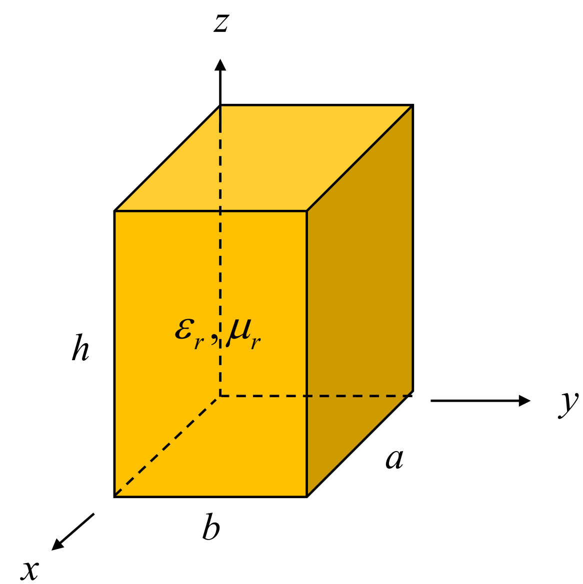
We use a Transverse Equivalent Network (TEN) to model any one of the waveguide modes:



$$k_z = k_z^{mn}$$

$$Z_0 = Z_{\text{TE,TM}}^{(m,n)}$$

(Choose Z_0 to be the wave impedance.)



We choose a reference plane at $z = 0^+$:

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

$$\overleftarrow{Z}_{in} = 0 \;\; \text{(PEC bottom)}$$

Hence $$\overrightarrow{Z}_{in} = 0$$

Rectangular Resonator (cont.)

Hence

$$\overrightarrow{Z}_{in} = jZ_0 \tan(\beta h) = 0$$

$$\Downarrow$$

$$jZ_{\text{TE,TM}}^{(m,n)} \tan\left(k_z^{(m,n)} h\right) = 0$$

$$\Downarrow$$

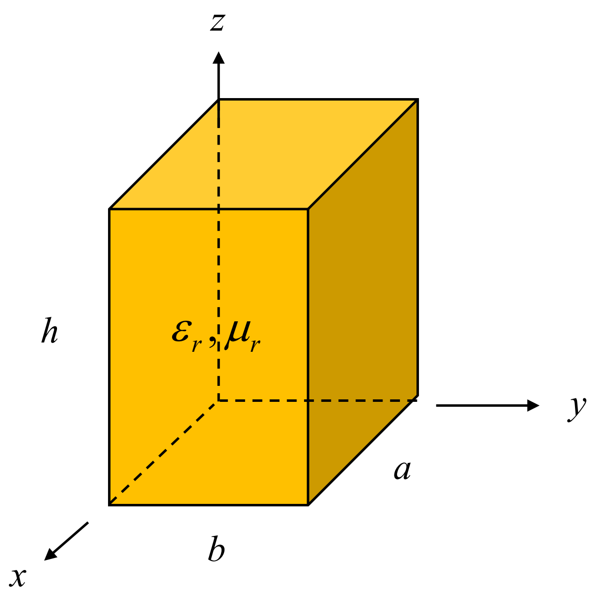
$$\tan\left(k_z^{(m,n)} h\right) = 0$$

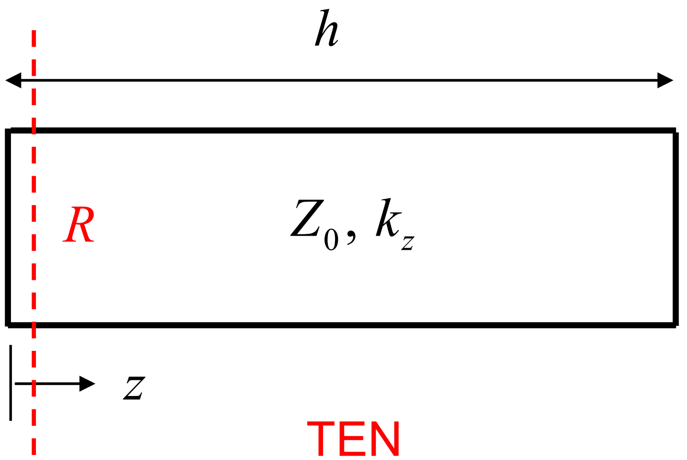
$$\Downarrow$$

$$k_z^{(m,n)} h = p\pi, \quad p = 0,1,2\ldots$$

$$\Downarrow$$

$$k_z^{(m,n)} = \frac{p\pi}{h} \quad p = 0,1,2\ldots$$





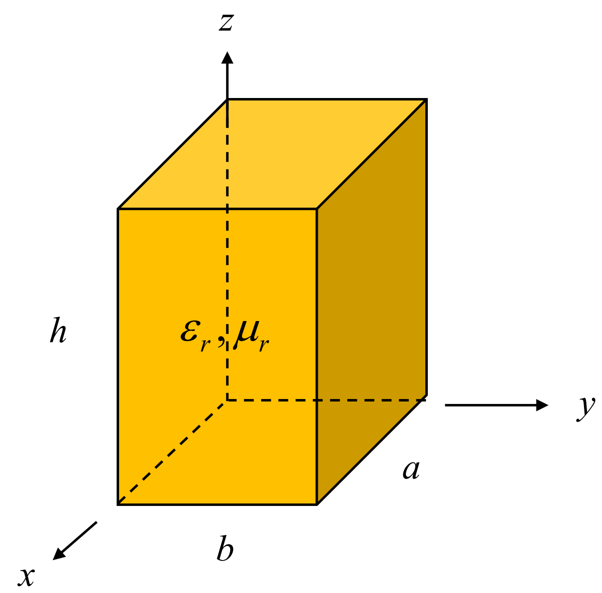
Note: $p \neq 0$ for TE modes (H_z must be zero on the top and bottom walls).

Rectangular Resonator (cont.)

$$k_z^{(m,n)} = \frac{p\pi}{h}, \quad p = 0,1,2\ldots$$

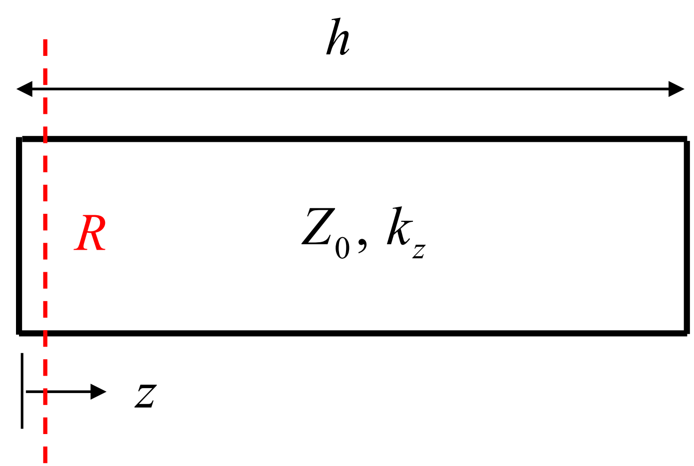
Also, we have

$$k_z^{(m,n)} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$



Hence, we have

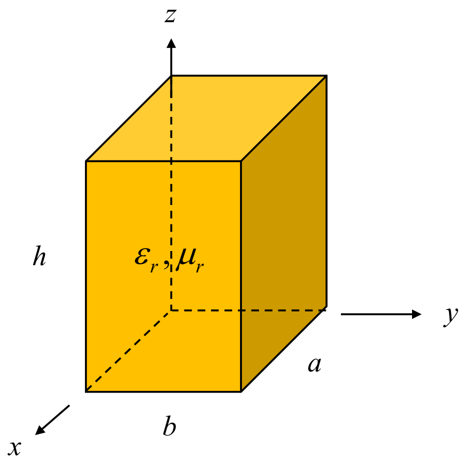
$$\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \frac{p\pi}{h}$$



Rectangular Resonator (cont.)

Solving for the wavenumber k we have:

$$k = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$



Hence

$$2\pi f_{mnp}\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

Note:
The TM_z and TE_z modes have the same resonance frequency.

or

$$f_{mnp} = \frac{c}{2\pi}\frac{1}{\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

$$c = 2.99792458 \times 10^8 \text{ [m/s]}$$

Rectangular Resonator (cont.)

Summary

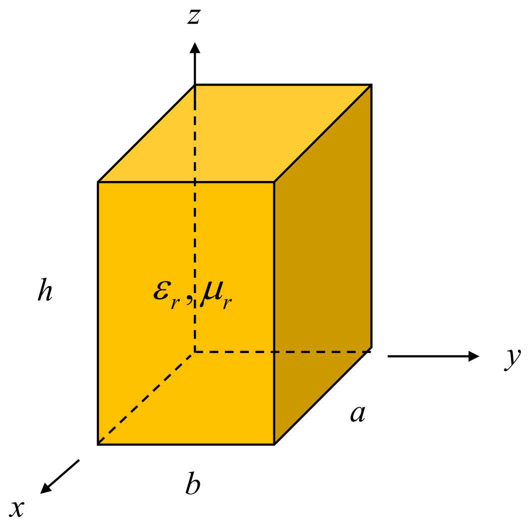
$$f_{mnp} = \frac{c}{2\pi} \frac{1}{\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

TE$_{mnp}$ mode:

$$m = 0, 1, 2, \ldots$$
$$n = 0, 1, 2, \ldots$$
$$p = 1, 2, \ldots$$
$$(m, n) \neq (0, 0)$$

TE$_{mnp}$ mode:

$$m = 1, 2, \ldots$$
$$n = 1, 2, \ldots$$
$$p = 0, 1, 2, \ldots$$



Note: p cannot be zero for a TE$_z$ mode, since H_z must vanish at the top and bottom walls.

The lowest mode is the TE$_{101}$ mode.

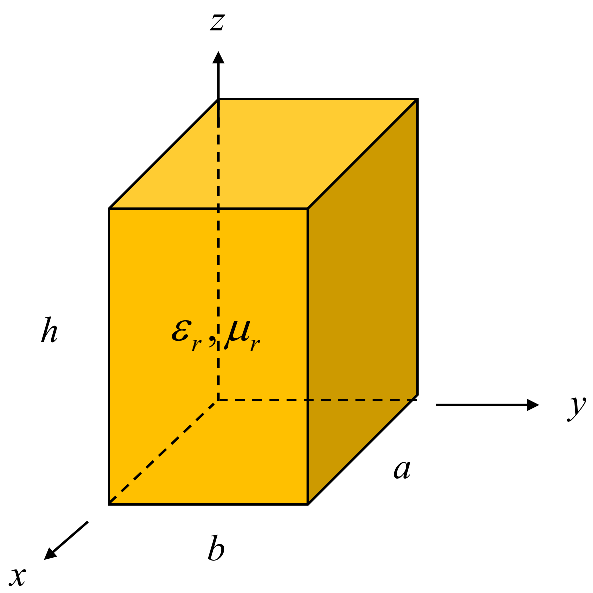
$$b < a < h$$

Rectangular Resonator (cont.)

TE$_{101}$ mode:

$$f_{101} = \frac{c}{2}\frac{1}{\sqrt{\mu_r \varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{h}\right)^2}$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{h}\right)$$



Note: The sin choice ensures the boundary condition on the PEC top and bottom plates:

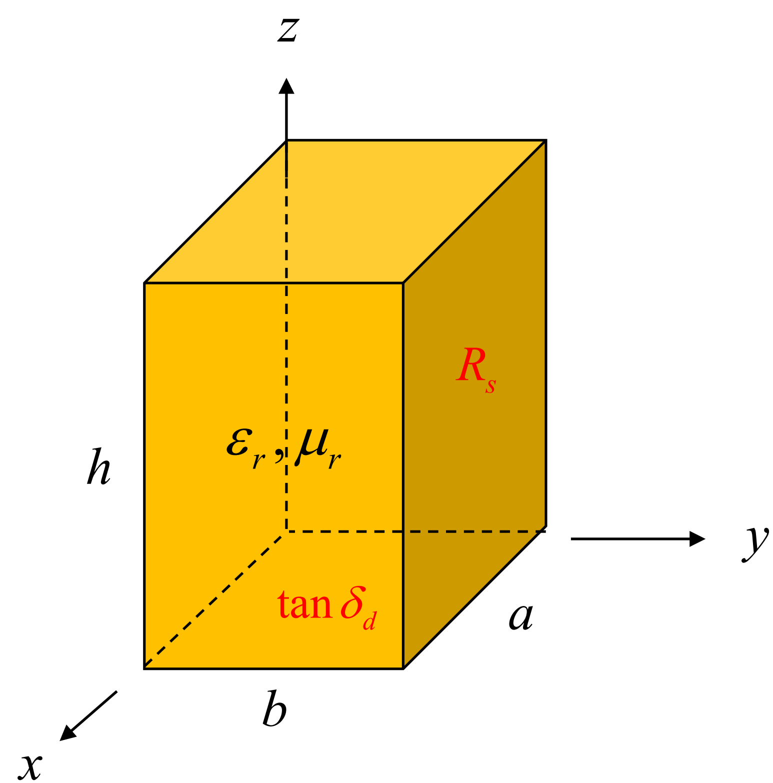
$$\underline{H} \cdot \underline{\hat{n}} = H_n = \pm H_z = 0$$

The other field components, E_y and H_x, can be found from H_z.

Rectangular Resonator (cont.)

Here we examine the Q of the resonator:

$$Q \equiv \omega_0' \frac{U}{P_{diss}^{ave}}$$



$$P_{diss}^{ave} = P_d + P_c$$ (dielectric and conductor loss)

$$\Rightarrow \frac{1}{Q} = \frac{1}{\omega_0'}\frac{P_{diss}^{ave}}{U} = \frac{1}{\omega_0'}\frac{P_d}{U} + \frac{1}{\omega_0'}\frac{P_c}{U}$$

$$\Rightarrow \frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c}$$

We now allow for dielectric and conductor loss in the resonator.

$$Q_d \equiv \omega_0' \frac{U}{P_d} \quad , \quad Q_c \equiv \omega_0' \frac{U}{P_c}$$

Q_d = dielectric Q
Q_c = conductor Q

Rectangular Resonator (cont.)

Q of TE$_{101}$ mode

$$U = U_E + U_H = 2U_E$$

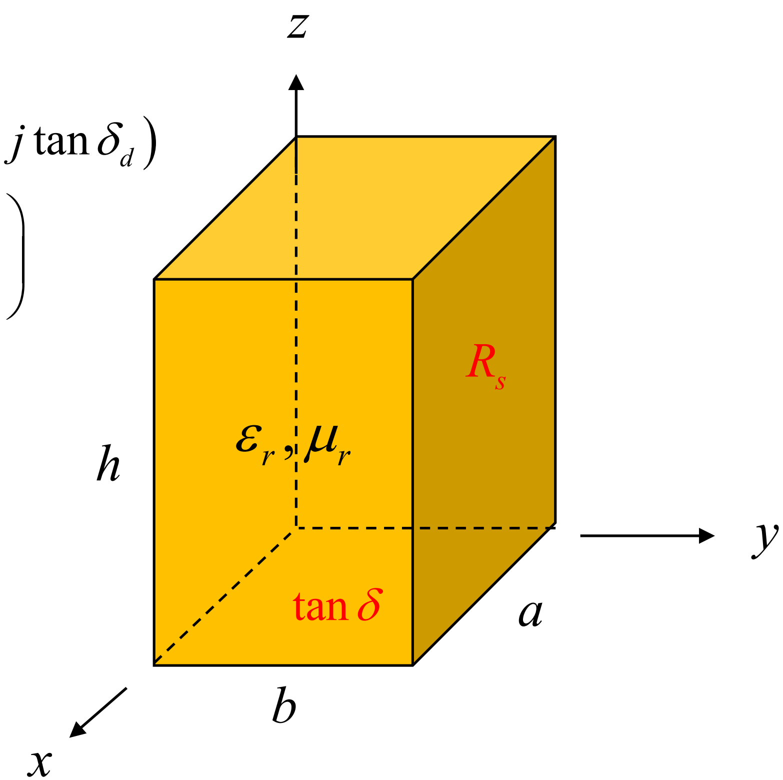
$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c'' = \varepsilon_0 \varepsilon_r \left(1 - j\tan\delta_d\right)$$

$$\left(\frac{\varepsilon_c''}{\varepsilon_c'} = \tan\delta_d\right)$$

$$U_E = \frac{1}{4}\varepsilon_c' \int_V \left|\underline{E}\right|^2 dV = \frac{1}{4}\varepsilon_c' \int_V \left|E_y^2\right| dV$$

$$P_d = \frac{1}{2}\left(\omega\varepsilon_c''\right)\int_V \left|\underline{E}\right|^2 dV = \frac{1}{2}\left(\omega\varepsilon_c''\right)\int_V \left|E_y^2\right| dV$$

$$P_c = \frac{1}{2} R_s \int_S \left|J_s^2\right| dS$$



$$E_y(x,y,z) = E_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{h}\right)$$

$$H_z(x,y,z) = H_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi z}{h}\right)$$

$$H_x(x,y,z) = -j\frac{E_0}{Z_{TE}^{(1,0)}} \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi z}{h}\right)$$

$$H_0 = \left(\frac{j\pi}{\omega\mu a}\right)E_0$$

Rectangular Resonator (cont.)

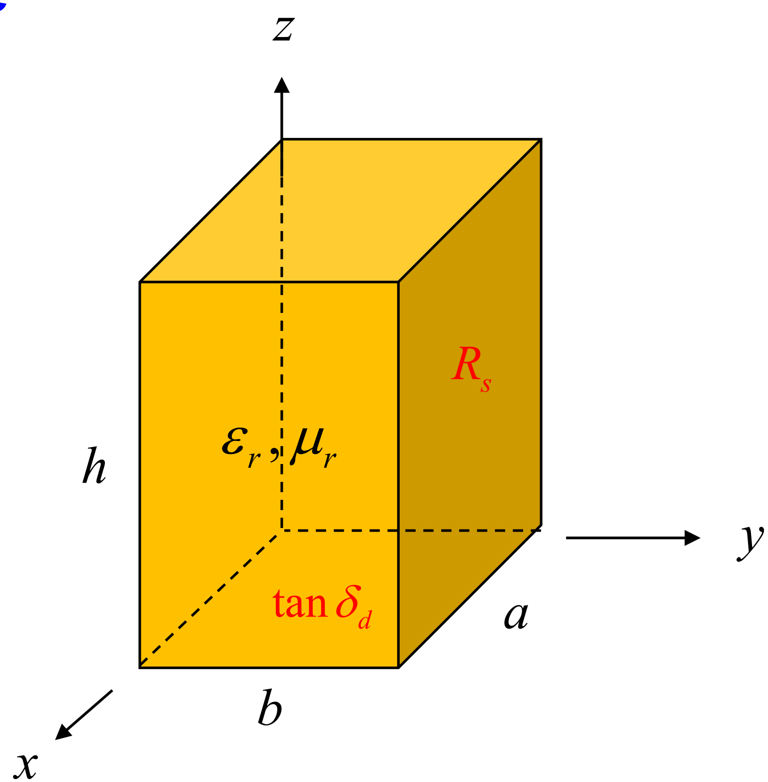
Q of TE_{101} mode

Results (from Pozar book):

$$U = \frac{1}{8}\varepsilon_c'(abh)|E_0|^2$$

$$P_d = \frac{1}{8}(\omega\varepsilon_c'')(abh)|E_0|^2$$

$$P_c = R_s|E_0|^2\frac{\lambda^2}{8\eta^2}\left(\frac{ab}{h^2}+\frac{bh}{a^2}+\frac{a}{2h}+\frac{h}{2a}\right)$$



This gives:

$$Q_d = \frac{1}{\tan\delta_d}$$

(This holds for any mode.)

$$Q_c = \frac{1}{R_s}\frac{(kah)^3 b\eta}{2\pi^2}\frac{1}{2a^3b+2bh^3+a^3h+ah^3}$$

(This is for the TE_{101} mode.)

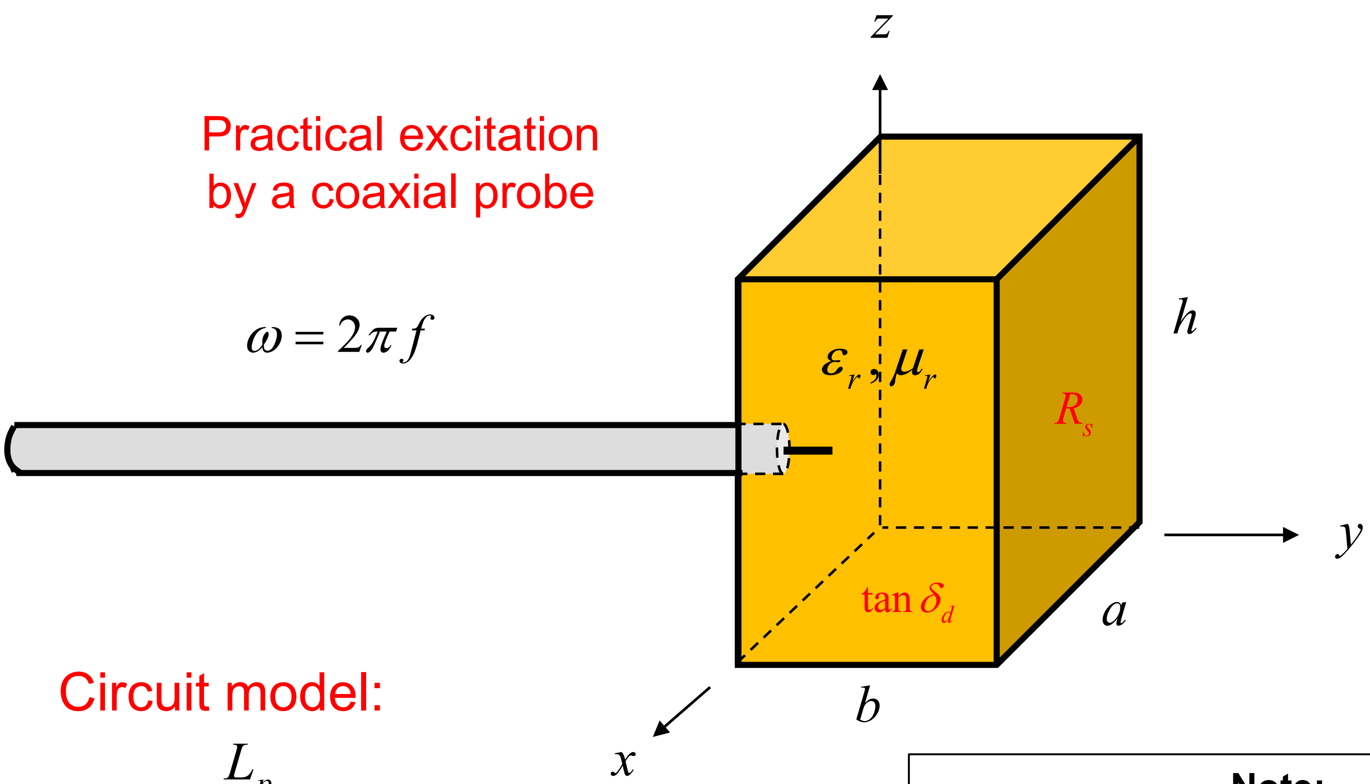
$$k = k_0\sqrt{\varepsilon_r\mu_r}$$

$$k_0 = \omega_0'\sqrt{\mu_0\varepsilon_0}$$

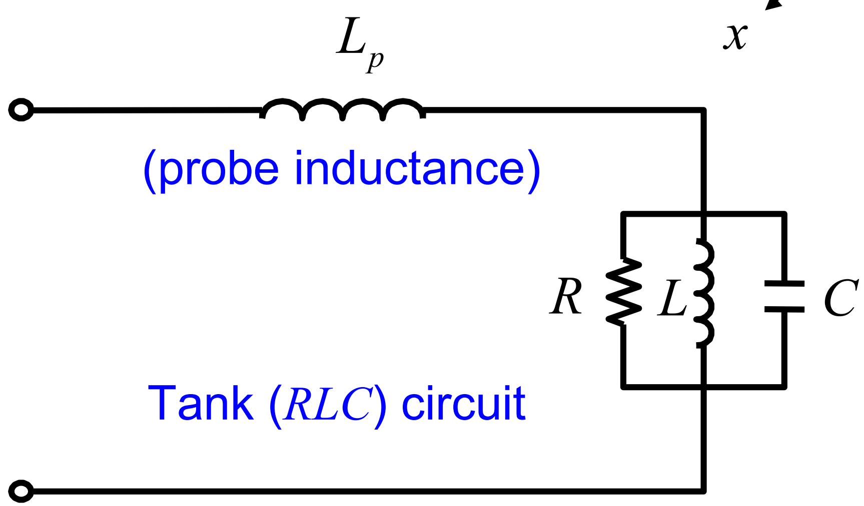
$$\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Excitation of Resonator

Practical excitation
by a coaxial probe



Circuit model:



Note:
The value of the circuit elements will depend on where the resonator is fed, and also the size of the probe. The values of Q and ω_0 do not depend on the feed.

$$Z_{in} = Z_{probe} + Z_{RLC}$$

Excitation of Resonator (cont.)

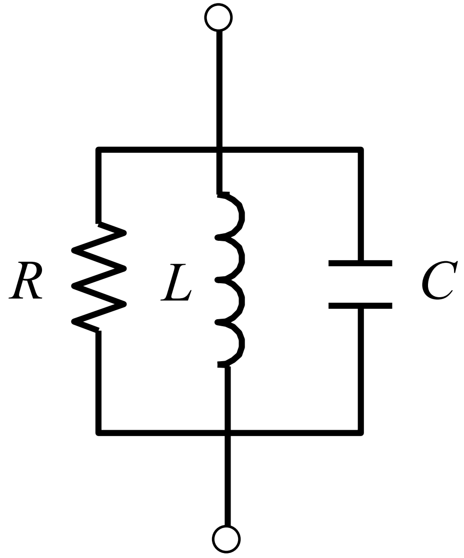
$$Z_{RLC} = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_0'} - \frac{\omega_0'}{\omega}\right)}$$

(from circuit theory – see next slide)

$$\omega = 2\pi f$$

where

$$\omega_0' = \frac{1}{\sqrt{LC}} \qquad Q = \frac{R}{\omega_0' L}$$



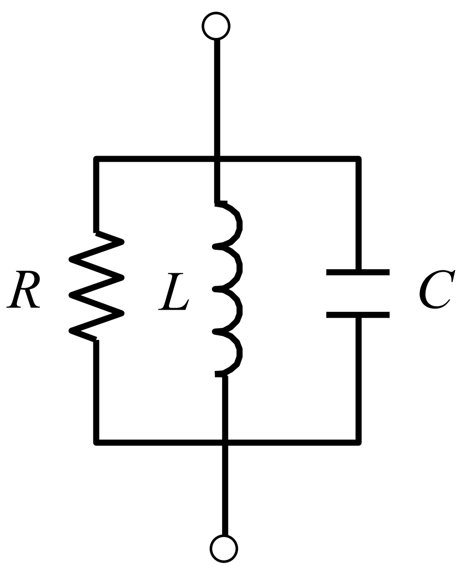
Approximate form (see slide 39):

$$Z_{RLC} \approx \frac{R}{1 + j2Q\left(\frac{\omega}{\omega_0'} - 1\right)}$$

Excitation of Resonator (cont.)

Derivation of Z_{RLC} formula:

Note:
Red color highlights where equivalent substitutions have been made. Blue color highlights where terms combine to unity.



$$Y_{RLC} = G + j\omega C - j\frac{1}{\omega L}$$

$$= G\left(1 + j\left(\frac{R}{\omega_0' L}\right)\frac{\omega_0' L}{R}\left(\omega\frac{C}{G} - \frac{1}{\omega}\frac{1}{LG}\right)\right)$$

$$= G\left(1 + j\left(\frac{R}{\omega_0' L}\right)\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0' L}{R}\frac{1}{\omega LG}\right)\right)$$

$$= G\left(1 + jQ\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0'}{\omega}\frac{L}{R}\frac{1}{LG}\right)\right)$$

$$= G\left(1 + jQ\left(\frac{\omega_0' L}{R}\omega\frac{C}{G} - \frac{\omega_0'}{\omega}\right)\right)$$

$$= G\left(1 + jQ\left(\frac{\omega_0' L}{R}\omega\frac{C}{G}\frac{\omega_0'^2}{\omega_0'^2} - \frac{\omega_0'}{\omega}\right)\right)$$

$$= G\left(1 + jQ\left(\frac{L\omega_0'^2}{R}\frac{\omega}{\omega_0'}\frac{C}{G} - \frac{\omega_0'}{\omega}\right)\right)$$

$$= G\left(1 + jQ\left(\frac{\omega}{\omega_0'} - \frac{\omega_0'}{\omega}\right)\right)$$

Excitation of Resonator (cont.)

Derivation of approximate form:

$$Z_{RLC} = \frac{R}{1 + jQ\left(\dfrac{\omega}{\omega_0'} - \dfrac{\omega_0'}{\omega}\right)}$$

$$\frac{\omega}{\omega_0'} - \frac{\omega_0'}{\omega} = \frac{\omega_0'}{\omega}\left(\left(\frac{\omega}{\omega_0'}\right)^2 - 1\right)$$

$$\approx \left(\frac{\omega}{\omega_0'}\right)^2 - 1 \qquad \omega \approx \omega_0'$$

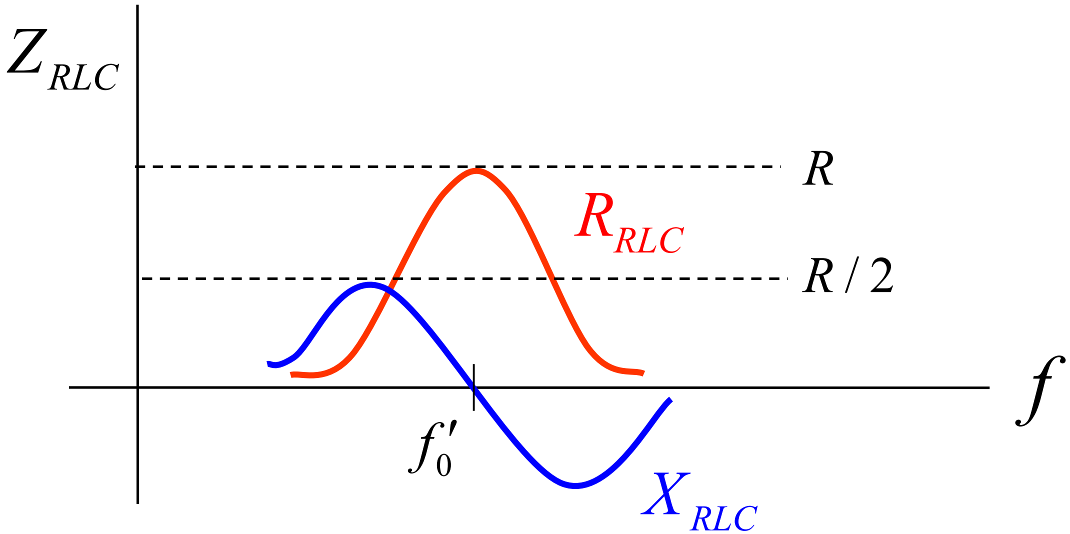
$$= \left(\frac{\omega}{\omega_0'} - 1\right)\left(\frac{\omega}{\omega_0'} + 1\right)$$

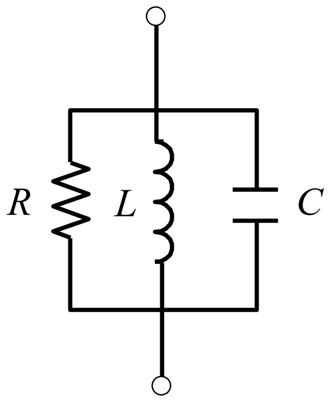
$$\approx \left(\frac{\omega}{\omega_0'} - 1\right) 2$$

$$Z_{RLC} \approx \frac{R}{1 + j2Q\left(\dfrac{\omega}{\omega_0'} - 1\right)}$$

Excitation of Resonator (cont.)

$$Z_{RLC} \approx \frac{R}{1 + j2Q\left(\dfrac{\omega}{\omega_0'} - 1\right)}$$





A larger Q of the resonator (less loss), means a more sharply peaked response, and a larger R value:

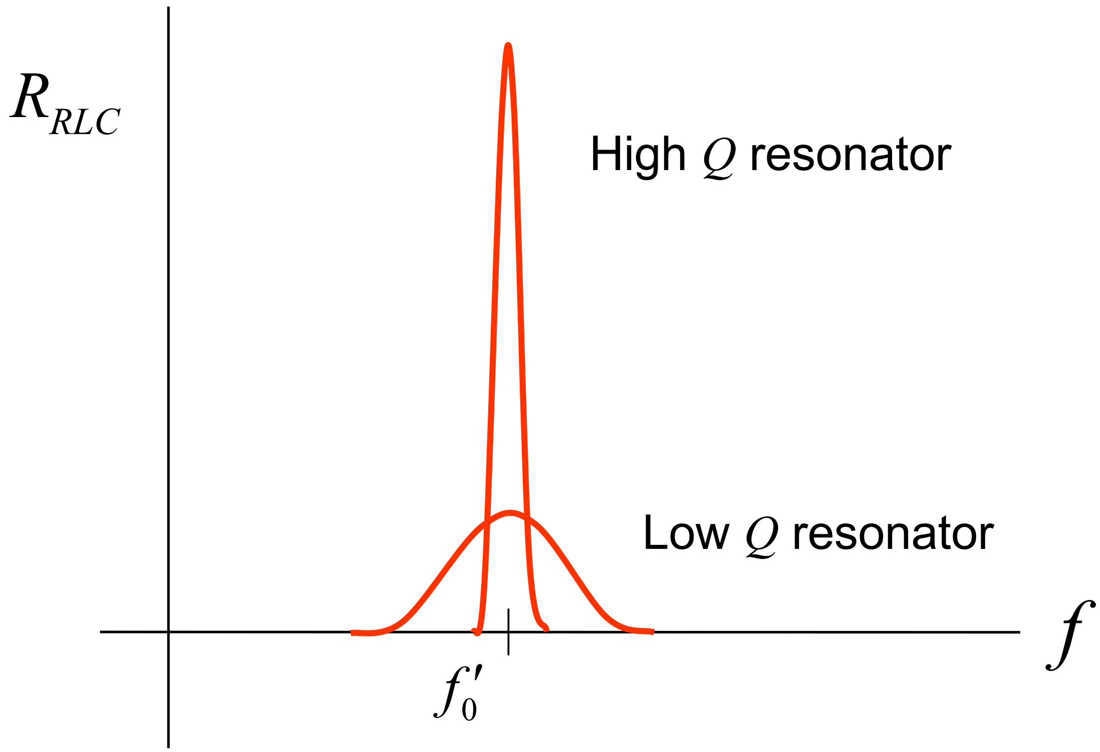
$$R = Q\omega_0' L$$

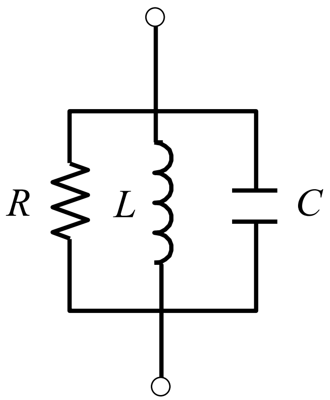
Lossless resonator:

$$R \to \infty$$

$$Q \to \infty$$

Excitation of Resonator (cont.)

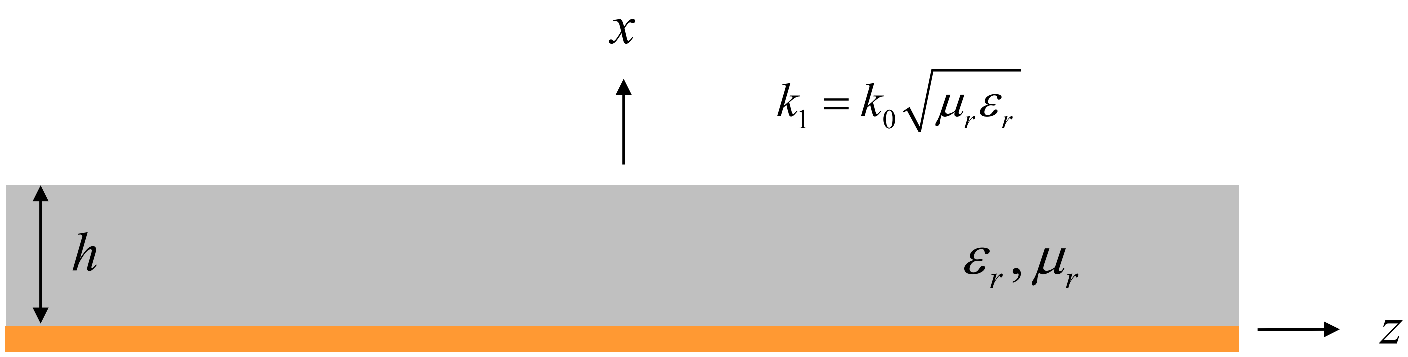




Note:
A larger Q of the resonator (less loss),
means a more sharply peaked response, and a larger R value.

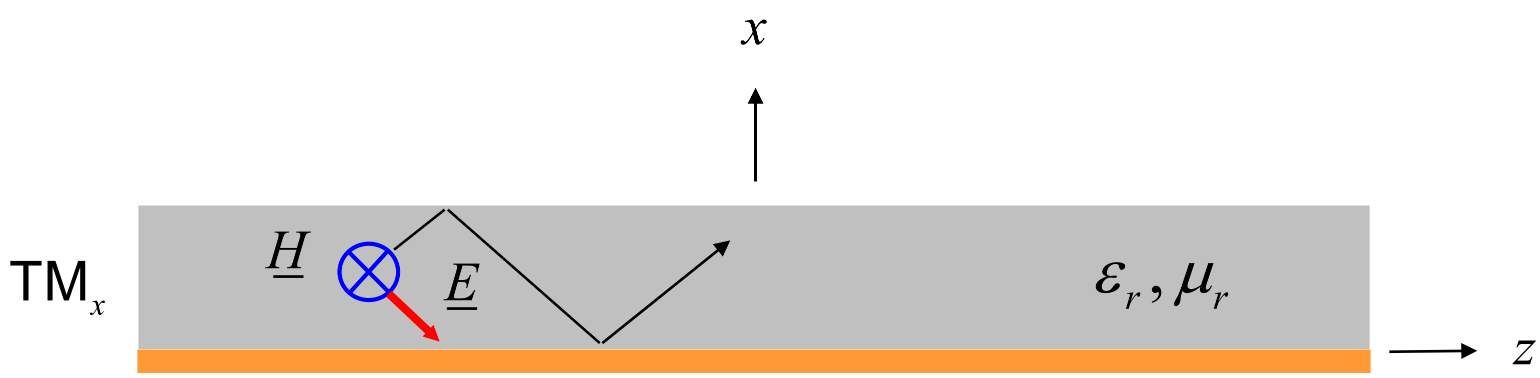
Grounded Dielectric Slab

Derive a transcendental equation for wavenumber of the TM_x surface waves by using the TRE.



Assumption: There is no variation of the fields in the y direction, and propagation is along the z direction.

Grounded Dielectric Slab (cont.)



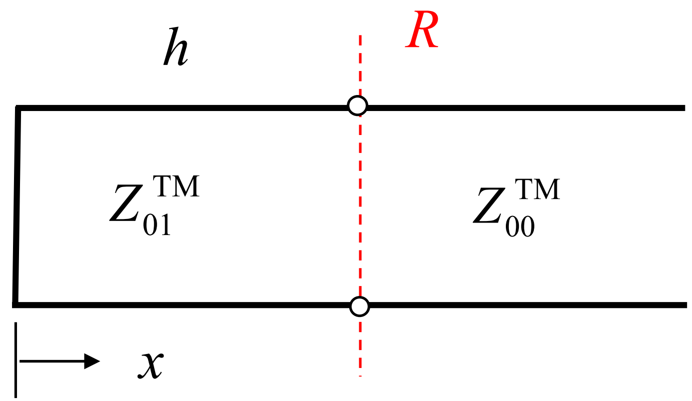
$$Z^{\mathrm{TM}_x} \equiv \frac{E_z}{-H_y} \quad \text{(defined for a wave going in the } + x \text{ direction)}$$

$$Z_{01}^{\mathrm{TM}} = \frac{k_{x1}}{\omega\varepsilon_1} \qquad Z_{00}^{\mathrm{TM}} = \frac{k_{x0}}{\omega\varepsilon_0}$$

We think of the transmission lines in the TEN as running in the x direction.

TM$_x$ Surface-Wave Solution

TEN:



The reference plane R is chosen at the interface.

$$Z_{01}^{\text{TM}} = \frac{k_{x1}}{\omega \varepsilon_1} \qquad Z_{00}^{\text{TM}} = \frac{k_{x0}}{\omega \varepsilon_0}$$

$$k_{x1} = \sqrt{k_1^{\,2} - k_z^{\,2}}$$

$$k_{x0} = (k_0^{\,2} - k_z^{\,2})^{1/2} = -j\sqrt{k_z^{\,2} - k_0^{\,2}} = -j\alpha_{x0}$$

$$\overleftarrow{Z}_{in} = jZ_{01}^{\text{TM}} \tan(k_{x1} h), \qquad \overrightarrow{Z}_{in} = Z_{00}^{TM}$$

TM_x Surface-Wave Solution (cont.)

TRE:

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

$$\Downarrow$$

$$jZ_{01}^{\mathrm{TM}} \tan(k_{x1}h) = -Z_{00}^{\mathrm{TM}}$$

$$\Downarrow$$

$$j\frac{k_{x1}}{\omega\varepsilon_1}\tan(k_{x1}h) = -\frac{k_{x0}}{\omega\varepsilon_0}$$

$$\Downarrow$$

$$\varepsilon_r = -j\left(\frac{k_{x1}}{k_{x0}}\right)\tan(k_{x1}h)$$

TM_x Surface-Wave Solution (cont.)

Letting

$$k_{x0} = -j\alpha_{x0} \qquad \left(\alpha_{x0} = \sqrt{k_z^{\,2} - k_0^{\,2}}\right)$$

we have

$$\varepsilon_r = \left(\frac{k_{x1}}{\alpha_{x0}}\right)\tan(k_{x1}h)$$

or

$$\varepsilon_r\sqrt{k_z^2 - k_0^2} = \sqrt{k_1^2 - k_z^2}\,\tan\left[\left(h\sqrt{k_1^2 - k_z^2}\right)\right]$$

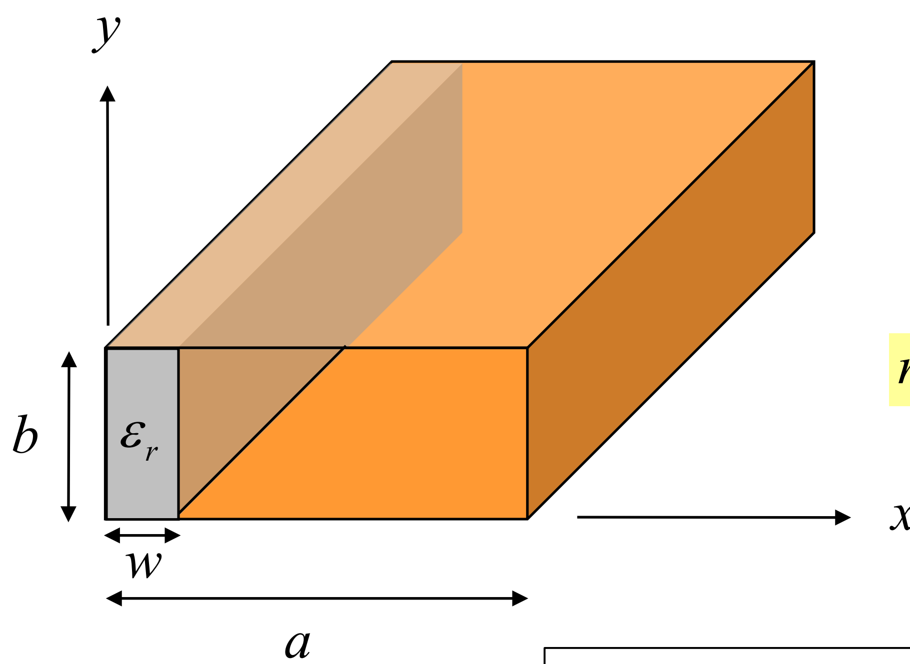
Note:
This method (TRE) is a lot simpler than doing the EM analysis and applying the boundary conditions!

Waveguide With Slab

Choose this representation:

TE_x^{mn} modes

TM_x^{mn} modes



$$k_y = \frac{n\pi}{b}$$

$$n = 0, 1, 2 \ldots$$

Note:
The modes are <u>hybrid</u> in the z direction (not TE_z or TM_z).

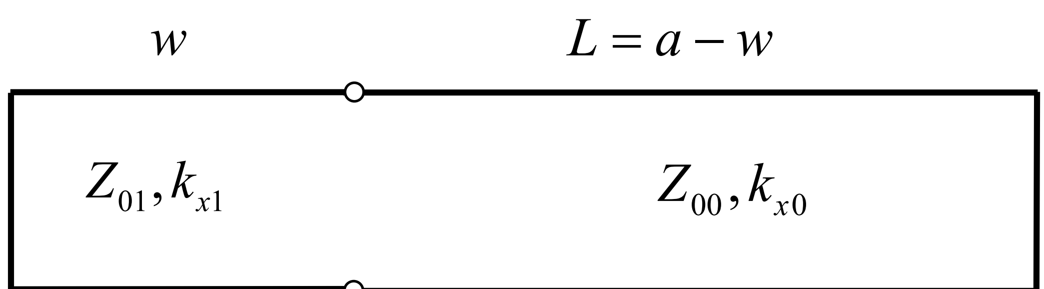
$$Z_0^{TM_x} = \frac{k_x}{\omega\varepsilon_c}$$

$$Z_0^{TE_x} = \frac{\omega\mu}{k_x}$$

$k_x = k_{x0}$ or k_{x1}

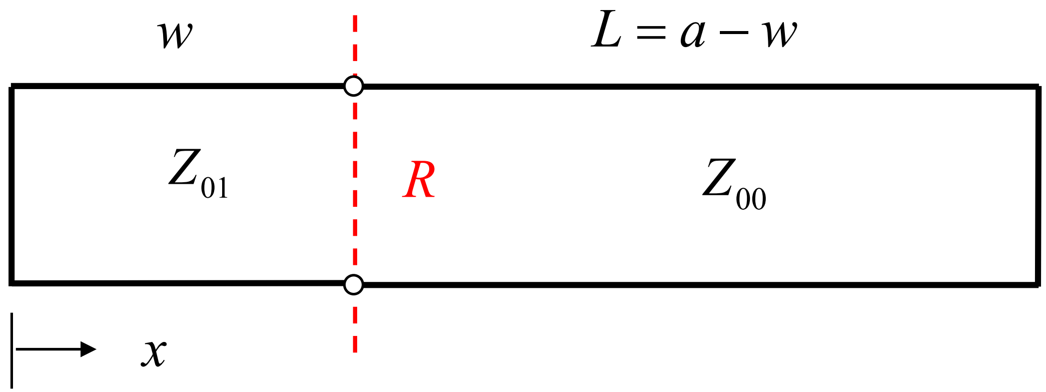
$Z_0 = Z_0^{TM_x}$ or $Z_0^{TE_x}$

TEN:



Waveguide With Slab (cont.)

TEN:



TRE:

$$\overleftarrow{Z}_{in} = -\overrightarrow{Z}_{in}$$

$$jZ_{01}\tan\left(k_{x1}w\right) = -jZ_{00}\tan\left(k_{x0}L\right)$$

$$Z_{01}\tan\left(k_{x1}w\right) + Z_{00}\tan\left(k_{x0}L\right) = 0$$

Waveguide With Slab (cont.)

$$Z_{01} \tan\left(k_{x1} w\right) + Z_{00} \tan\left(k_{x0} L\right) = 0$$

Choose TE$_x$:

$$\left(\frac{\omega\mu_0}{k_{x1}}\right)\tan\left(k_{x1} w\right) + \left(\frac{\omega\mu_0}{k_{x0}}\right)\tan\left(k_{x0} L\right) = 0$$

$$Z_0^{\mathrm{TE}_x} = \frac{\omega\mu}{k_x}$$

Separation equations:

$$k_{x0}^2 + k_y^2 + k_z^2 = k_0^2$$

$$k_{x1}^2 + k_y^2 + k_z^2 = k_1^2 = k_0^2 \varepsilon_r$$

so

$$k_{x0}^2 = k_0^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$

$$k_{x1}^2 = k_0^2 \varepsilon_r - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$

Waveguide With Slab (cont.)

Final transcendental equation for the unknown wavenumber k_z:

TE$_x$:

$$\left(\frac{1}{k_{x1}}\right)\tan\left(k_{x1}w\right)+\left(\frac{1}{k_{x0}}\right)\tan\left(k_{x0}L\right)=0$$

with

$$k_{x0}^2 = k_0^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$

$$k_{x1}^2 = k_0^2\varepsilon_r - \left(\frac{n\pi}{b}\right)^2 - k_z^2$$

Note:
The integer n is arbitrary but fixed.

The equation has an infinite number of solutions for k_z for a given n: $m = 1, 2, 3, \ldots$

$$\mathrm{TE}_{mn}^{(x)}:\ \underbrace{\mathrm{TE}_{10}^{(x)},\ \mathrm{TE}_{20}^{(x)},\ \mathrm{TE}_{30}^{(x)}}_{n=0},\ \ldots\underbrace{\mathrm{TE}_{01}^{(x)},\ \mathrm{TE}_{11}^{(x)},\mathrm{TE}_{21}^{(x)}}_{n=1}\ldots$$

Waveguide With Slab (cont.)

Limiting case: $w \to 0$:

$$\left(\frac{1}{k_{x1}}\right)\tan\left(k_{x1}w\right)+\left(\frac{1}{k_{x0}}\right)\tan\left(k_{x0}L\right)=0$$

$$\Downarrow$$

$$\tan\left(k_{x0}L\right)=0$$

$$\Downarrow$$

$$\tan\left(k_{x0}a\right)=0$$

$$\mathrm{TE}_{10}^{(x)} \to \mathrm{TE}_{10}^{(z)}$$

This mode becomes the usual TE_{10} mode of the hollow waveguide.

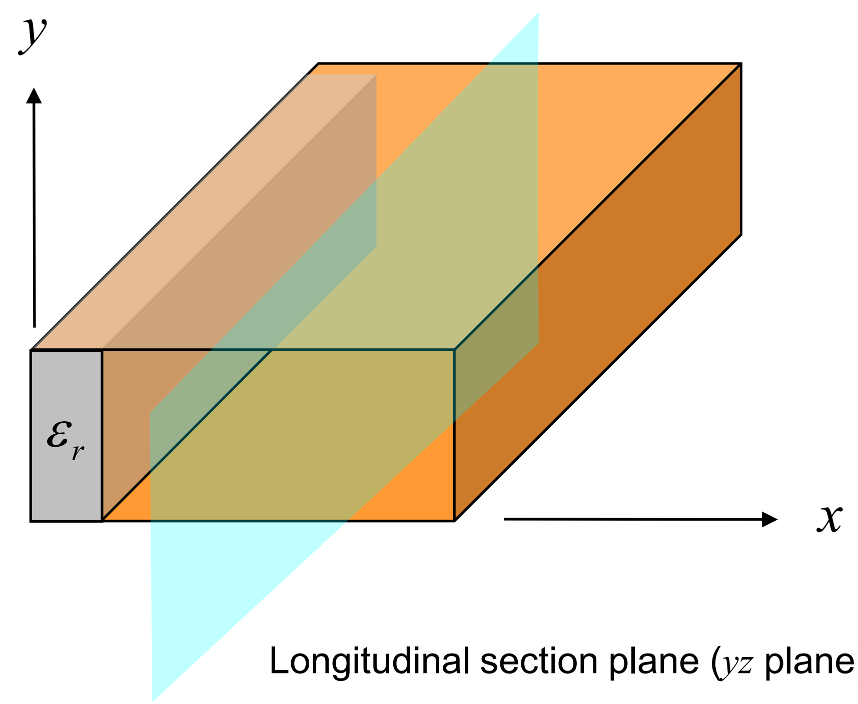
$$\Downarrow$$

Hence, we have

$$k_{x0}=\frac{m\pi}{a}$$

$$k_z=\sqrt{k_0^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$$

Waveguide With Slab (cont.)



Alternative notation:

Another notation that is common in the literature for slab-loaded waveguides is the designation of "LSE" and "LSM" modes.

LSE mode: "longitudinal section electric" mode
LSM mode: "longitudinal section magnetic" mode

LSE: TE_x^{mn} (The electric field vector stays in the "longitudinal section".)

LSM: TM_x^{mn} (The magnetic field vector stays in the "longitudinal section".)