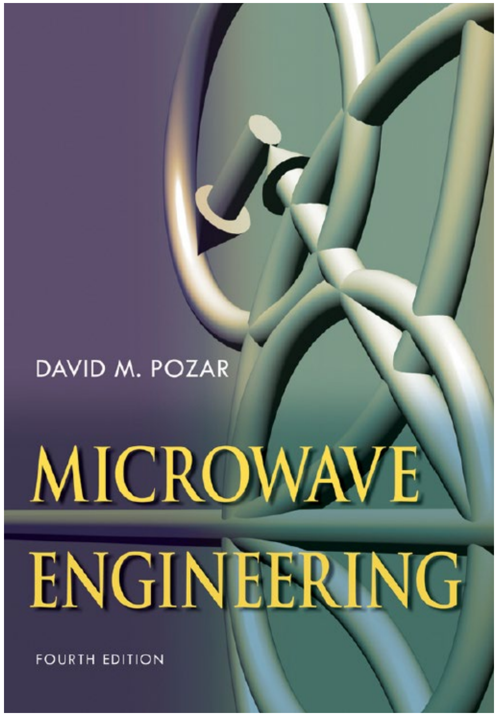
Adapted from notes by
Prof. Jeffery T. Williams

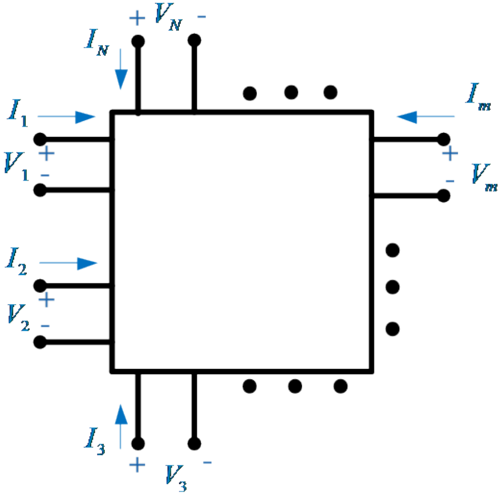
ECE 5317-6351
Microwave Engineering

Fall 2019

Prof. David R. Jackson
Dept. of ECE



Notes 16
Network Analysis



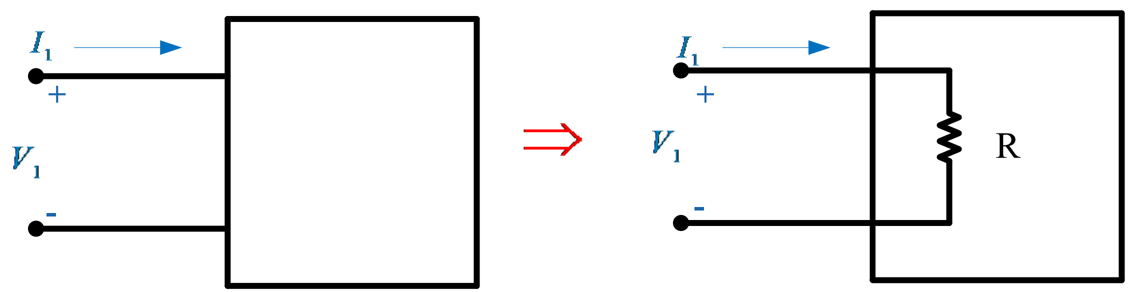
Multiport Networks

A general circuit can be represented by a multi-port network, where the "ports" are defined as access terminals at which we can define voltages and currents.

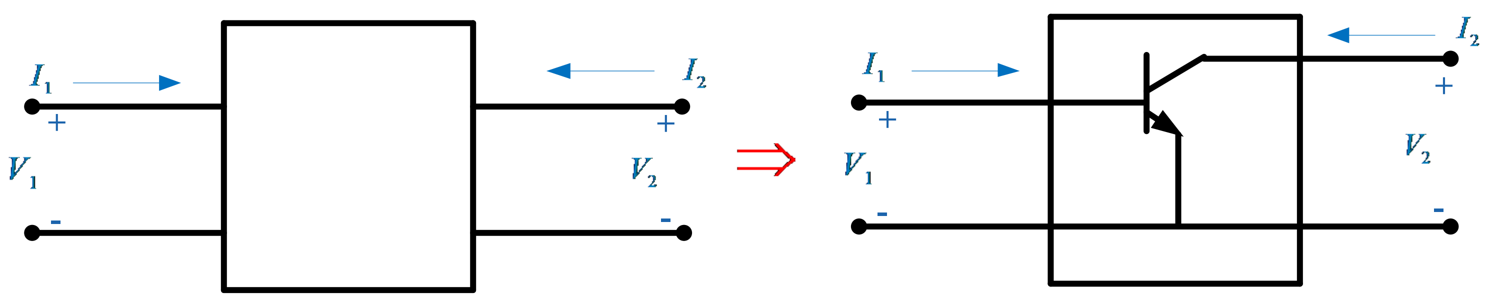
Note:
Equal and opposite currents are assumed on the two wires of a port.

Examples:

1) One-port network

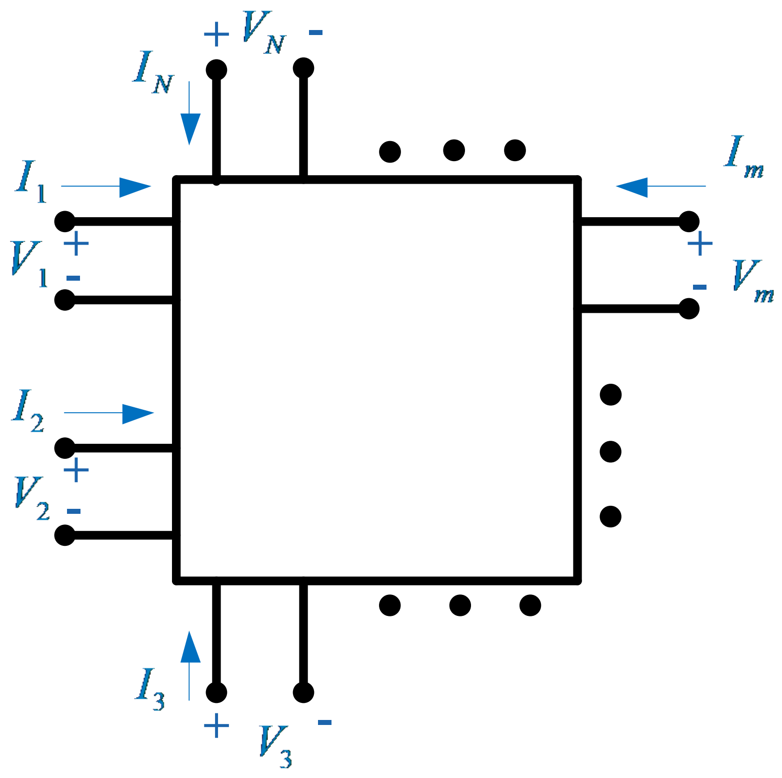


2) Two-port network



Multiport Networks (cont.)

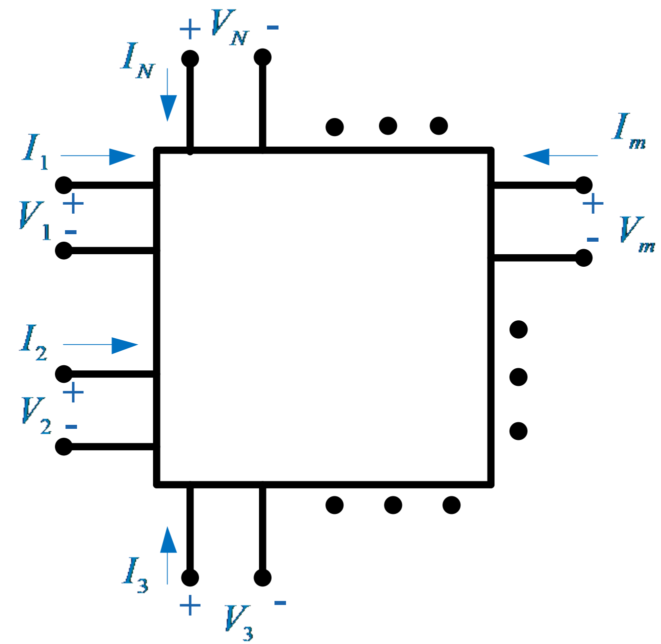
3) N-port network



Note: Passive sign convention is used at the ports.

Multiport Networks (cont.)

***N*-port network**



To represent multi-port networks we use:

- Z (impedance) parameters
- Y (admittance) parameters
- $ABCD$ parameters

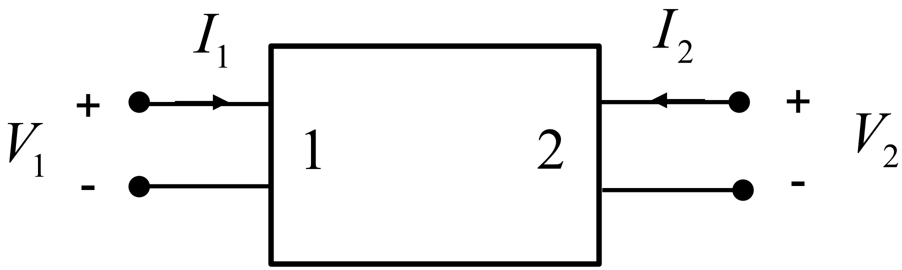
Not easily measurable at high frequency

- S (scattering) parameters

Measurable at high frequency

Two-Port Networks

Consider a 2-port linear network:



In terms of Z-parameters, we have (from superposition):

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Impedance (Z) matrix

Therefore

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \Rightarrow [V] = [Z][I]$$

Elements of Z-Matrix: Z-Parameters

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Port 2 open circuited

$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0}$$

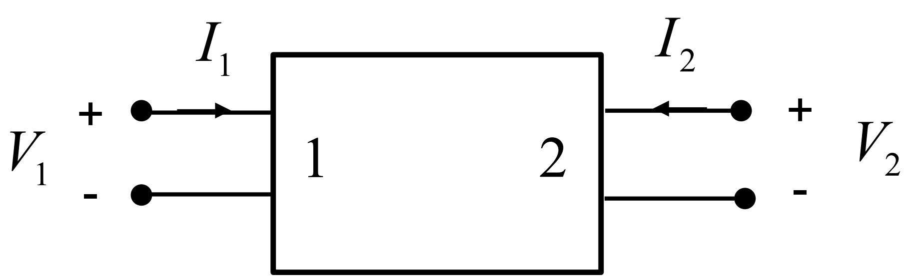
$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0}$$

$$\boxed{Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k = 0 \quad k \neq j}}$$

Port 1 open circuited

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1 = 0}$$

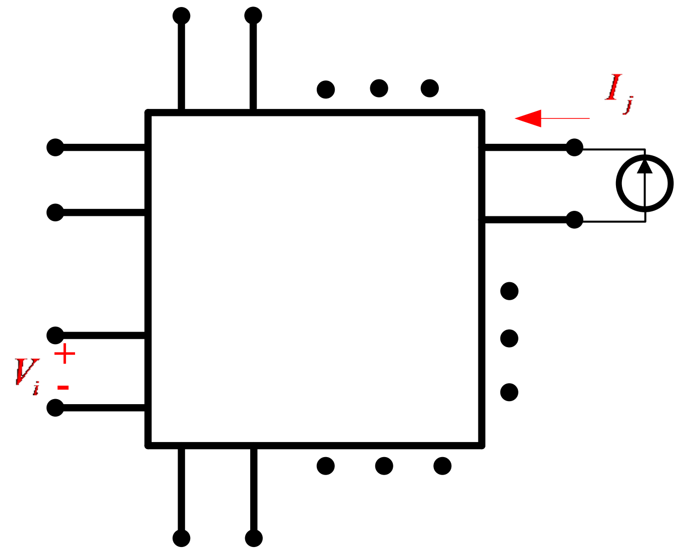
$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1 = 0}$$



Z-Parameters (cont.)

$$Z_{ij} = \left.\frac{V_i}{I_j}\right|_{I_k=0 \quad k\neq j}$$

N-port network



We inject a current into port j and measure the voltage (with an ideal voltmeter) at port i. All ports are open-circuited except j.

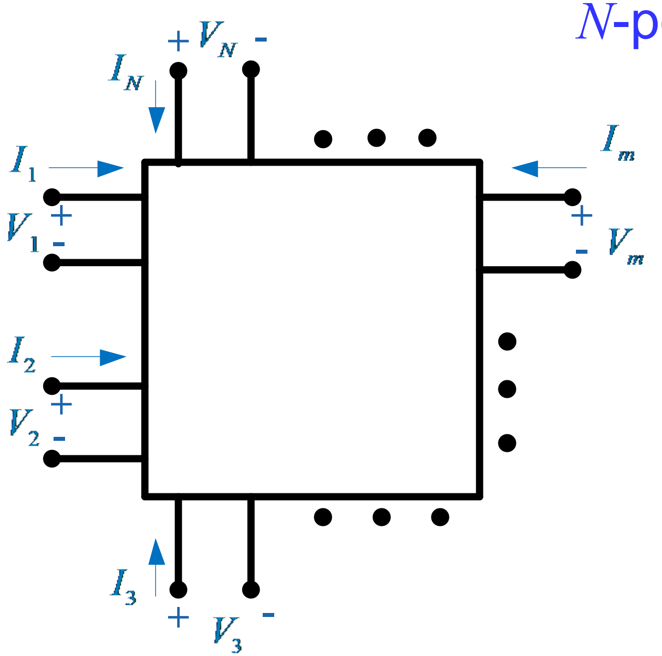
Summary of Z Parameters

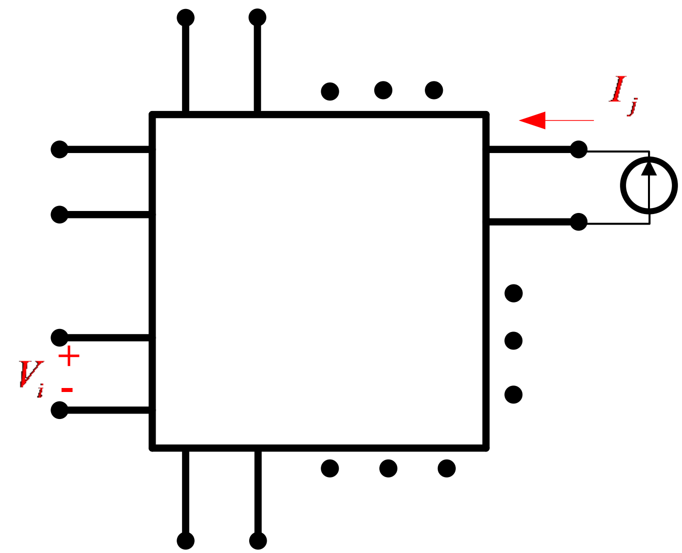
Summary of Z Parameters

$$[V]=[Z][I]$$

$$Z_{ij}=\left.\frac{V_i}{I_j}\right|_{I_k=0\quad k\neq j}$$

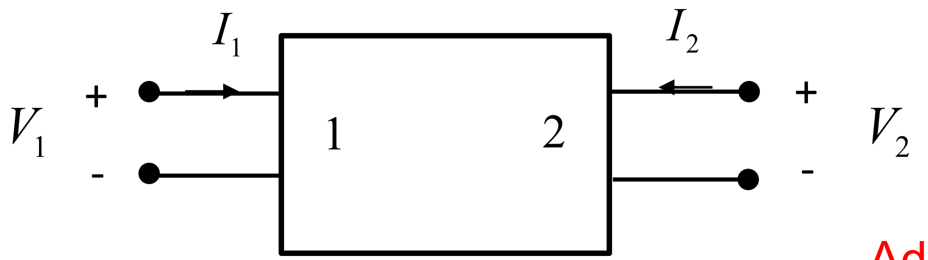
N-port network





Admittance (Y) Parameters

Consider a 2-port linear network:



$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Admittance matrix

or

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Rightarrow [I] = [Y][V]$$

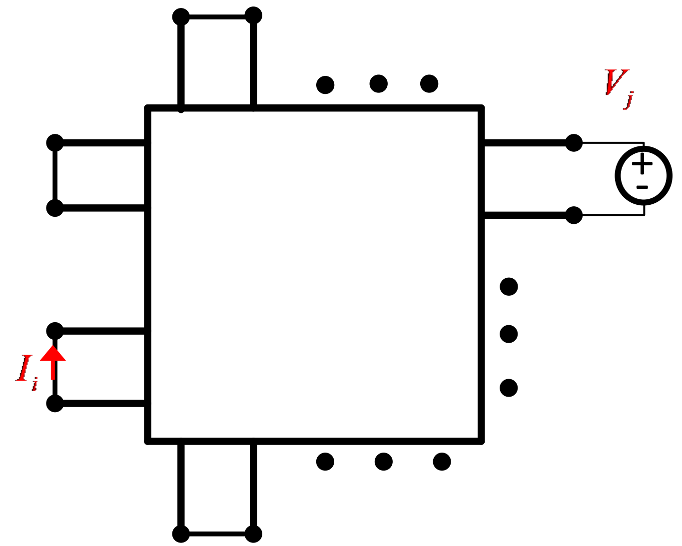
$$\boxed{Y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_k=0 \quad k\neq j}}$$

Y-Parameters (cont.)

N-port network

$$[I] = [Y][V]$$

$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \quad k \neq j}$$



We apply a voltage across port j and measure the current (with an ideal current meter) at port i. All ports are short-circuited except j.

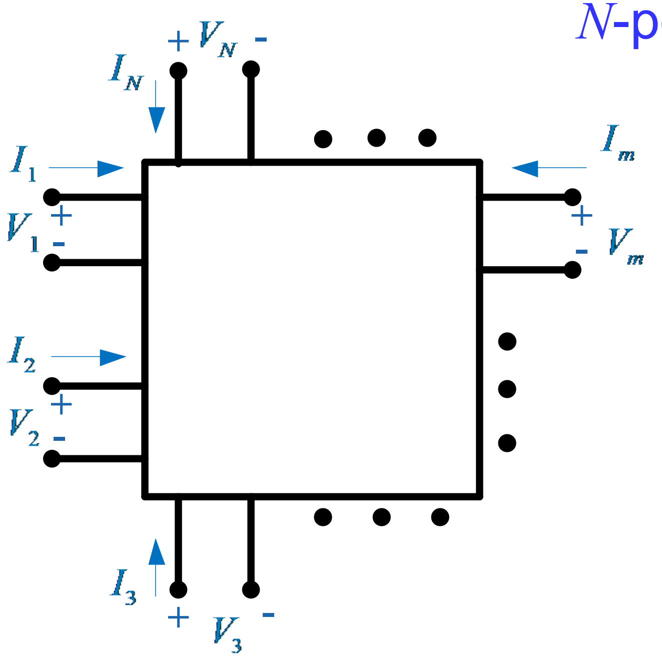
Summary of Y Parameters

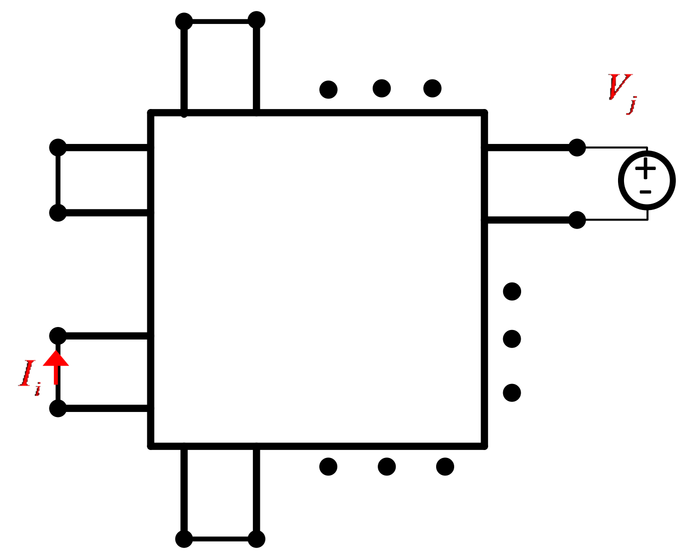
Summary of Y Parameters

$$[I]=[Y][V]$$

$$Y_{ij}=\left.\frac{I_i}{V_j}\right|_{V_k=0\quad k\neq j}$$

N-port network





Relation Between Z and Y Parameters

Relation between [Z] and [Y] matrices:

$$[V]=[Z][I]$$

$$[I]=[Y][V]$$

Hence:

$$\begin{aligned}[V]&=[Z]\left([Y][V]\right)\\&=\left([Z][Y]\right)[V]\end{aligned}$$

It follows that

$$[Z][Y]=[U]=\text{Identity Matrix}$$

Therefore $$\boxed{[Y]=[Z]^{-1}}$$

Reciprocal Networks

If a network does not contain non-reciprocal devices or materials*
(i.e. ferrites, or active devices), then the network is "reciprocal", which means that the Z and Y matrices are symmetric.

$$\Rightarrow \; Z_{ij} = Z_{ji} \quad \left(Y_{ij} = Y_{ji} \right)$$

$\Rightarrow \; \left[Z \right]$ and $\left[Y \right]$ are symmetric matrices

(proof omitted)

Note:
The inverse of a symmetric matrix is symmetric.

* A reciprocal material is one that has symmetric permittivity and permeability matrices. A reciprocal device is one that is made from reciprocal materials.

Example of a nonreciprocal material: a biased ferrite
(This is very useful for making isolators and circulators.)

Reciprocal Materials

$$\underline{D} = \underline{\underline{\varepsilon}} \cdot \underline{E} \qquad \underline{B} = \underline{\underline{\mu}} \cdot \underline{H}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \qquad \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Reciprocal: $\varepsilon_{ij} = \varepsilon_{ji}, \;\; \mu_{ij} = \mu_{ji}$

Ferrite: $$\underline{\underline{\mu}} = \mu_0 \begin{bmatrix} \alpha & j\gamma & 0 \\ -j\gamma & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{\underline{\mu}}$ is <u>not</u> symmetric!

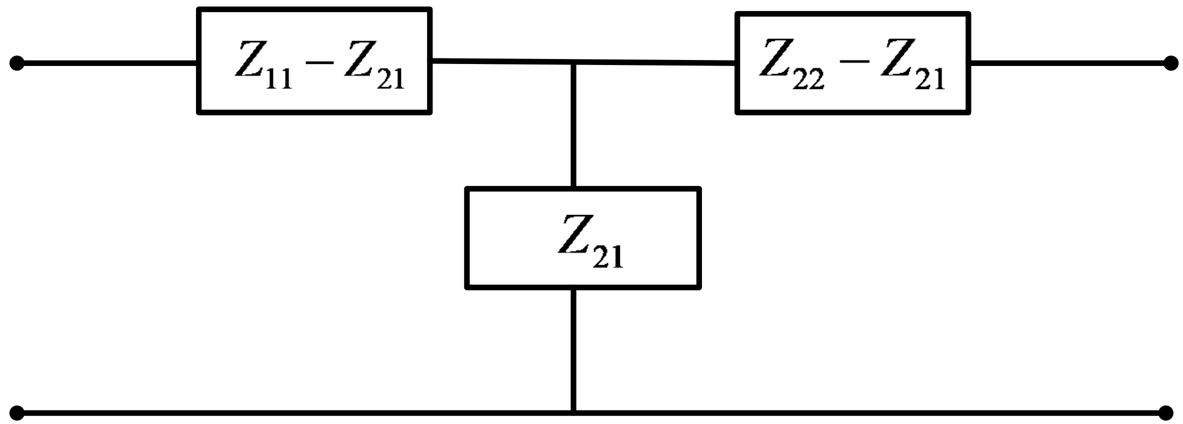
(not a reciprocal material)

Reciprocal Networks (cont.)

We can show that the equivalent circuits for reciprocal 2-port networks are:

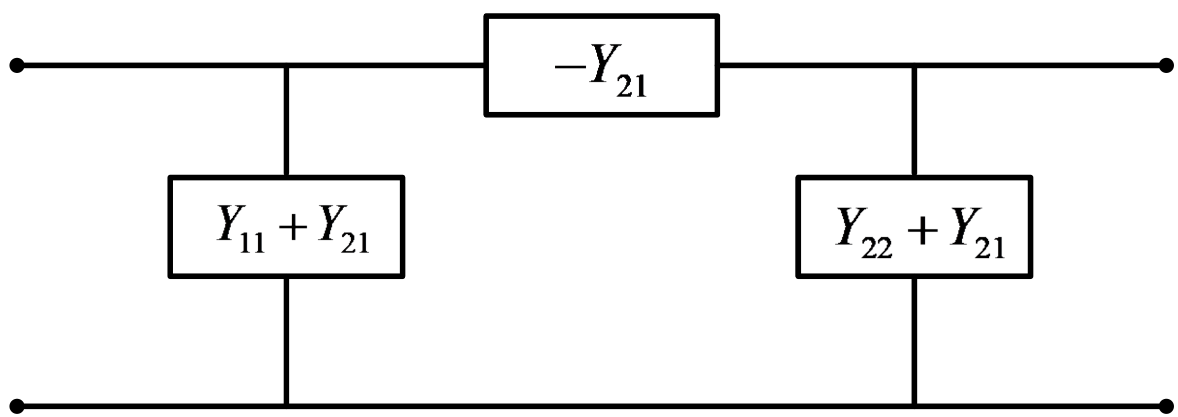
T-equivalent

"T network"



Pi-equivalent

"Π network"



ABCD-Parameters

There are defined only for 2-port networks.

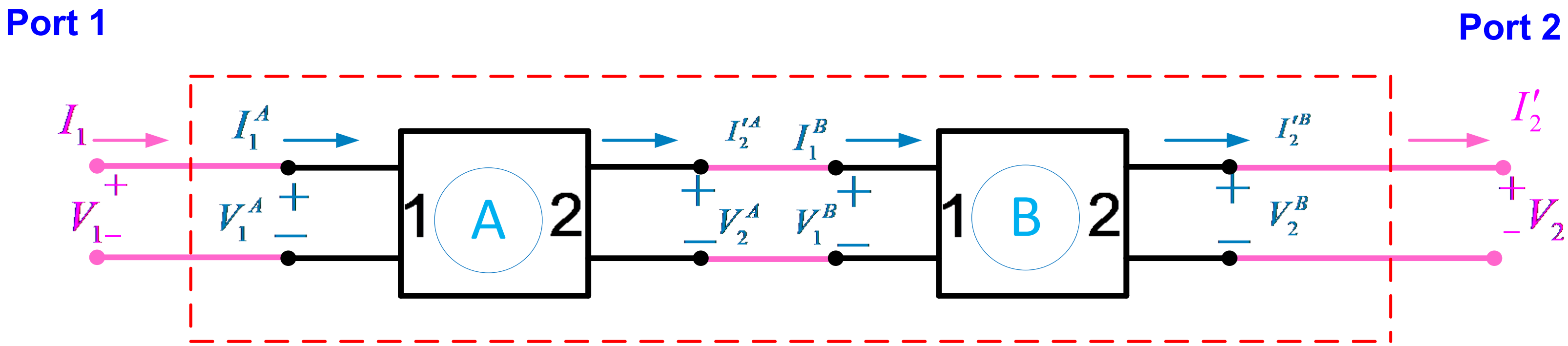
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2' \end{bmatrix}$$

$$I_2' = -I_2$$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2'=0} \qquad C = \left.\frac{I_1}{V_2}\right|_{I_2'=0}$$

$$B = \left.\frac{V_1}{I_2'}\right|_{V_2=0} \qquad D = \left.\frac{I_1}{I_2'}\right|_{V_2=0}$$

Cascaded Networks



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_1^A \\ I_1^A \end{bmatrix} = \left[ABCD^A \right] \begin{bmatrix} V_2^A \\ I_2'^A \end{bmatrix}$$

$$= \left[ABCD^A \right] \begin{bmatrix} V_1^B \\ I_1^B \end{bmatrix}$$

$$= \underbrace{\left[ABCD^A \right] \left[ABCD^B \right]} \begin{bmatrix} V_2^B \\ I_2'^B \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \left[ABCD^{AB} \right] \begin{bmatrix} V_2 \\ I_2' \end{bmatrix}$$

A nice property of the ABCD matrix is that it is easy to use with cascaded networks: you simply multiply the ABCD matrices together.

Scattering Parameters

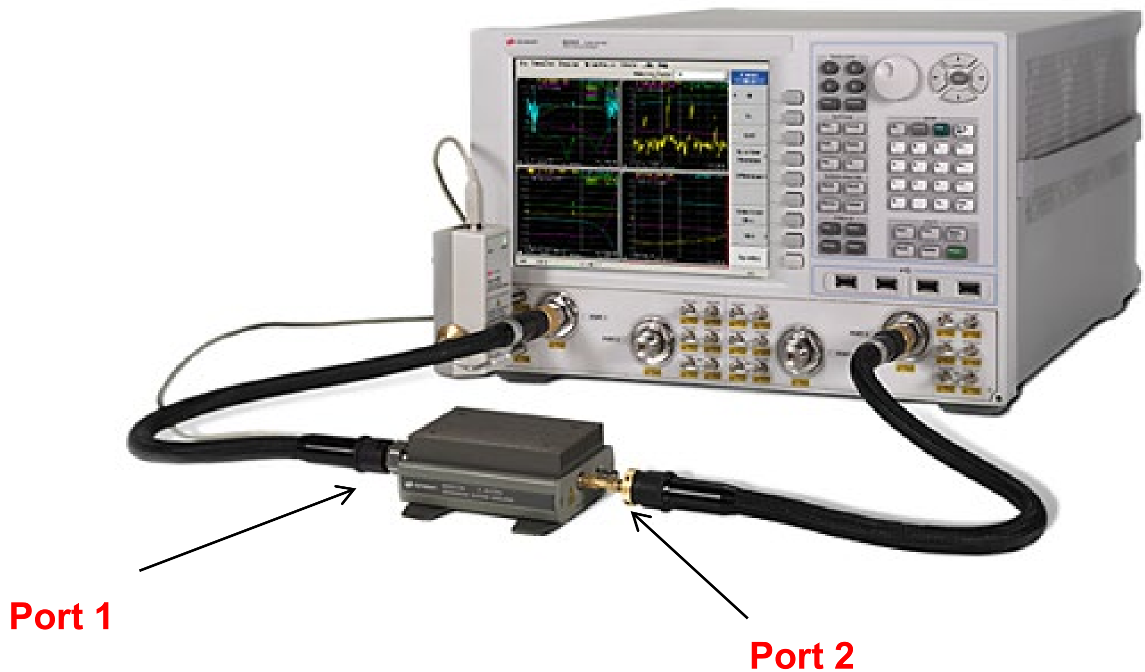
- At high frequencies, Z, Y, and $ABCD$ parameters are difficult (if not impossible) to measure.
  - V and I are not always uniquely defined (e.g., microstrip, waveguides).
  - Even if defined, V and I are extremely difficult to measure (particularly I).
  - Required open and short-circuit conditions are often difficult to achieve.

- Scattering (S) parameters are often the best representation for multi-port networks at high frequency.

Note: We can always convert from S parameters to Z, Y, or $ABCD$ parameters.

Scattering Parameters (cont.)

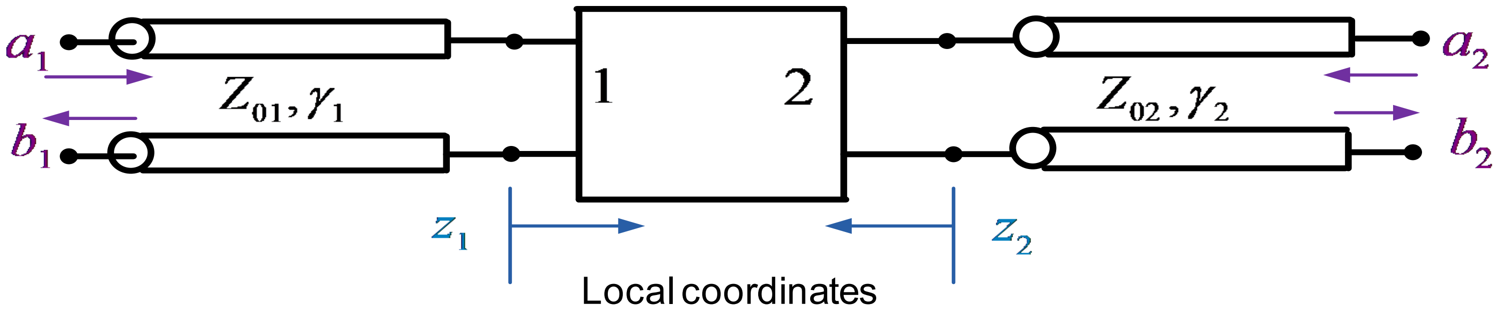
A Vector Network Analyzer (VNA) is usually used to measure S parameters.



Keysight (formerly Agilent) VNA shown performing a measurement.

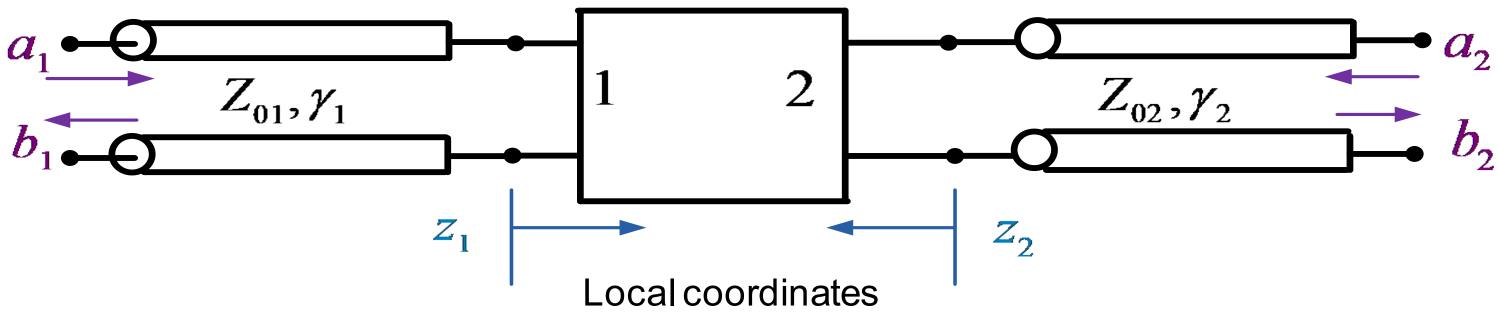
Scattering Parameters (cont.)

For scattering parameters, we think in terms of incident and reflected <u>waves</u> on transmission lines connected to a device.



- The a coefficients represent <u>incident</u> waves.
- The b coefficients represent <u>reflected</u> waves.

Scattering Parameters (cont.)



On each transmission line:

$$V_i(z_i) = V_{i0}^{+} e^{-\gamma_i z_i} + V_{i0}^{-} e^{+\gamma_i z_i} = V_i^{+}(z_i) + V_i^{-}(z_i)$$

$$I_i(z_i) = \frac{V_i^{+}(z_i)}{Z_{0i}} - \frac{V_i^{-}(z_i)}{Z_{0i}} \qquad i = 1,\, 2$$

We define normalized voltage wave functions:

Incoming wave function $\equiv a_i(z_i) \equiv V_i^{+}(z_i) \big/ \sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i) \equiv V_i^{-}(z_i) \big/ \sqrt{Z_{0i}}$

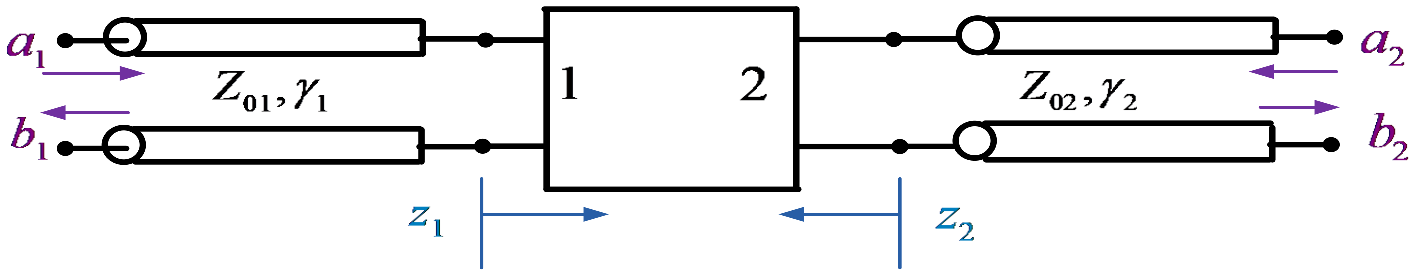
Scattering Parameters (cont.)

Why are the wave functions (a and b) defined as they are?

- The normalization makes power calculation easy (see next slide).
- The $[S]$ matrix is unitary (discussed later).

Scattering Parameters (cont.)

Power Calculations



$i = 1, 2$

$$P_i^+(0) = \frac{1}{2}\,\mathrm{Re}\left[V_i^+(0)\, I_i^{+*}(0)\right] = \frac{1}{2}\frac{\left|V_i^+(0)\right|^2}{Z_{0i}}$$

(assuming lossless lines, so Z_0 is real)

Recall: $a_i(0) \equiv V_i^+(0)\big/\sqrt{Z_{0i}}$

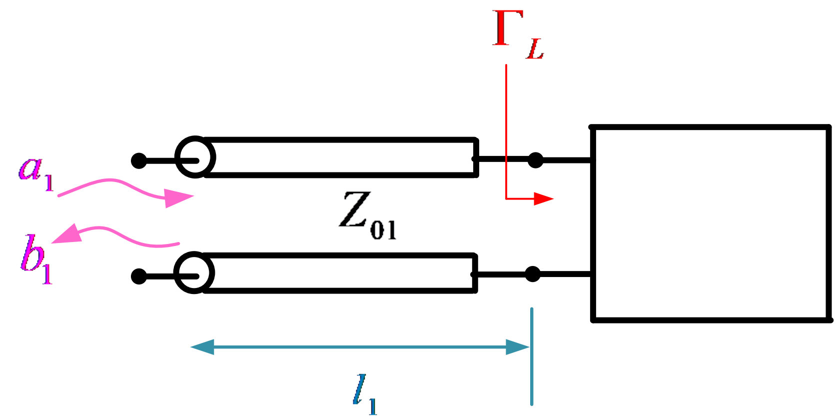
$$\Rightarrow \quad P_i^+(0) = \frac{1}{2}\left|a_i(0)\right|^2$$

Similarly,

$$P_i^-(0) = \frac{1}{2}\frac{\left|V_i^-(0)\right|^2}{Z_{0i}} = \frac{1}{2}\left|b_i(0)\right|^2$$

A One-Port Network

$$\Gamma_L = \frac{V_1^-(0)}{V_1^+(0)} = \frac{V_1^-(0)\big/\sqrt{Z_{01}}}{V_1^+(0)\big/\sqrt{Z_{01}}} = \frac{b_1(0)}{a_1(0)}$$



$$b_1(0) = S_{11} a_1(0)$$

Definition of S_{11} for a one-port

$$\boxed{S_{11} = \Gamma_L}$$

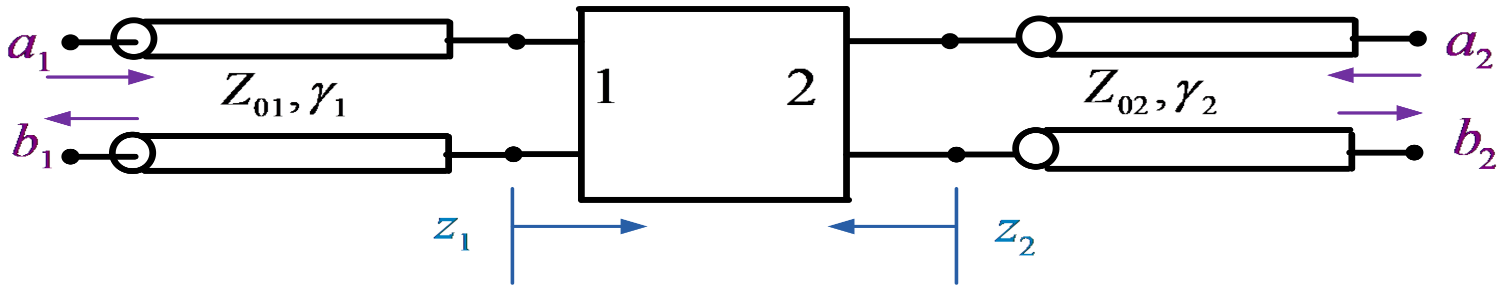
For a one-port network, S_{11} is the same as Γ_L.

Recall:

Incoming wave function $\equiv a_i(z_i) \equiv V_i^+(z_i)\big/\sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i) \equiv V_i^-(z_i)\big/\sqrt{Z_{0i}}$

A Two-Port Network



From linearity:

$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

Scattering matrix

or

$$\begin{bmatrix} b_1(0) \\ b_2(0) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix} \Rightarrow [b] = [S][a]$$

A Two-Port Network (cont.)

$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0)$$
$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0)$$

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0}$$ Output is matched ← input reflection coef. w/ output matched

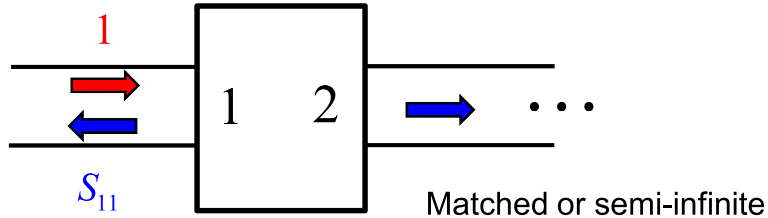
$$S_{12} = \left.\frac{b_1(0)}{a_2(0)}\right|_{a_1=0}$$ Input is matched ← reverse transmission coef. w/ input matched

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$ Output is matched ← forward transmission coef. w/ output matched

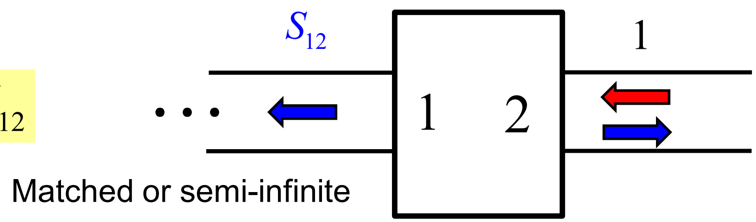
$$S_{22} = \left.\frac{b_2(0)}{a_2(0)}\right|_{a_1=0}$$ Input is matched ← output reflection coef. w/ input matched

A Two-Port Network (cont.)

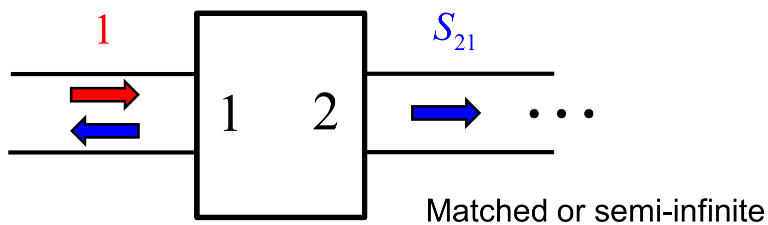
S_{11}



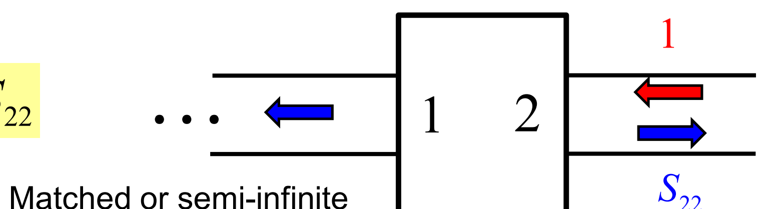
S_{12}



S_{21}



S_{22}



$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0}$$

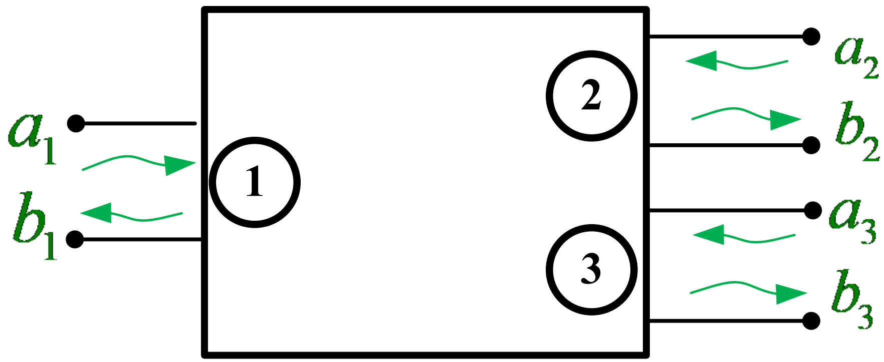
$$S_{12} = \left.\frac{b_1(0)}{a_2(0)}\right|_{a_1=0}$$

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

$$S_{22} = \left.\frac{b_2(0)}{a_2(0)}\right|_{a_1=0}$$

Three-Port Network

Illustration of a three-port network



$$b_1(0) = S_{11}a_1(0) + S_{12}a_2(0) + S_{13}a_3(0)$$

$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0) + S_{23}a_3(0)$$

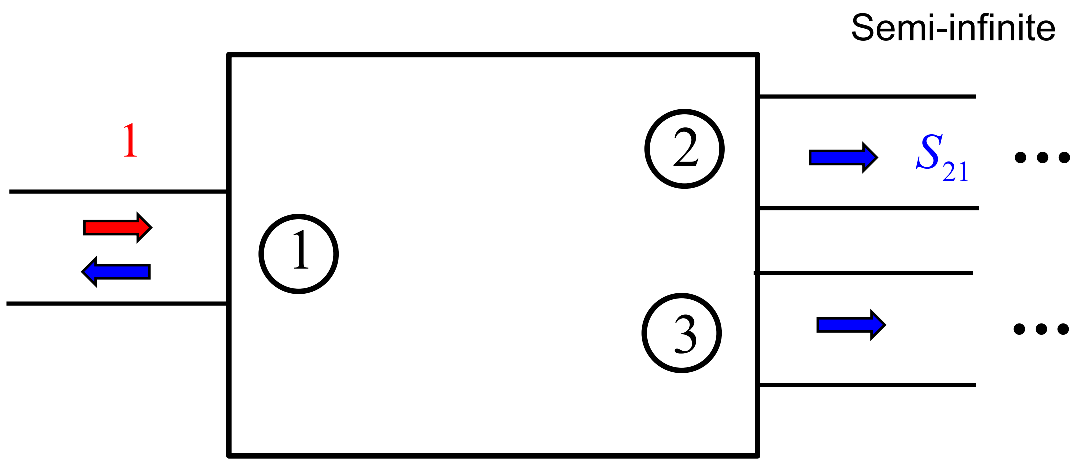
$$b_3(0) = S_{31}a_1(0) + S_{23}a_2(0) + S_{33}a_3(0)$$

To Illustrate:

$$b_2(0) = S_{21}a_1(0) + S_{22}a_2(0) + S_{23}a_3(0) \quad \Rightarrow \quad S_{21} = \frac{b_2(0)}{a_1(0)} \quad \text{when } a_2 = a_3 = 0$$

Three-Port Network (cont.)

Illustration of S_{21}:



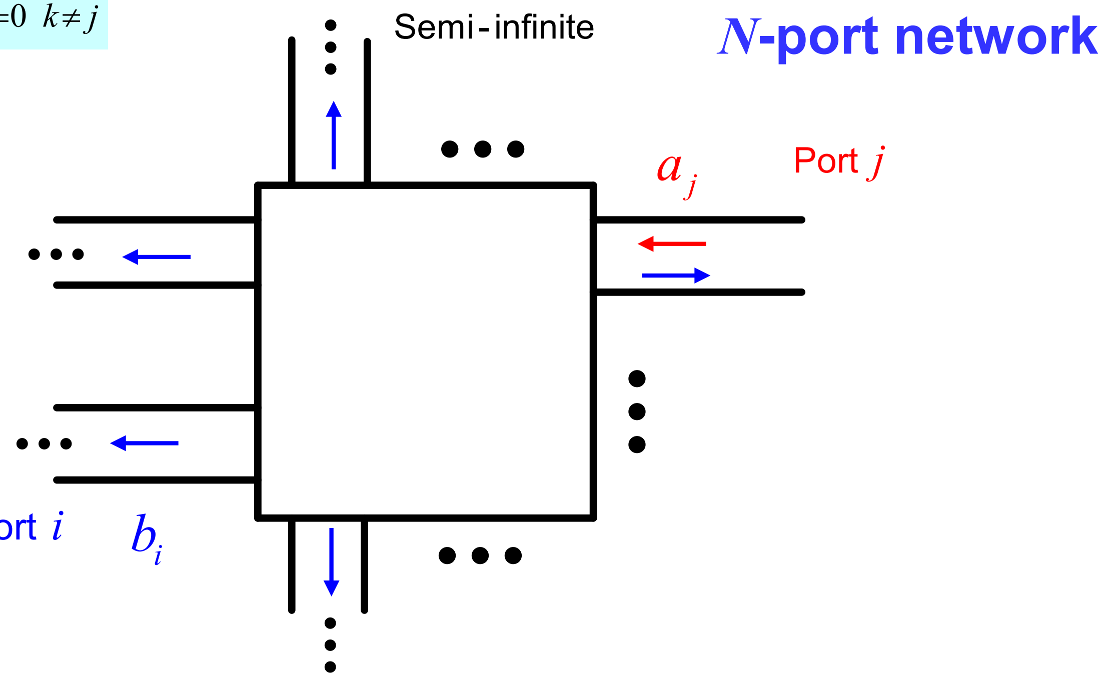
$$S_{21} = \frac{b_2(0)}{a_1(0)} \quad \text{when } a_2 = a_3 = 0$$

N-Port Network

For a general multiport network:

$$S_{ij} = \left.\frac{b_i(0)}{a_j(0)}\right|_{a_k=0\;\; k\neq j}$$

All ports except port j are semi-infinite (or with matched load at ports) with no incident wave.



We send in an incident wave on port j and measure the outgoing wave on port i, when all lines except j are semi-infinite (or terminated in a matched load), and thus there is an incident wave only on port j.

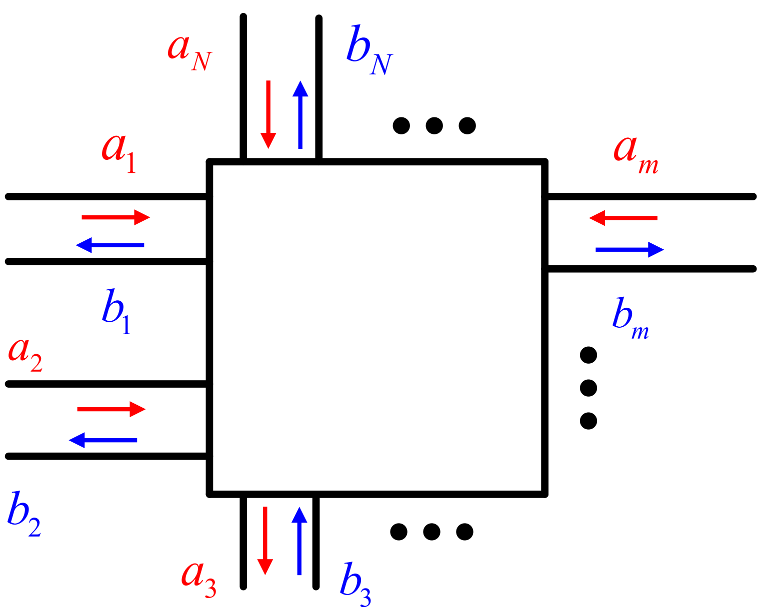
Summary of S Parameters

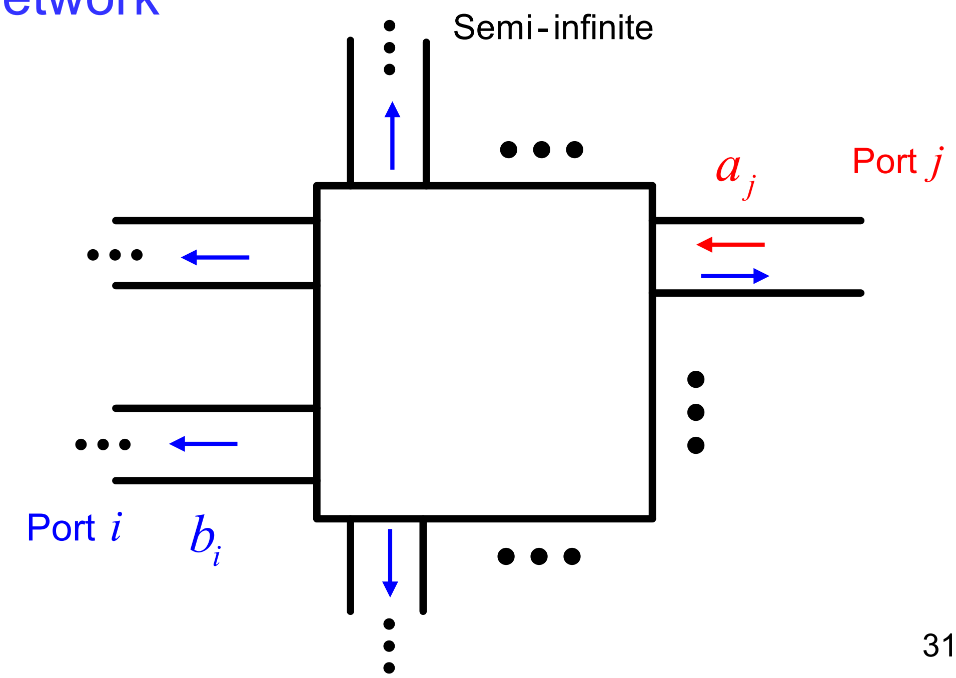
Summary of S Parameters

$$[b]=[S][a]$$

$$S_{ij}=\left.\frac{b_i(0)}{a_j(0)}\right|_{a_k=0\ \ k\neq j}$$

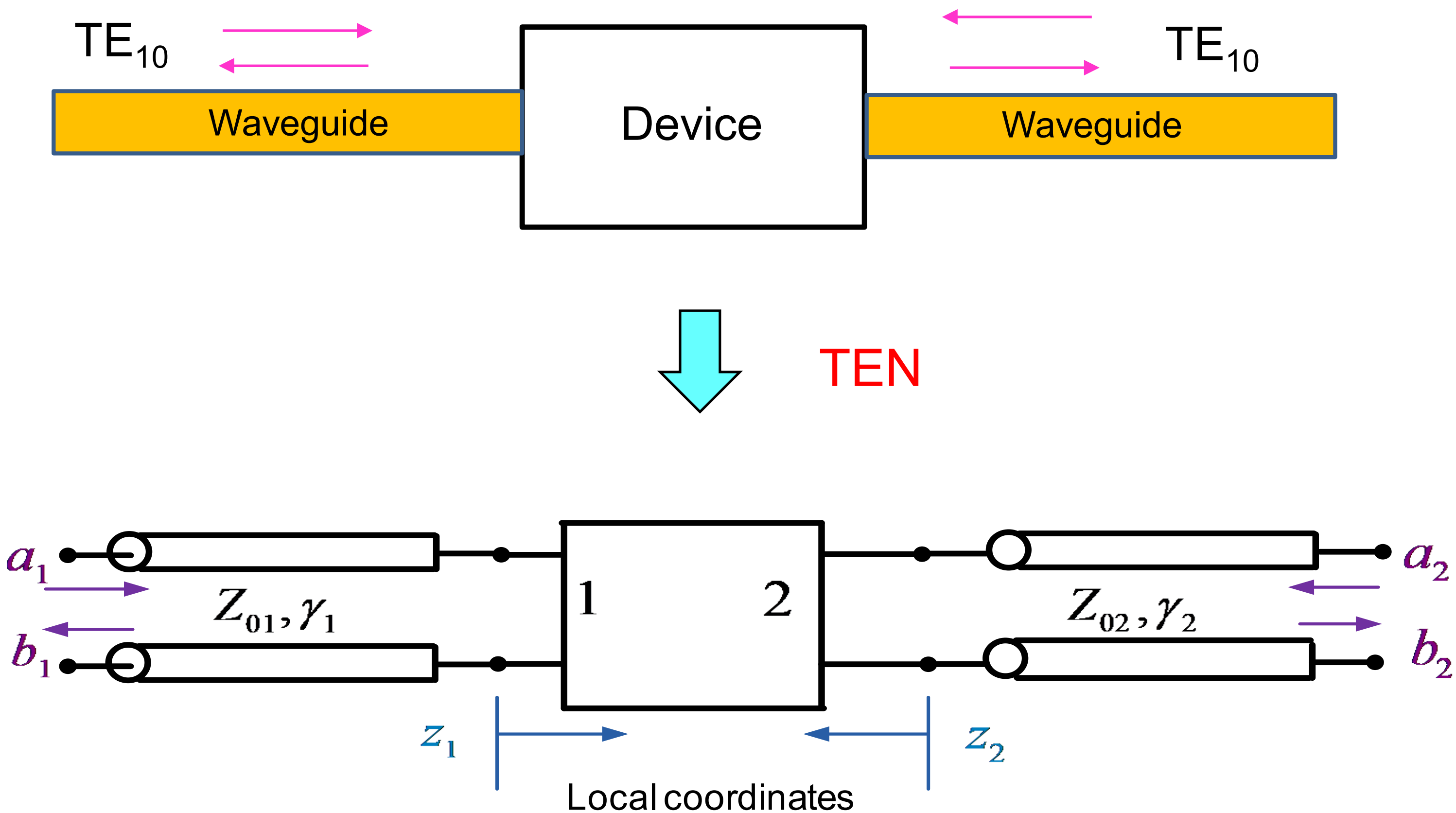
N-port network





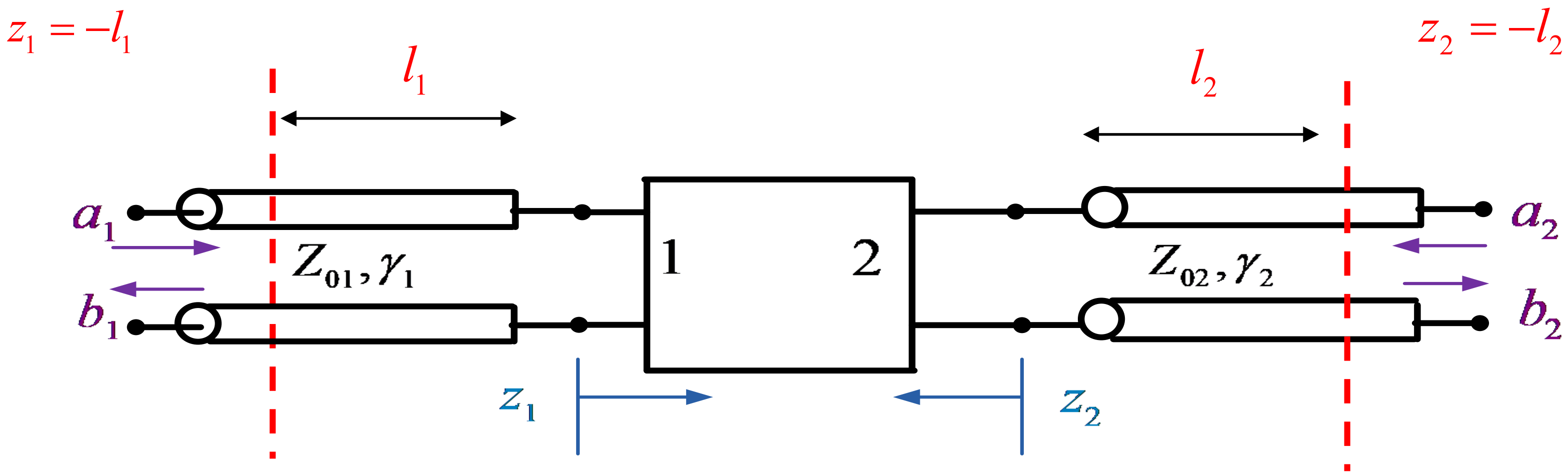
Scattering Parameters with Waveguides

A microwave system may have waveguides entering a device. In this case, the transmission lines are TEN models for the waveguides.



Shift in Reference Planes

(illustrated for a two-port)



$i, j = 1$ or 2

$$S'_{ij} = \left. \frac{b_i(z_i)}{a_j(z_j)} \right|_{a_k=0 \;\; k \neq j} \Rightarrow S'_{ij} = \left. \frac{b_i(0)\, e^{+\gamma_i z_i}}{a_j(0)\, e^{-\gamma_j z_j}} \right|_{a_k=0 \;\; k \neq j} \Rightarrow S'_{ij} = \left. \frac{b_i(0)\, e^{-\gamma_i l_i}}{a_j(0)\, e^{+\gamma_j l_j}} \right|_{a_k=0 \;\; k \neq j}$$

$$S'_{ij} = S_{ij}\, e^{-\gamma_i l_i} e^{-\gamma_j l_j}$$

Properties of the S Matrix

For reciprocal networks (networks containing only reciprocal materials), the S-matrix is symmetric:

$$S_{ij} = S_{ji} \qquad i \neq j$$

$$\Rightarrow \quad [S] = [S]^T$$

Example of a nonreciprocal material: a biased ferrite

Properties of the S Matrix (cont.)

❖ For lossless networks, the S-matrix is unitary†.

$$\Rightarrow \quad [S]^T [S]^* = [S]^* [S]^T = [U]$$

Identity matrix

Note :

If $[A][B] = [U]$

then $[B][A] = [U]$

Hence,

$$[S]^T = \left[S^*\right]^{-1} = [S]^{-1\,*}$$

$$\Rightarrow \quad [S]^{T*} = [S]^{-1}$$

(a "unitary" matrix)

Alternate notation:

$$[S]^{\dagger} = [S]^{H} \equiv [S]^{T*}$$

Therefore, $[S]^{\dagger} = [S]^{-1}$

("Hermetian conjugate" or "Hermetian transpose")

†A proof is in the Pozar book.

Properties of the S Matrix (cont.)

Start with the first part of the unitary equation:

$$[S]^T [S]^* = [U]$$

N-port network

Take (i, j) element $\Rightarrow \sum_{k=1}^{N} S_{ik}^T \; S_{kj}^* = \sum_{k=1}^{N} S_{ki} \; S_{kj}^* = \delta_{ij}$

$$\delta_{ij} = \begin{cases} 1\,;\, i = j \\ 0\,;\, i \neq j \end{cases}$$

Properties of the S Matrix (cont.)

$$\sum_{k=1}^{N} S_{ki}\, S_{kj}^{*} = \delta_{ij}$$

Interpretation: The inner product of columns i and j is zero unless $i = j$.

$$\begin{array}{cc} & \begin{array}{ccc} \underline{S}_1 \text{ vector} & & \underline{S}_3 \text{ vector} \end{array} \\ [S] = & \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{bmatrix} \end{array}$$

The rows also form orthogonal sets (this follows from starting with the second part of the unitary equation).

$$\underline{S}_i \cdot \underline{S}_j^{*} = \delta_{ij}$$

Physical interpretation: All of the power outgoing on the ports is equal to all of the power incident on the ports.

Comment on Normalization

Note: If all lines entering the network have the same characteristic impedance, then

$$S_{ij} = \frac{b_i(0)}{a_j(0)} = \left.\frac{V_i^-(0)}{V_j^+(0)}\right|_{V_k^+=0 \;\; k\neq j}$$

In this case, there is no difference between normalized and unnormalized scattering parameters.

In general (different characteristic impedances):

$$S_{ij} \equiv \left.\frac{V_i^-(0)}{V_j^+(0)}\right|_{V_k^+=0 \;\; k\neq j}$$

"unnormalized" scattering parameters

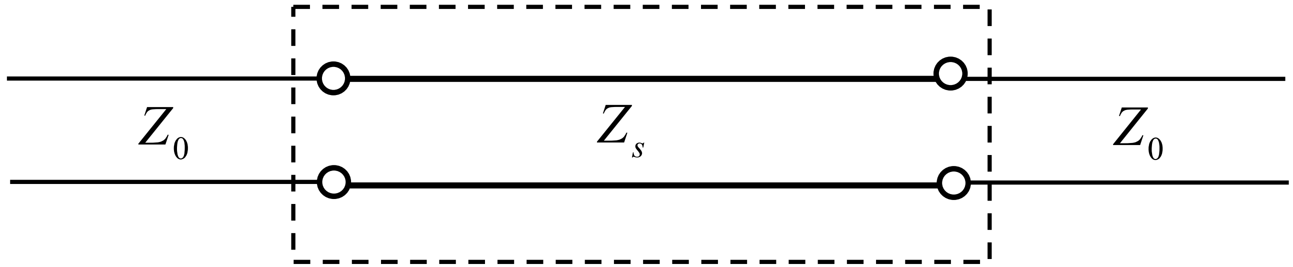
$$S_{ij} \equiv \left.\frac{b_i(0)}{a_j(0)}\right|_{a_k=0 \;\; k\neq j}$$

"normalized" scattering parameters

Note:
The unitary property of the scattering matrix requires normalized parameters.
We use normalized parameters in these notes.

Note on Z_0

Important Note:
The S parameters depend on Z_0.
(The Z and Y parameters do not.)



Example: The device is a section of transmission line.

$$S_{11} = 0 \quad \text{when} \quad Z_0 = Z_s$$

$$S_{11} \neq 0 \quad \text{when} \quad Z_0 \neq Z_s \; (\text{in general})$$

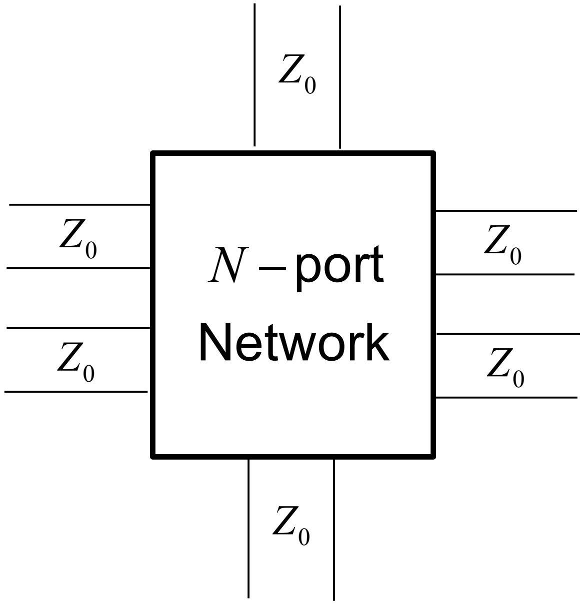
Scattering Parameters (cont.)

A general formula for converting from Z parameters to S parameters is:

$$[S]=\left(\frac{1}{Z_0}[Z]-[U]\right)\left([U]+\frac{1}{Z_0}[Z]\right)^{-1}$$

$$[Z]=Z_0\left([U]+[S]\right)\left(\left([U]-[S]\right)\right)^{-1}$$

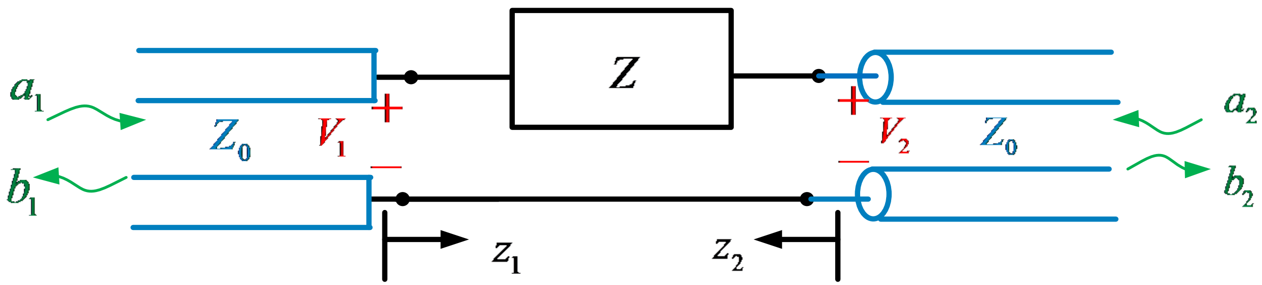
(This assumes all transmission lines are identical with characteristic impedance Z_0.)



Note:
The derivation is in the Pozar book.

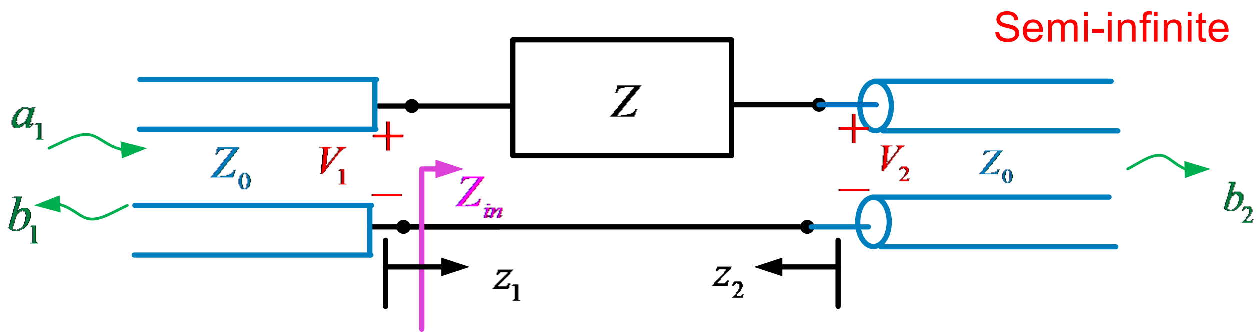
Example

Find the S parameters for a series impedance Z.



Note that two different coordinate systems are being used here!

Example (cont.)



S_{11} Calculation:

$$S_{11} = \left.\frac{b_1(0)}{a_1(0)}\right|_{a_2=0} = \frac{V_1^-(0)}{V_1^+(0)} = \left.\frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right|_{a_2=0} = \frac{(Z+Z_0) - Z_0}{(Z+Z_0) + Z_0}$$

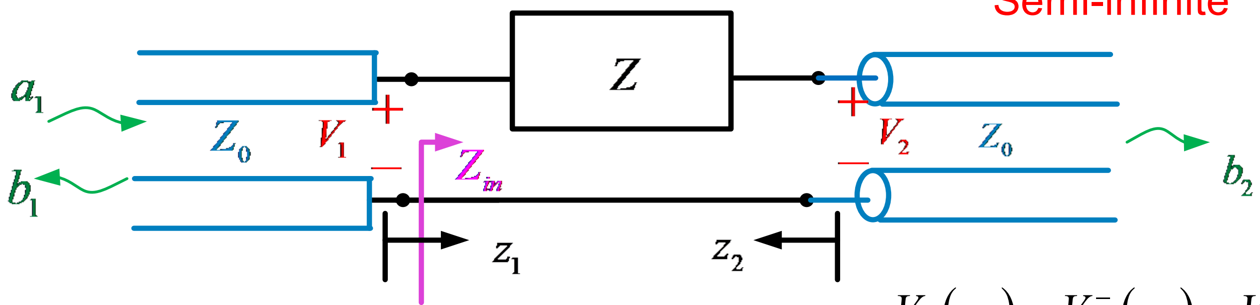
$$\Rightarrow \quad \boxed{S_{11} = \frac{Z}{Z+2Z_0}}$$

By symmetry:

$$\boxed{S_{22} = S_{11}}$$

Example (cont.)

S_{21} Calculation:



$$V_2(z_2) = V_2^-(z_2) = V_2^-(0)e^{+jk_{z2}z_2}$$

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0}$$

$$= \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0}$$

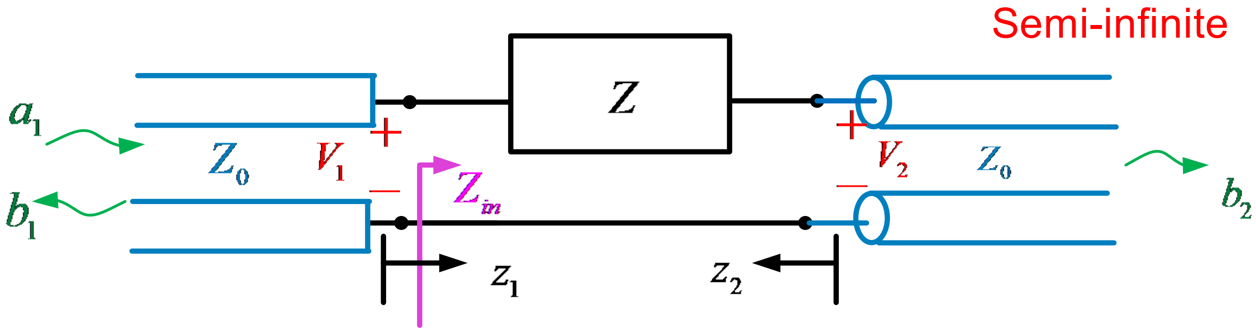
Voltage divider: $V_2^-(0) = V_2(0) = V_1(0)\left(\dfrac{Z_0}{Z+Z_0}\right)$

Input voltage: $V_1(0) = V_1^+(0)(1+S_{11})$

From first equation: $V_2^-(0) = V_1^+(0)(1+S_{11})\left(\dfrac{Z_0}{Z+Z_0}\right)$

$$\Rightarrow \quad S_{21} = \frac{V_2^-(0)}{V_1^+(0)} = (1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right)$$

Example (cont.)



Hence

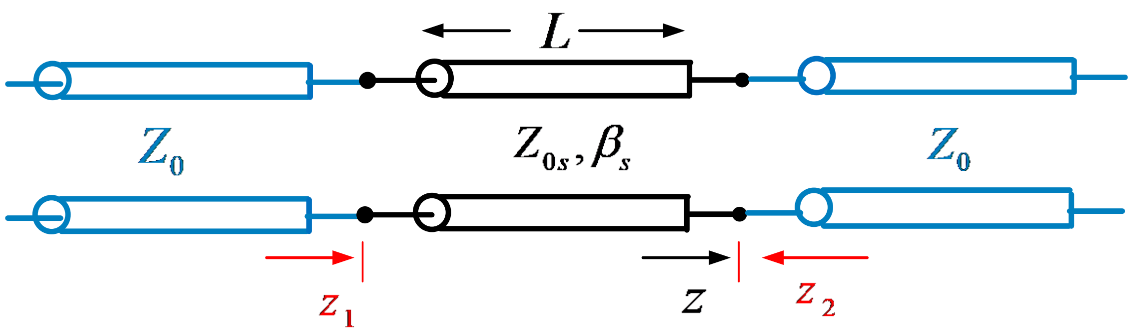
$$S_{21} = (1+S_{11})\left(\frac{Z_0}{Z+Z_0}\right) = \left(1+\frac{Z}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right) = \left(\frac{2Z+2Z_0}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right) = 2\left(\frac{Z+Z_0}{Z+2Z_0}\right)\left(\frac{Z_0}{Z+Z_0}\right)$$

We then have:

$$\boxed{S_{21} = \frac{2Z_0}{Z+2Z_0}} \qquad \boxed{S_{12} = S_{21}}$$

Example

Find the S parameters for a length L of transmission line.



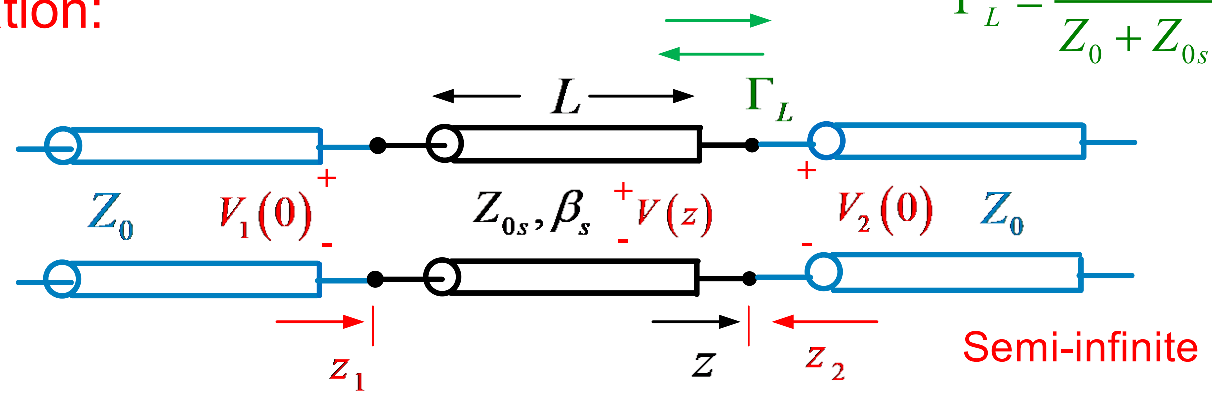
Note that three different coordinate systems are being used here!

(The subscript "s" denotes the middle section.)

Example (cont.)

S_{11} Calculation:

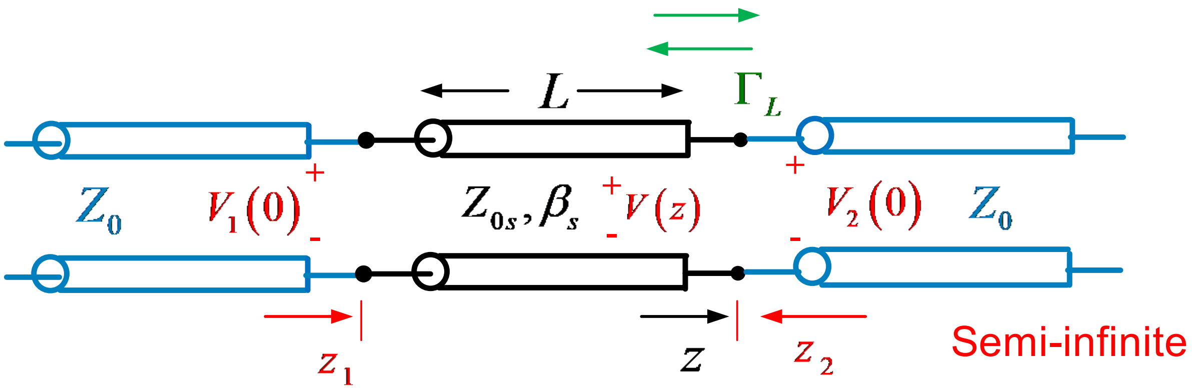
$$\Gamma_L = \frac{Z_0 - Z_{0s}}{Z_0 + Z_{0s}}$$



$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{\left.Z_{in}\right|_{a_2=0} - Z_0}{\left.Z_{in}\right|_{a_2=0} + Z_0} = S_{22} \;\; \text{(by symmetry)}$$

$$\left.Z_{in}\right|_{a_2=0} = Z_{0s}\frac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} = Z_{0s}\frac{\left(1+\Gamma_L e^{-j2\beta_s L}\right)}{\left(1-\Gamma_L e^{-j2\beta_s L}\right)}$$

Example (cont.)



Hence

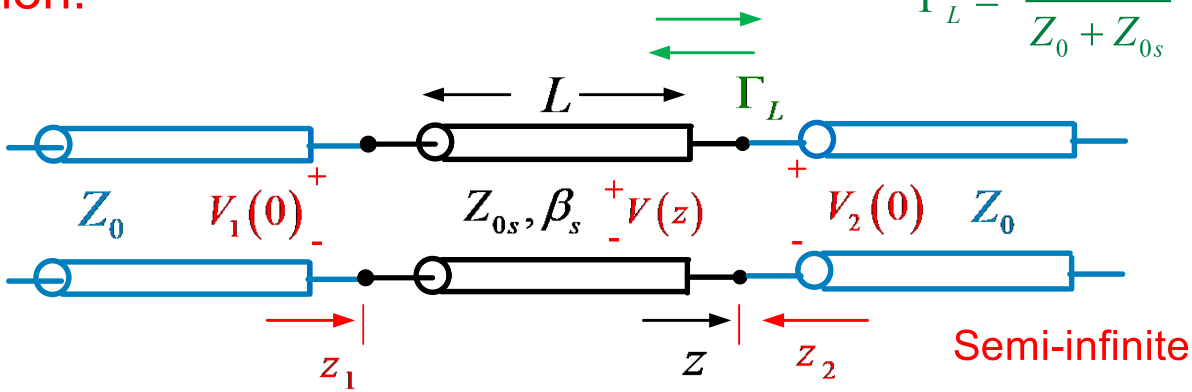
$$S_{11} = S_{22} = \frac{Z_{0s}\dfrac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} - Z_0}{Z_{0s}\dfrac{\left(Z_0 + jZ_{0s}\tan\beta_s L\right)}{\left(Z_{0s} + jZ_0\tan\beta_s L\right)} + Z_0}$$

Note: If $Z_{0s} = Z_0 \Rightarrow \left. Z_{in}\right|_{a_2=0} = Z_0 \Rightarrow S_{11} = S_{22} = 0$

Example (cont.)

S_{21} Calculation:

$$\Gamma_L = \frac{Z_0 - Z_{0s}}{Z_0 + Z_{0s}}$$



$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_2^-(0)/\sqrt{Z_0}}{V_1^+(0)/\sqrt{Z_0}}\right|_{a_2=0} = \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{V_2^-=0}$$

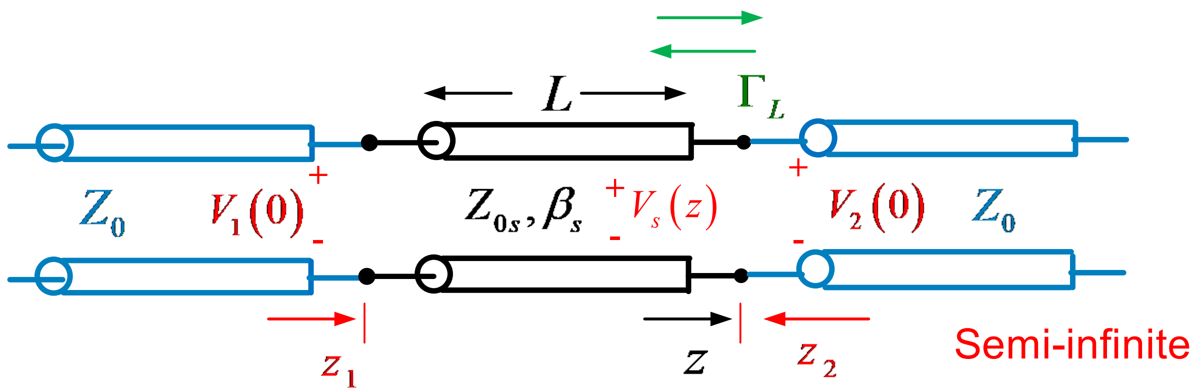
Total voltage at port 1: $V_1(0) = V_1^+(0)(1+S_{11})$

Hence, for the denominator of the S_{21} equation we have

$$V_1^+(0) = \frac{V_1(0)}{1+S_{11}}$$

We now try to put the numerator of the S_{21} equation in terms of $V_1(0)$.

Example (cont.)



$$V_2^-(0) = V_2(0) = V_s(0) = V_s^+(0)(1+\Gamma_L)$$

Next, we need to get $V_s^+(0)$ in terms of $V_1(0)$:

$$V_s(z) = V_s^+(0)e^{-j\beta_s z}\left(1+\Gamma_L e^{+j2\beta_s z}\right)$$

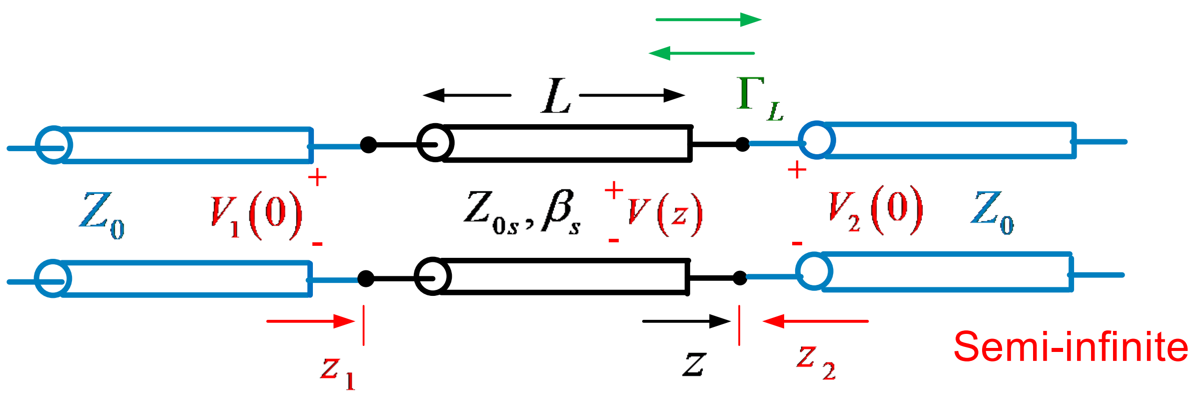
$$\Rightarrow V_1(0) = V_s(-L) = V_s^+(0)e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)$$

$$\Rightarrow V_s^+(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}$$

Hence, from the top equation we have

$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}(1+\Gamma_L)$$

Example (cont.)



$$V_2^-(0) = \frac{V_1(0)}{e^{+j\beta_s L}\left(1+\Gamma_L e^{-j2\beta_s L}\right)}\left(1+\Gamma_L\right)$$

$$V_1^+(0) = \frac{V_1(0)}{1+S_{11}}$$

Therefore, we have

$$S_{21} = \left.\frac{V_2^-(0)}{V_1^+(0)}\right|_{a_2=0} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}}$$

so

$$S_{21} = \frac{\left(1+S_{11}\right)\left(1+\Gamma_L\right)e^{-j\beta_s L}}{1+\Gamma_L e^{-j2\beta_s L}} = S_{12}$$ (the S matrix is symmetric)

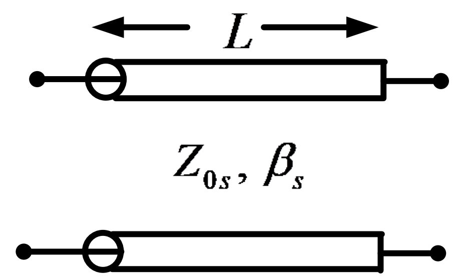
Example (cont.)

Special cases:

$$a)\;\; Z_{0s} = Z_0$$

$$Z_{0s} = Z_0 \Rightarrow S_{11} = S_{22} = 0, \quad \Gamma_L = 0$$

$$S_{21} = S_{12} = e^{-j\beta_s L}$$



$$a) \quad [S] = \begin{bmatrix} 0 & e^{-j\beta_s L} \\ e^{-j\beta_s L} & 0 \end{bmatrix}$$

$$b)\;\; L = \frac{\lambda_g}{2}$$

$$L = \frac{\lambda_g}{2} \;\Rightarrow\; \beta_s L = \frac{2\pi}{\lambda_g}\frac{\lambda_g}{2} = \pi$$

$$\Rightarrow Z_{in}\big|_{a_2=0} = Z_0 \quad \Rightarrow S_{11} = S_{22} = 0$$

$$e^{-j\beta_s L} = -1, \;\; e^{-j2\beta_s L} = +1 \quad \Rightarrow \quad S_{21} = -1$$

$$b) \quad [S] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

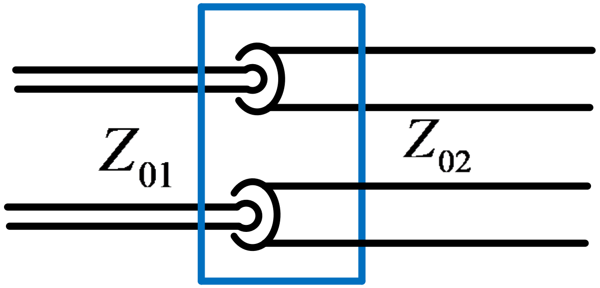
Example

Find the S parameters for a step-impedance discontinuity.

S_{11} Calculation:

$$S_{11} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

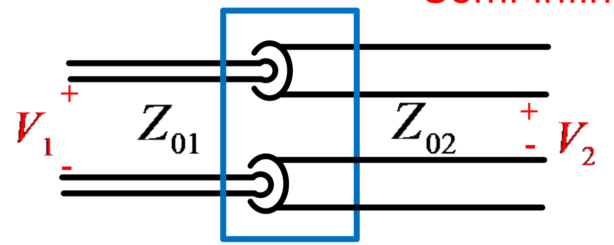
$$S_{22} = \frac{Z_{01} - Z_{02}}{Z_{02} + Z_{01}} = -S_{11}$$



S_{21} Calculation:

$$S_{21} = \left.\frac{b_2(0)}{a_1(0)}\right|_{a_2=0} = \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

Example (cont.)



Because of continuity of the voltage across the junction, we have:

$$V_2^-(0)\Big|_{a_2=0} = V_2(0)\Big|_{a_2=0} = V_1(0)\Big|_{a_2=0} = V_1^+(0)(1+S_{11})$$

$$S_{21} = \left.\frac{\dfrac{V_2^-(0)}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0} = \left.\frac{\dfrac{V_1^+(0)(1+S_{11})}{\sqrt{Z_{02}}}}{\dfrac{V_1^+(0)}{\sqrt{Z_{01}}}}\right|_{a_2=0}$$

$$1+S_{11} = 1+\frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} = \frac{2Z_{02}}{Z_{02}+Z_{01}}$$

so

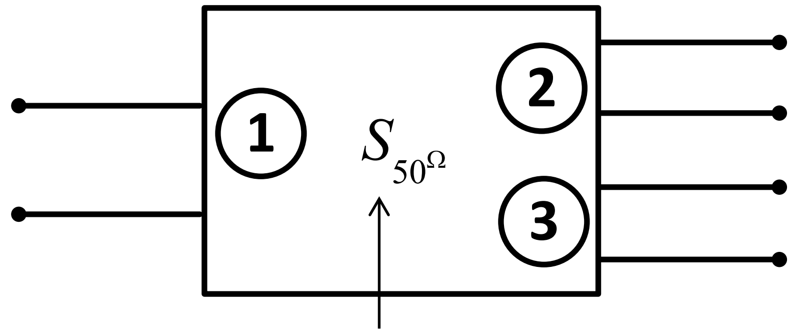
$$S_{21} = (1+S_{11})\sqrt{\frac{Z_{01}}{Z_{02}}}$$

Hence

$$\boxed{S_{21} = S_{12} = 2\frac{\sqrt{Z_{01}Z_{02}}}{Z_{01}+Z_{02}}}$$

Example

$$\left[S_{50\Omega}\right] = \begin{bmatrix} 0 & \dfrac{-j}{\sqrt{2}} & \dfrac{-j}{\sqrt{2}} \\ \dfrac{-j}{\sqrt{2}} & 0 & 0 \\ \dfrac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$



These are the S parameters assuming 50 Ω lines entering the device.

Not unitary $\rightarrow$ lossy

(For example, column 2 doted with the conjugate of column 3 is not zero.)

1) Find the input impedance looking into port 1 when ports 2 and 3 are terminated in 50 [Ω] loads.

2) Find the input impedance looking into port 1 when port 2 is terminated in a 75 [Ω] load and port 3 is terminated in a 50 [Ω] load.

Example (cont.)

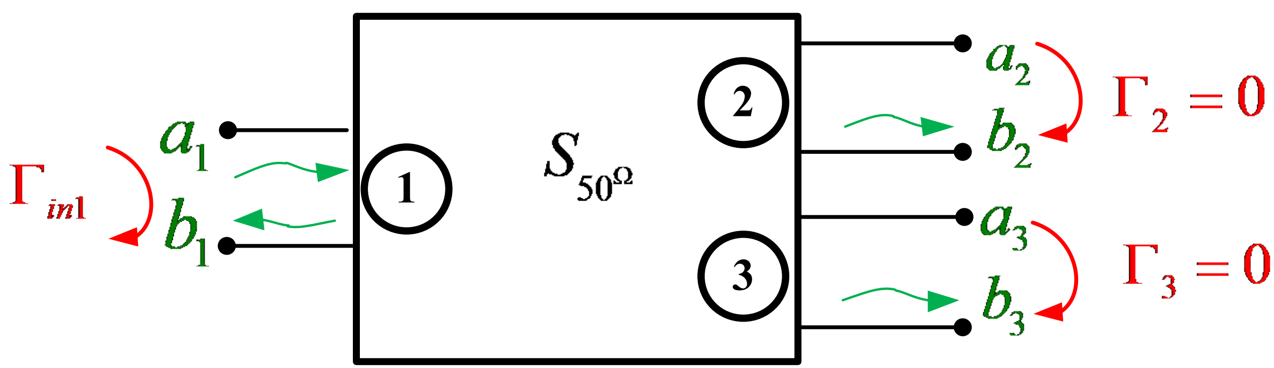
1) Z_{in} if ports 2 and 3 are terminated in 50 [Ω]: $a_2 = a_3 = 0$

$$b_1 = S_{11}a_1 + S_{12}\cancel{a_2} + S_{13}\cancel{a_3}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = S_{11} = 0 \quad \Rightarrow \quad Z_{in1} = Z_{01}$$

$$\boxed{Z_{in1} = 50\ [\Omega]}$$



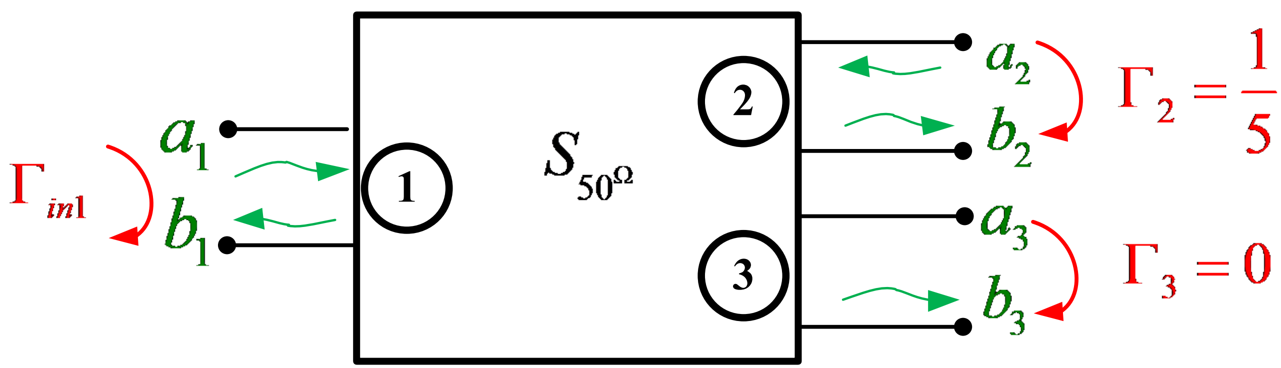
Example (cont.)

2) Z_{in} if port 2 is terminated in 75 [Ω] and port 3 in 50 [Ω]:

$$\Gamma_2 = \frac{a_2}{b_2} = \frac{75-50}{75+50} = \frac{1}{5}$$

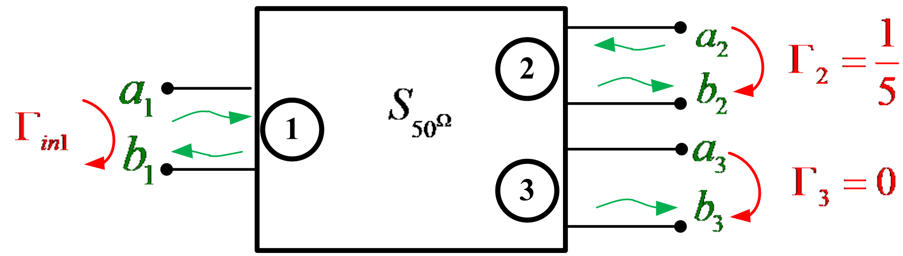
$$\Gamma_3 = \frac{a_3}{b_3} = \frac{50-50}{50+50} = 0$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



Example (cont.)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & 0 & 0 \\ \frac{-j}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



$$b_2 / a_1 = S_{21} + \cancel{S_{22}\left(\frac{a_2}{a_1}\right)} + \cancel{S_{23}\left(\frac{a_3}{a_1}\right)}$$

$$\Rightarrow \Gamma_{in1} = \frac{b_1}{a_1} = \cancel{S_{11}} + S_{12}\frac{a_2}{a_1} + \cancel{S_{13}\frac{a_3}{a_1}}$$

$$= S_{12}\left(\frac{\Gamma_2 b_2}{a_1}\right) = S_{12}\left(\Gamma_2 S_{21}\right) = \left(\frac{-j}{\sqrt{2}}\right)\left(\frac{1}{5}\right)\left(\frac{-j}{\sqrt{2}}\right) = -\frac{1}{10}$$

$$a_2 = \Gamma_2 b_2$$

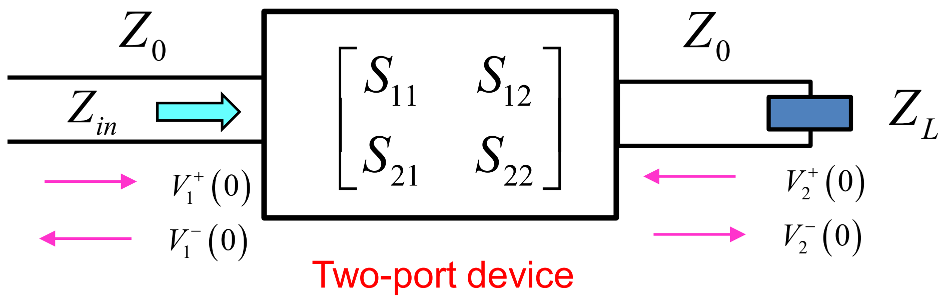
Hence

$$\boxed{Z_{in1} = 50\left(\frac{1+\Gamma_{in1}}{1-\Gamma_{in2}}\right) = 44.55\ [\Omega]}$$

Example

Find Γ_{in} for the general two-port system shown below.

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$



$$\Gamma_L = \frac{Z_L - Z_0}{Z_L - Z_0}$$

Assume : $V_1^+(0) = 1V$

so $\Gamma_{in} = V_1^-(0) = S_{11}(1) + S_{12}V_2^+(0)$

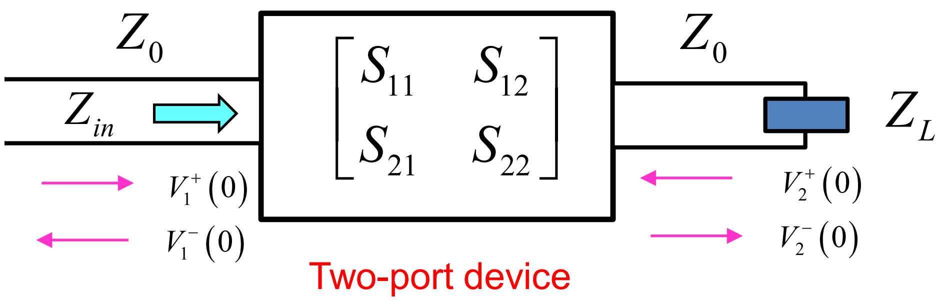
We also have:

$$\left. \begin{aligned} V_2^+(0) &= \Gamma_L V_2^-(0) \\ V_2^-(0) &= S_{21}(1) + S_{22}V_2^+(0) \end{aligned} \right\} \quad V_2^+(0) = \Gamma_L\left(S_{21}(1) + S_{22}V_2^+(0)\right)$$

Solve for $V_2^+(0)$

$$V_2^+(0) = \frac{\Gamma_L S_{21}}{1 - S_{22}\Gamma_L}$$

Example (cont.)



Hence

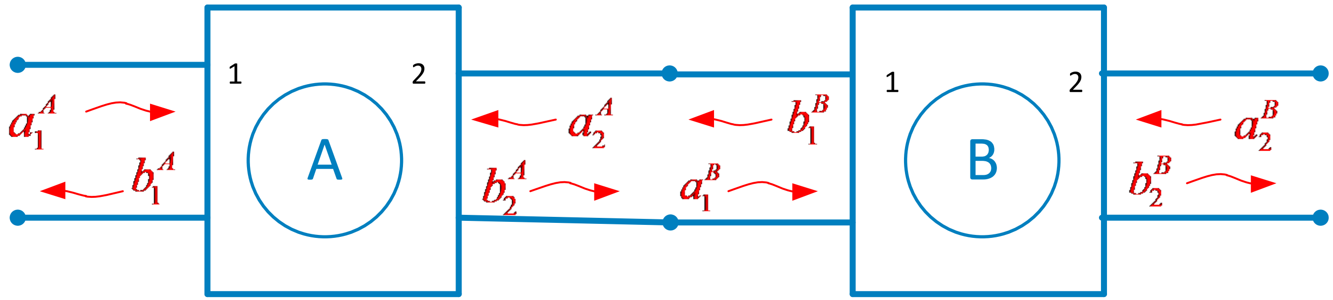
$$\Gamma_{in} = S_{11} + S_{12}V_2^+(0)$$
$$= S_{11} + S_{12}\left(\frac{\Gamma_L S_{21}}{1 - S_{22}\Gamma_L}\right)$$

so

$$\boxed{\Gamma_{in} = S_{11} + \left(\frac{\Gamma_L S_{12} S_{21}}{1 - S_{22}\Gamma_L}\right)} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L - Z_0}$$

Transfer (T) Matrix

For cascaded 2-port networks:



T Matrix:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$= [T] \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \dfrac{1}{S_{21}} & \dfrac{-S_{22}}{S_{21}} \\ \dfrac{S_{11}}{S_{21}} & S_{12} - \dfrac{S_{11}S_{22}}{S_{21}} \end{bmatrix}$$

(derivation omitted)

$$[S] = \begin{bmatrix} \dfrac{-T_{21}}{T_{22}} & \dfrac{1}{T_{22}} \\ T_{11} - \dfrac{T_{12}^2}{T_{22}} & \dfrac{T_{12}}{T_{22}} \end{bmatrix}$$

Transfer (T) Matrix (cont.)

Cascading property:

$$\begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} b_2^A \\ a_2^A \end{bmatrix}$$

(definition of T matrix)

$$= \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} a_1^B \\ b_1^B \end{bmatrix}$$

$$= \begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} T^B \end{bmatrix} \begin{bmatrix} b_2^B \\ a_2^B \end{bmatrix}$$

so that

$$\begin{bmatrix} a_1^A \\ b_1^A \end{bmatrix} = \underbrace{\begin{bmatrix} T^A \end{bmatrix} \begin{bmatrix} T^B \end{bmatrix}}_{\begin{bmatrix} T^{AB} \end{bmatrix}} \begin{bmatrix} b_2^B \\ a_2^B \end{bmatrix}$$

Conclusion:

$\boxed{\text{The } T \text{ matrices can be multiplied together, just as for ABCD matrices.}}$

Conversion Between Parameters
(Two-Ports)

TABLE 4.2 Conversions Between Two-Port Network Parameters

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\lvert Y\rvert}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\lvert Y\rvert}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\lvert Y\rvert}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\lvert Y\rvert}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\lvert Z\rvert}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\lvert Z\rvert}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\lvert Z\rvert}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\lvert Z\rvert}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\lvert Z\rvert}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\lvert Y\rvert}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

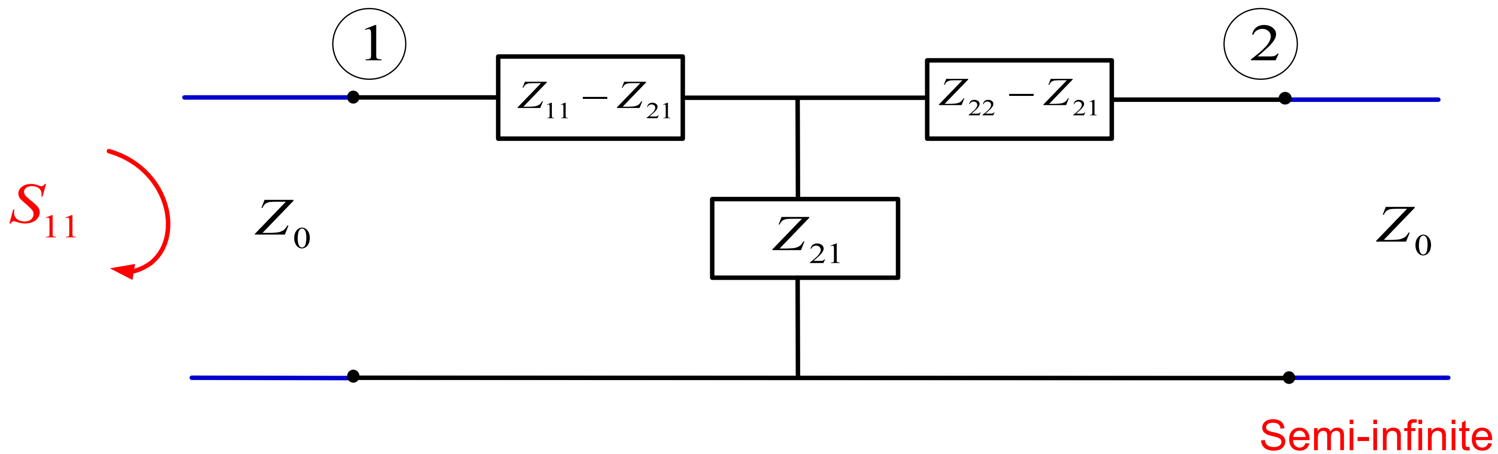
$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$.

Example

Derive S_{ij} from the Z parameters for a 2 port network.

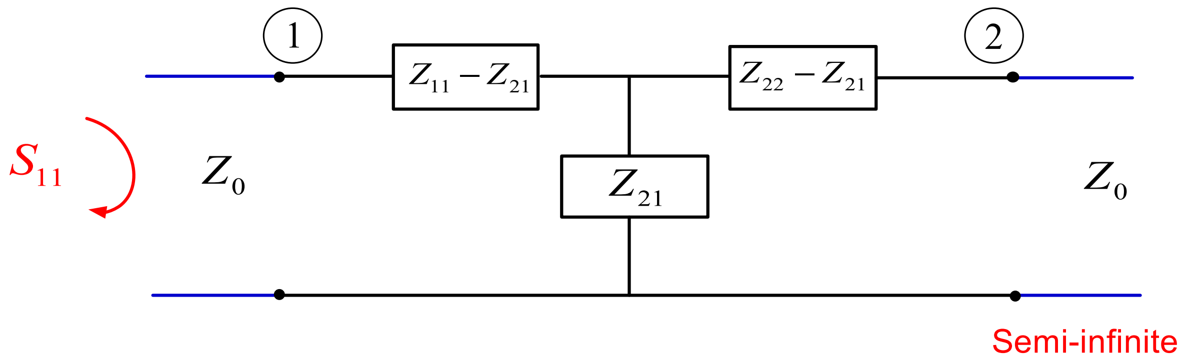
(The result is given inside row 1, column 2, of the previous table.)

S_{11} Calculation:



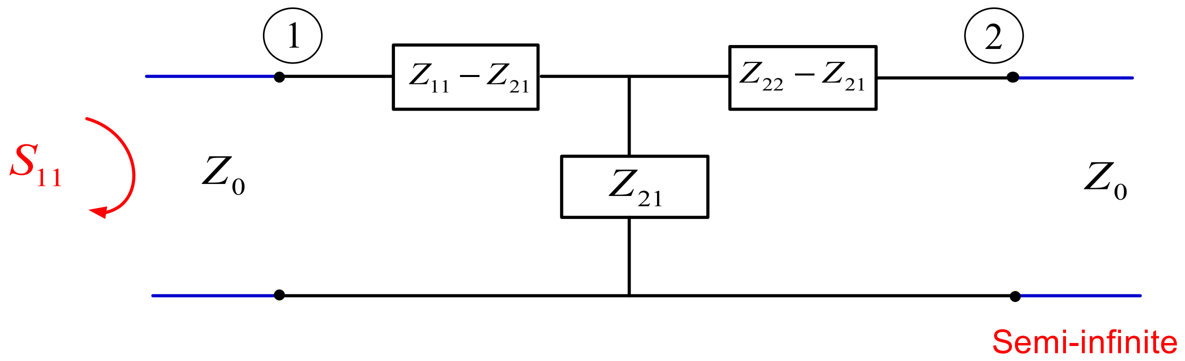
$$S_{11} = \Gamma_{in1} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \;, \qquad Z_{in} = (Z_{11} - Z_{21}) + \left((Z_{21}) \parallel \left[(Z_{22} - Z_{21}) + Z_0\right]\right)$$

Example (cont.)



$$
\begin{aligned}
Z_{in} &= \left(Z_{11} - Z_{21}\right) + \left(\left(Z_{21}\right) \parallel \left[\left(Z_{22} - Z_{21}\right) + Z_0\right]\right) \\
&= \left(Z_{11} - Z_{21}\right) + \frac{Z_{21}\left(Z_{22} - Z_{21} + Z_0\right)}{\cancel{Z_{21}} + \left(\left(Z_{22} - \cancel{Z_{21}}\right) + Z_0\right)} \\
&= \frac{\left(Z_{11} - Z_{21}\right)\left(Z_{22} + Z_0\right) + Z_{21}\left(Z_{22} - Z_{21} + Z_0\right)}{Z_{22} + Z_0} \\
&= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - \cancel{Z_{21}Z_{22}} - \cancel{Z_{21}Z_0} + \cancel{Z_{21}Z_{22}} - Z_{21}^2 + \cancel{Z_{21}Z_0}}{Z_{22} + Z_0} \\
&= \frac{Z_{11}Z_{22} + Z_{11}Z_0 - Z_{21}^2}{Z_{22} + Z_0}
\end{aligned}
$$

Example (cont.)



From the last slide:

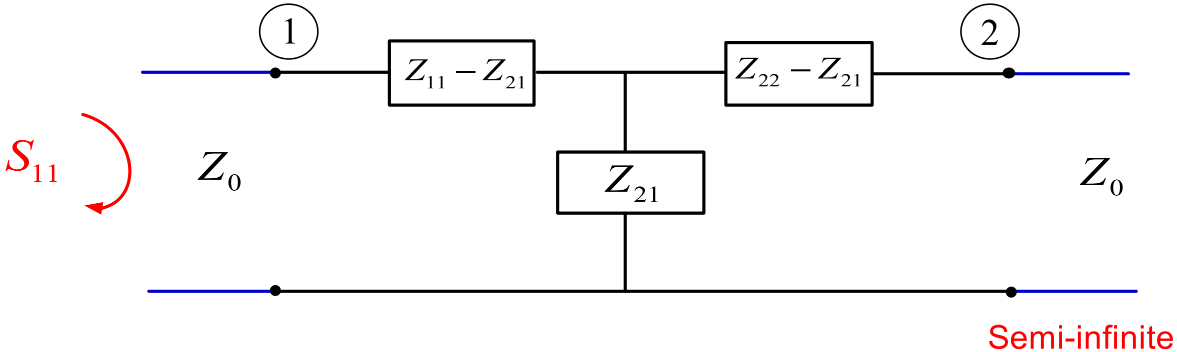
$$Z_{in}=\frac{Z_{11}\left(Z_0+Z_{22}\right)-Z_{21}^2}{Z_{22}+Z_0}$$

so

$$S_{11}=\frac{Z_{in}-Z_0}{Z_{in}+Z_0}=\frac{Z_{in}\left(Z_0+Z_{22}\right)-Z_0\left(Z_0+Z_{22}\right)}{Z_{in}\left(Z_0+Z_{22}\right)+Z_0\left(Z_0+Z_{22}\right)}=\frac{\left(Z_{11}\left(Z_0+Z_{22}\right)-Z_{21}^2\right)-Z_0\left(Z_0+Z_{22}\right)}{\left(Z_{11}\left(Z_0+Z_{22}\right)-Z_{21}^2\right)+Z_0\left(Z_0+Z_{22}\right)}$$

We next simplify this.

Example (cont.)

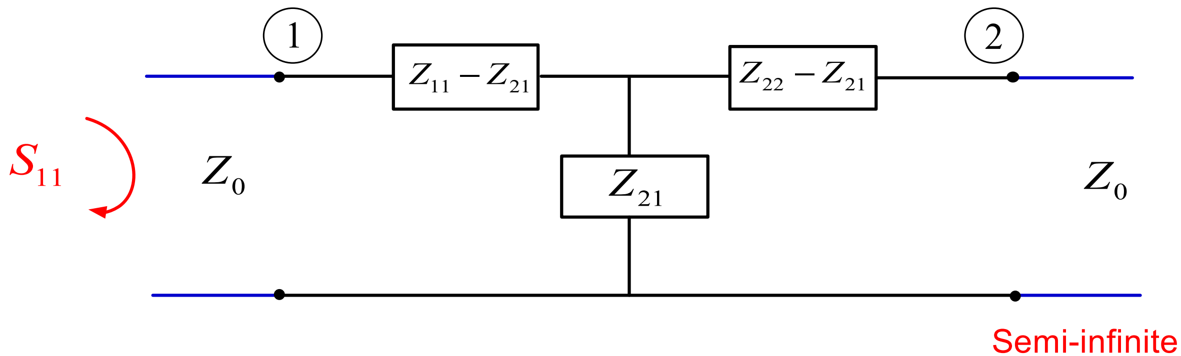


$$S_{11} = \frac{Z_{11}\left(Z_0 + Z_{22}\right) - Z_{21}^2 - Z_0\left(Z_0 + Z_{22}\right)}{Z_{11}\left(Z_0 + Z_{22}\right) - Z_{21}^2 + Z_0\left(Z_0 + Z_{22}\right)}$$

$$= \frac{Z_{11}Z_0 + Z_{11}Z_{22} - Z_{21}^2 - Z_0^2 - Z_0 Z_{22}}{Z_{11}Z_0 + Z_{11}Z_{22} - Z_{21}^2 + Z_0^2 + Z_0 Z_{22}}$$

$$= \frac{\left(Z_0 + Z_{22}\right)\left(Z_{11} - Z_0\right) - Z_{21}^2}{\left(Z_0 + Z_{22}\right)\left(Z_{11} + Z_0\right) - Z_{21}^2}$$

Example (cont.)



Hence

$$S_{11}=\frac{(Z_0+Z_{22})(Z_{11}-Z_0)-Z_{21}^2}{(Z_0+Z_{22})(Z_{11}+Z_0)-Z_{21}^2}$$

This agrees with the table.

Z
$\dfrac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$
$\dfrac{2Z_{12}Z_0}{\Delta Z}$
$\dfrac{2Z_{21}Z_0}{\Delta Z}$
$\dfrac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$

$\Delta Z=(Z_{11}+Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}$

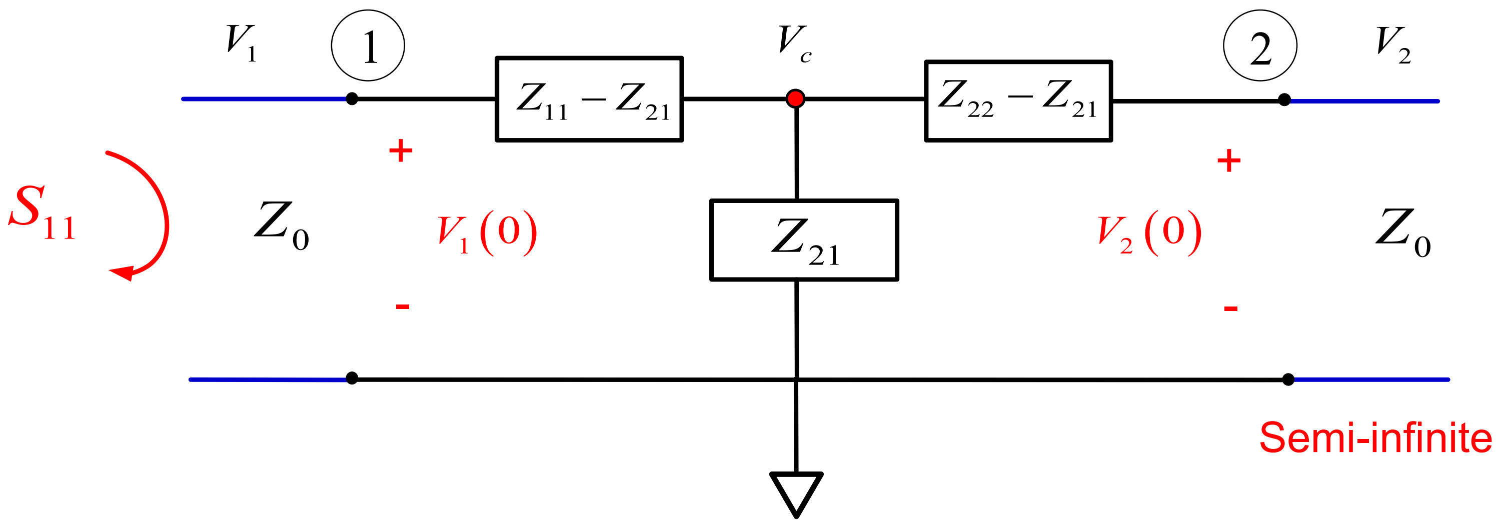
To get S_{22}, simply let $Z_{11}\rightarrow Z_{22}$ in the previous result. Hence, we have:

$$S_{22}=\frac{(Z_0+Z_{11})(Z_{22}-Z_0)-Z_{21}^2}{(Z_0+Z_{11})(Z_{22}+Z_0)-Z_{21}^2}$$

This agrees with the table.

Example (cont.)

S_{21} Calculation:



Assume $V_1^+(0) = 1\ [\mathrm{V}]$

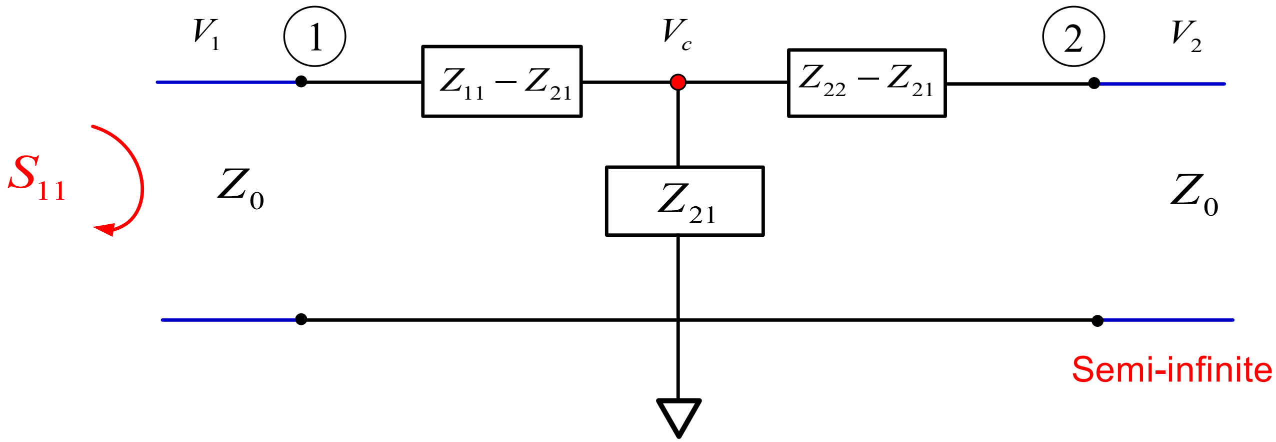
$$V_1(0) = 1 + S_{11}$$

$$\Rightarrow \quad S_{21} = V_2^-(0) = V_2(0)$$

Use voltage divider equation twice to get $V_2(0)$: $V_1(0) \rightarrow V_c \rightarrow V_2(0)$

Example (cont.)

$V_1(0) = 1 + S_{11}$



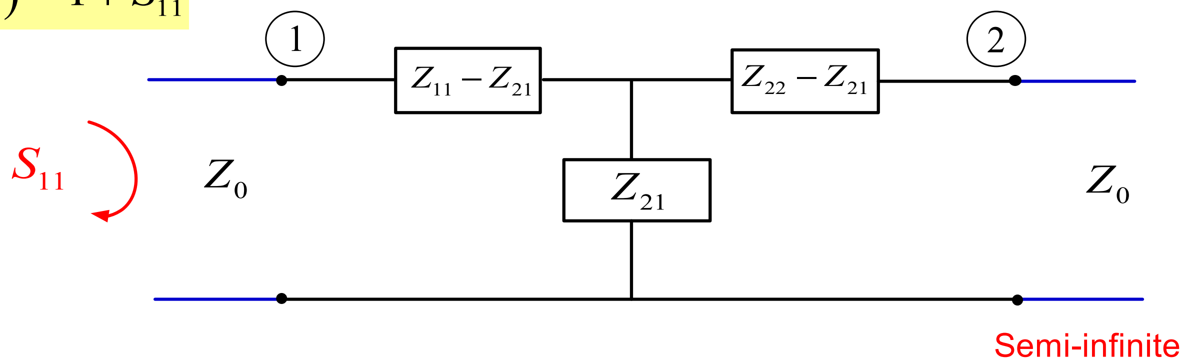
Use voltage divider equation twice:

$$V_c = V_1(0)\left(\frac{(Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}{(Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}\right)$$

$$V_2(0) = V_c\left(\frac{Z_0}{(Z_{22} - Z_{21}) + Z_0}\right)$$

Example (cont.)

$$V_1(0) = 1 + S_{11}$$



Hence

$$S_{21} = V_2(0) = (1 + S_{11})\left(\frac{(Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}{(Z_{11} - Z_{21}) + (Z_{21}) \,||\, \left[(Z_{22} - Z_{21}) + Z_0\right]}\right)\left(\frac{Z_0}{(Z_{22} - Z_{21}) + Z_0}\right)$$

where

$$1 + S_{11} = 1 + \frac{(Z_0 + Z_{22})(Z_{11} - Z_0) - Z_{21}^2}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

Example (cont.)

Our result:

$$S_{21} = (1 + S_{11}) \left(\frac{(Z_{21}) \| \left[(Z_{22} - Z_{21}) + Z_0 \right]}{(Z_{11} - Z_{21}) + (Z_{21}) \| \left[(Z_{22} - Z_{21}) + Z_0 \right]} \right) \left(\frac{Z_0}{(Z_{22} - Z_{21}) + Z_0} \right)$$

where

$$1 + S_{11} = 1 + \frac{(Z_0 + Z_{22})(Z_{11} - Z_0) - Z_{21}^2}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

After simplifying, we should get the result in the table:

(You are welcome to check it!)

$$S_{21} = S_{21} = \frac{2 Z_{12} Z_0}{(Z_0 + Z_{22})(Z_{11} + Z_0) - Z_{21}^2}$$

This is the result in the table.

Example (cont.)

Different approach: Use the formula on slide 40:

$$[S] = \left(\frac{1}{Z_0}[Z] - [U]\right)\left([U] + \frac{1}{Z_0}[Z]\right)^{-1}$$

so

$$[S] = \left(\left[\bar{Z}\right] - [U]\right)\left([U] + \left[\bar{Z}\right]\right)^{-1}$$

where

$$\left[\bar{Z}\right] \equiv \frac{1}{Z_0}[Z]$$

Hence, for a two-port, we have

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left(\begin{bmatrix} \bar{Z}_{11} - 1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} - 1 \end{bmatrix}\right)\left(\begin{bmatrix} \bar{Z}_{11} + 1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} + 1 \end{bmatrix}\right)^{-1}$$

Example (cont.)

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left(\begin{bmatrix} \bar{Z}_{11} - 1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} - 1 \end{bmatrix} \right) \left(\begin{bmatrix} \bar{Z}_{11} + 1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} + 1 \end{bmatrix} \right)^{-1}$$

Hence,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left(\begin{bmatrix} \bar{Z}_{11} - 1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} - 1 \end{bmatrix} \right) \left(\begin{bmatrix} \bar{Z}_{22} + 1 & -\bar{Z}_{12} \\ -\bar{Z}_{21} & \bar{Z}_{11} + 1 \end{bmatrix} \frac{1}{\Delta} \right)$$

where

$$\Delta = \text{determinant} = \left(\bar{Z}_{11} + 1\right)\left(\bar{Z}_{22} + 1\right) - \bar{Z}_{21}\bar{Z}_{12}$$

This gives us directly the components of the S matrix.

Example (cont.)

Examining the components, we have:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left(\begin{bmatrix} \bar{Z}_{11}-1 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22}-1 \end{bmatrix}\right)\left(\begin{bmatrix} \bar{Z}_{22}+1 & -\bar{Z}_{12} \\ -\bar{Z}_{21} & \bar{Z}_{11}+1 \end{bmatrix}\frac{1}{\Delta}\right)$$

$$\Downarrow$$

$$S_{11} = \frac{1}{\Delta}\left(\left(\bar{Z}_{11}-1\right)\left(\bar{Z}_{22}+1\right)-\bar{Z}_{12}\bar{Z}_{21}\right)$$

$$S_{12} = \frac{1}{\Delta}\left(-\left(\bar{Z}_{11}-1\right)\bar{Z}_{12}+\bar{Z}_{12}\left(\bar{Z}_{11}+1\right)\right)$$

$$S_{21} = \frac{1}{\Delta}\left(-\left(\bar{Z}_{22}-1\right)\bar{Z}_{21}+\bar{Z}_{21}\left(\bar{Z}_{22}+1\right)\right)$$

$$S_{22} = \frac{1}{\Delta}\left(\left(\bar{Z}_{11}+1\right)\left(\bar{Z}_{22}-1\right)-\bar{Z}_{12}\bar{Z}_{21}\right)$$

Example (cont.)

Simplifying the terms S_{12} and S_{21}, we have:

$$S_{11} = \frac{1}{\Delta}\left(\left(\bar{Z}_{11} - 1\right)\left(\bar{Z}_{22} + 1\right) - \bar{Z}_{12}\bar{Z}_{21}\right)$$

$$S_{12} = \frac{1}{\Delta}\left(2\bar{Z}_{12}\right)$$

$$S_{21} = \frac{1}{\Delta}\left(2\bar{Z}_{21}\right)$$

$$S_{22} = \frac{1}{\Delta}\left(\left(\bar{Z}_{11} + 1\right)\left(\bar{Z}_{22} - 1\right) - \bar{Z}_{12}\bar{Z}_{21}\right)$$

This agrees with the table.

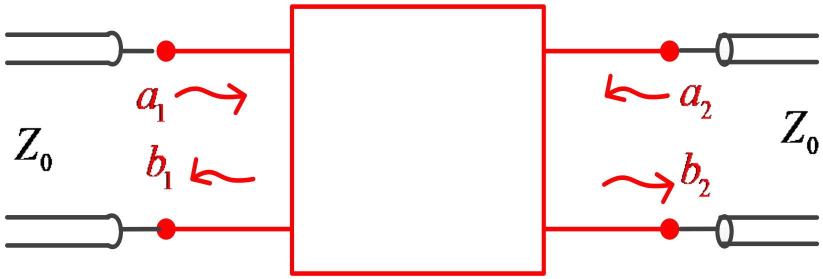
Z
$\dfrac{(Z_{11} - Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21}}{\Delta Z}$
$\dfrac{2Z_{12}Z_0}{\Delta Z}$
$\dfrac{2Z_{21}Z_0}{\Delta Z}$
$\dfrac{(Z_{11} + Z_0)(Z_{22} - Z_0) - Z_{12}Z_{21}}{\Delta Z}$

$$\Delta Z = (Z_{11} + Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21}$$

Signal-Flow Graph

This is a way to graphically represent the S parameters.

Please see the Pozar book for more details.

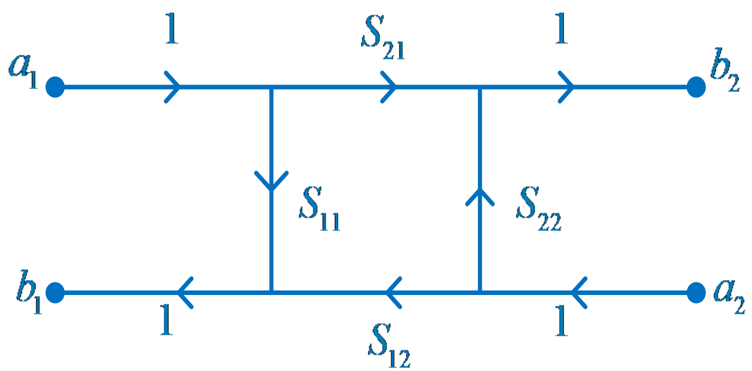


The wave amplitudes are represented as <u>nodes</u> in the single-flow graph.

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

(wave amplitudes evaluated at $z_i = 0$)

$\Rightarrow$



Rule: The value at each node is the sum of the values coming into the node from the various other nodes.