Adapted from notes by
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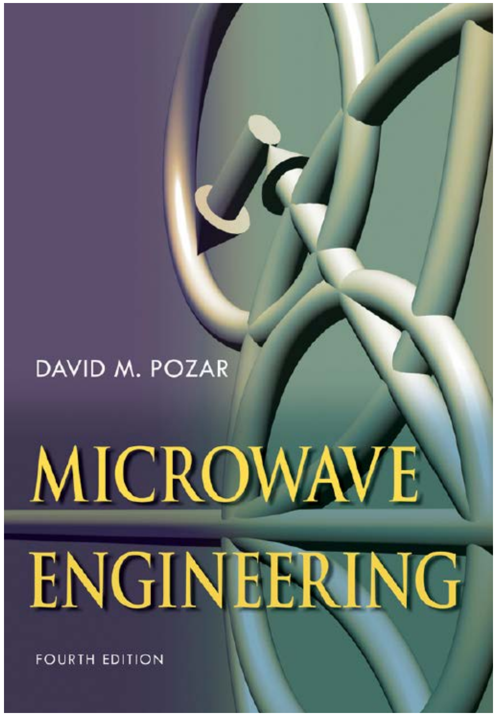
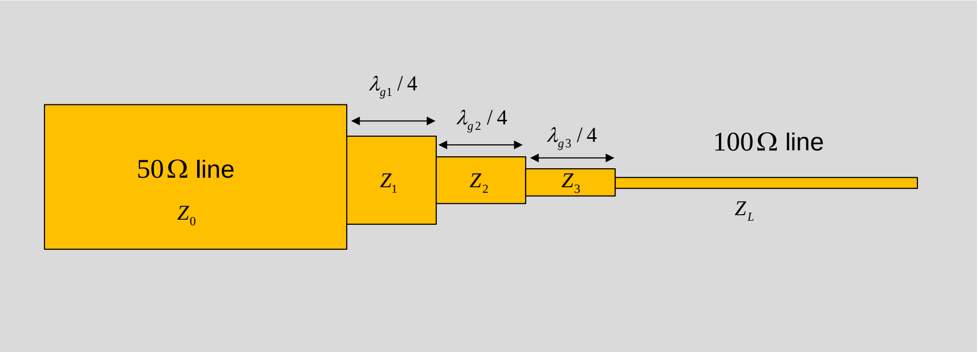


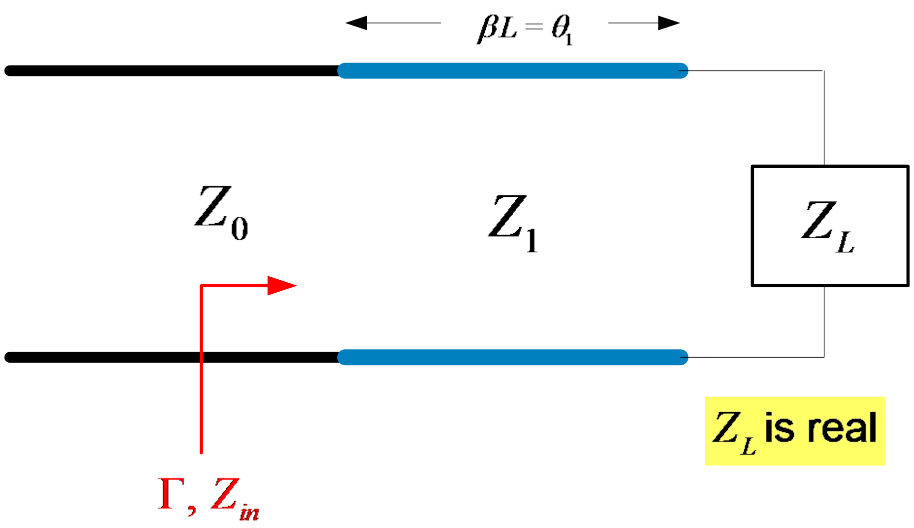
Notes 19

Multistage Transformers



Single-stage Transformer

A single-stage transformer length is shown below.



$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \qquad Z_{in} = Z_1 \left(\frac{1 + \Gamma_L e^{-j2\theta_1}}{1 - \Gamma_L e^{-j2\theta_1}} \right) \qquad \Gamma_L = \frac{Z_L - Z_1}{Z_L + Z_1}$$

Note: Normally $\theta_1 = \pi/2$ at the center frequency.

Single-stage Transformer (cont.)

Assume small reflections:

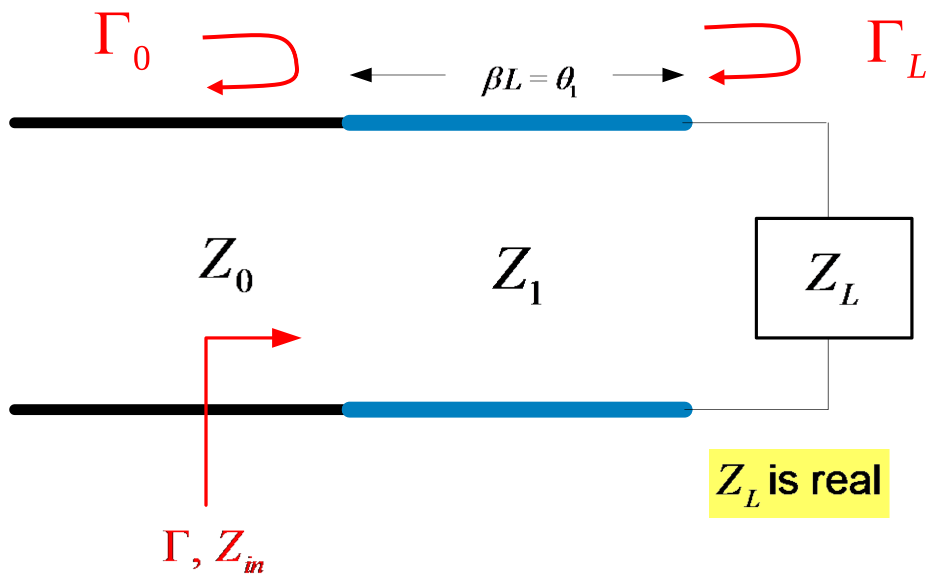
$$Z_0 \approx Z_1 \approx Z_L$$

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$

$$|\Gamma_0| << 1$$

$$\Gamma_L = \frac{Z_L - Z_1}{Z_L + Z_1}$$

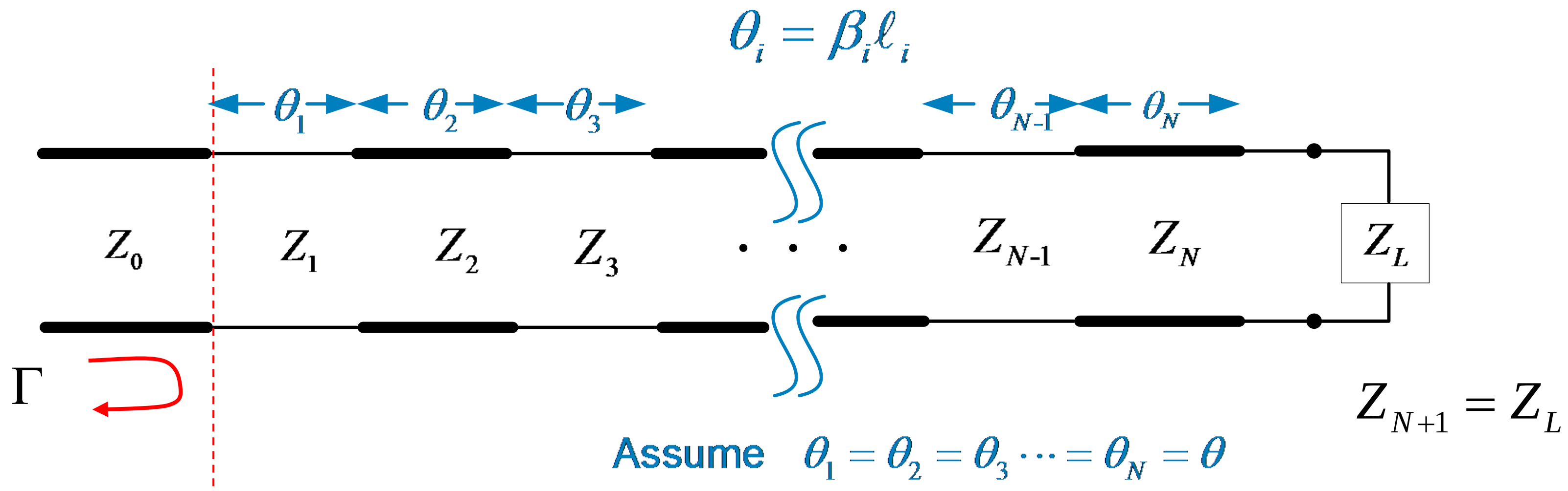
$$|\Gamma_L| << 1$$



$$\Gamma \approx \Gamma_0 + \Gamma_L e^{-j2\theta_1}$$

(This neglects terms that are quadratic and higher.)

Multistage Transformer



Assuming small reflections:

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + ... + \Gamma_N e^{-j2N\theta}$$

where $\Gamma_n = \dfrac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$

Note that this is a polynomial in powers of $z = \exp(-j2\theta)$.

Multistage Transformer (cont.)

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + \ldots.. + \Gamma_N e^{-j2N\theta}$$ (N+1 terms)

If we assume symmetric reflections of the sections about the center of the structure (not a symmetric layout of line impedances), we have:

$$\Gamma_0 = \Gamma_N, \quad \Gamma_1 = \Gamma_{N\text{-}1}, \quad \Gamma_2 = \Gamma_{N\text{-}2}, \ \ldots$$

$$\Rightarrow \Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + \ldots.. + \Gamma_1 e^{-j(2N-2)\theta} + \Gamma_0 e^{-j2N\theta}$$

or

$$\Gamma(\theta) \approx e^{-jN\theta}\left[\Gamma_0\left(e^{jN\theta} + e^{-jN\theta}\right) + \Gamma_1\left(e^{j(N-2)\theta} + e^{-j(N-2)\theta}\right) + \ldots\right]$$

Last term

$$N \text{ even} \rightarrow \text{last term} = \Gamma_{\frac{N}{2}}$$

$$N \text{ odd} \rightarrow \text{last term} = \Gamma_{\frac{N-1}{2}}\left(e^{j\theta} + e^{-j\theta}\right)$$

Multistage Transformer (cont.)

Hence, for symmetric reflections we can also write:

$$\Gamma(\theta) \approx \begin{cases} 2e^{-jN\theta}\left[\Gamma_0 \cos(N\theta) + \Gamma_1 \cos((N-2)\theta) + \ldots + \Gamma_n \cos((N-2n)\theta) + \ldots + \frac{1}{2}\Gamma_{\frac{N}{2}}\right]; & N \text{ even} \\ 2e^{-jN\theta}\left[\Gamma_0 \cos(N\theta) + \Gamma_1 \cos((N-2)\theta) + \ldots + \Gamma_n \cos((N-2n)\theta) + \ldots + \Gamma_{\frac{N-1}{2}} \cos\theta\right]; & N \text{ odd} \end{cases}$$

Note: This is a finite Fourier cosine series.

Multistage Transformer (cont.)

Design philosophy:

If we choose a response for $\Gamma(\theta)$ that is in the form of either a polynomial (in powers of $z = \exp(-j2\theta)$) or a Fourier cosine series, we can obtain the needed values of Γ_n and hence complete the design.

Design #1

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + \ldots\ldots + \Gamma_N e^{-j2N\theta}$$

(does not assume symmetric reflections)

or

Design #2

$$\Gamma(\theta) \approx \begin{cases} 2e^{-jN\theta}\left[\Gamma_0 \cos(N\theta) + \Gamma_1 \cos((N-2)\theta) + \ldots + \Gamma_n \cos((N-2n)\theta) + \ldots + \frac{1}{2}\Gamma_{\frac{N}{2}}\right]; \; N \text{ even} \\ 2e^{-jN\theta}\left[\Gamma_0 \cos(N\theta) + \Gamma_1 \cos((N-2)\theta) + \ldots + \Gamma_n \cos((N-2n)\theta) + \ldots + \Gamma_{\frac{N-1}{2}} \cos\theta\right]; \; N \text{ odd} \end{cases}$$

(assumes symmetric reflections)

Multistage Transformer (cont.)

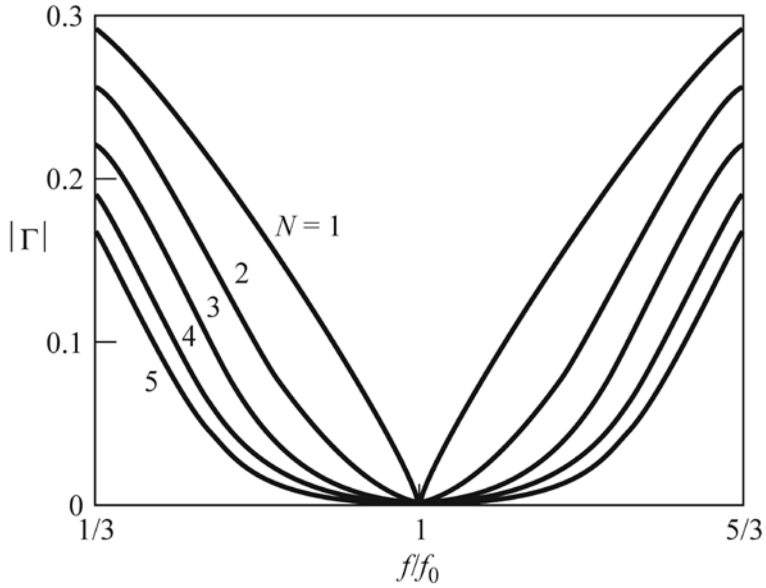
Two common designs:

- **Binomial (Butterworth):**
  - This design has the “flattest” response about the center frequency.
  - This is an example of Design 1.

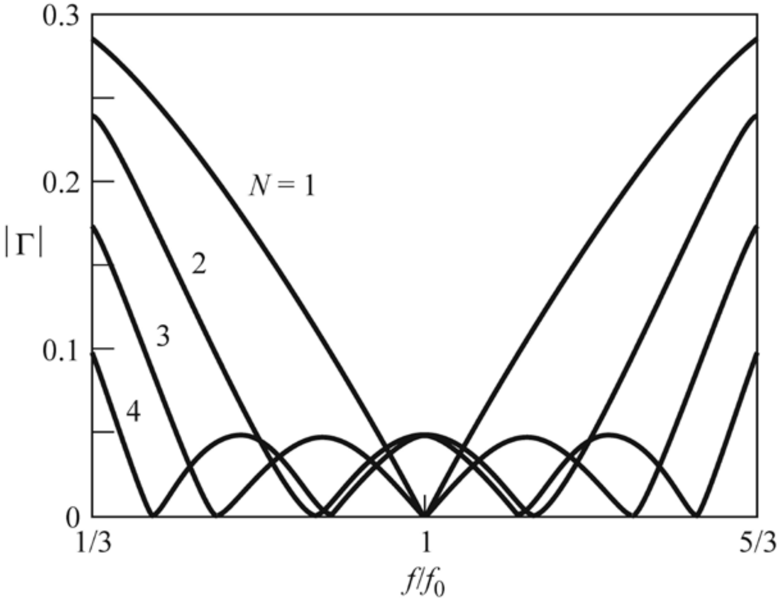
- **Chebyshev**:
  - This design has a constant ripple over the bandwidth.
  - This design has the largest bandwidth for a given acceptable level of reflection coefficient.
  - This is an example of Design 2.

Multistage Transformer (cont.)

Examples of Reflection Coefficient Responses



Binomial



Chebyshev

Binomial (Butterworth*) Multistage Transformer

This is an example of Design #1.

Choose:

$$\Gamma(\theta) = A\left(1+e^{-j2\theta}\right)^N$$

(A is real, could be positive or negative)

$$\Gamma(\theta) = Ae^{-jN\theta}\left(e^{j\theta}+e^{-j\theta}\right)^N = A\,e^{-jN\theta}\,2^N\cos^N\theta$$

$$\Rightarrow \quad \Gamma(\theta) = A\,e^{-jN\theta}\,2^N\cos^N\theta$$

(alternative useful form)

*The name comes from the British physicist/engineer Stephen Butterworth, who described the design of filters using the binomial principle in 1930.

Binomial Multistage Transformer (cont.)

Maximally flat property:

Use: $$\left|\Gamma(\theta)\right| = |A| \; 2^N \; |\cos\theta|^N$$

Choose all lines to be a quarter wavelength at the center frequency so that

$$f = f_0 \;\Rightarrow\; \theta = \beta\ell = \frac{\pi}{2} \;\Rightarrow\; \left|\Gamma\left(\frac{\pi}{2}\right)\right| = 0$$

(We have a perfect match at the center frequency.)

Also, $$\frac{d^n}{d\theta^n}\left|\Gamma(\theta)\right|_{\theta=\frac{\pi}{2}} = 0 \quad \text{for } n = 1, 2, \ldots, N-1$$

The first $(N-1)$ derivatives are zero $\Rightarrow$ maximally flat.

The reflection coefficient stay small for as wide a frequency as possible.

Binomial Multistage Transformer (cont.)

Using the binomial expansion, we can express the Butterworth response in terms of a polynomial series:

$$(1+z)^N = \sum_{n=0}^{N} C_n^N \, z^n \qquad \text{where} \quad C_n^N = \frac{N!}{(N-n)! \; n!} \qquad \text{Recall:} \quad z \equiv e^{-j2\theta}$$

A binomial type of response is obtained if we thus choose

$$\Gamma(\theta) = A\left(1+e^{-j2\theta}\right)^N = A\sum_{n=0}^{N} C_n^N \, e^{-j2n\theta}$$

We want to use a multistage transformer to realize this type of response.

$$\Rightarrow \Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + \ldots\ldots + \Gamma_N e^{-j2N\theta}$$

$$= A\sum_{n=0}^{N} C_n^N \, e^{-j2n\theta}$$

Set equal

(Both are now in the form of polynomials.)

First, we need to solve for A.

Binomial Multistage Transformer (cont.)

Solving for A:

Use: $$\Gamma(\theta) = A\, e^{-jN\theta}\, 2^N \cos^N \theta$$

Note that as $f \to 0 \;\Rightarrow\; \beta\ell = \theta \to 0$ (zero length transmisison lines)

Hence $\Gamma(\theta) \to A 2^N$

Also $\Gamma(\theta) \to \dfrac{Z_L - Z_0}{Z_L + Z_0}$ (A zero-length set of lines has no effect.)

Equating these two limiting results, we have

$$A = 2^{-N}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)$$

Binomial Multistage Transformer (cont.)

Then solving for Γ_n:

$$\Rightarrow \Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \Gamma_3 e^{-j6\theta} + \ldots + \Gamma_n e^{-j2n\theta} + \ldots + \Gamma_N e^{-j2N\theta}$$

$$= A \sum_{n=0}^{N} C_n^N e^{-j2n\theta}$$

Set equal

Equating responses for each term in the polynomial series gives us:

$$\Gamma_n = A C_n^N, \quad n = 1, 2, \ldots\ldots, N$$

Hence

$$\frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} = A C_n^N$$

or

$$\frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} = 2^{-N} \left(\frac{Z_L - Z_0}{Z_L + Z_0} \right) C_n^N$$

Binomial Multistage Transformer (cont.)

Solving for Z_{n+1}, we have:

$$Z_{n+1} = Z_n \left(\frac{1+R_n}{1-R_n} \right)$$

where

$$R_n = 2^{-N} \left(\frac{Z_L - Z_0}{Z_L + Z_0} \right) C_n^N$$

Note:
The Pozar book also introduces an approximate form of this equation involving $\ln(Z_L/Z_0)$. It is recommend to use this exact form.

This gives us a solution for the line impedances
(a recursive formula).

Note : $Z_0 =$ characteristic impedance of the incoming line, and $Z_{N+1} = Z_L$

Binomial Multistage Transformer (cont.)

Note on reflection coefficients

$$\Gamma_n = A C_n^N, \quad n = 1, 2, \ldots\ldots, N$$

$$C_n^N = \frac{N!}{(N-n)!\; n!}$$

Note that:

$$C_{N-n}^N = \frac{N!}{\left(N-(N-n)\right)!\;(N-n)!} = \frac{N!}{n!\;(N-n)!} = C_n^N$$

Hence $\quad \Gamma_n = \Gamma_{N-n}$

Although we did not assume that the reflection coefficients were symmetric in the design process (Design #1), they actually come out that way for the binominal design.

Binomial Multistage Transformer (cont.)

TABLE 5.1 Binomial Transformer Design

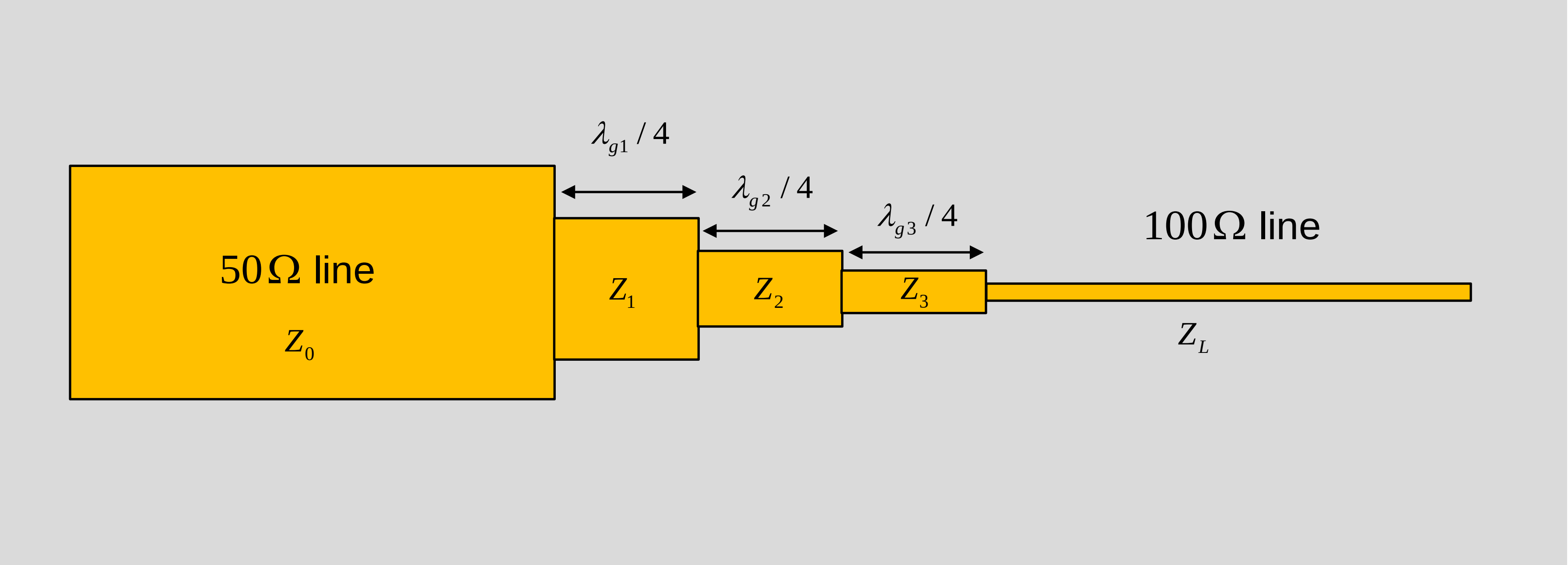
	N = 2		N = 3			N = 4			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1067	1.3554	1.0520	1.2247	1.4259	1.0257	1.1351	1.3215	1.4624
2.0	1.1892	1.6818	1.0907	1.4142	1.8337	1.0444	1.2421	1.6102	1.9150
3.0	1.3161	2.2795	1.1479	1.7321	2.6135	1.0718	1.4105	2.1269	2.7990
4.0	1.4142	2.8285	1.1907	2.0000	3.3594	1.0919	1.5442	2.5903	3.6633
6.0	1.5651	3.8336	1.2544	2.4495	4.7832	1.1215	1.7553	3.4182	5.3500
8.0	1.6818	4.7568	1.3022	2.8284	6.1434	1.1436	1.9232	4.1597	6.9955
10.0	1.7783	5.6233	1.3409	3.1623	7.4577	1.1613	2.0651	4.8424	8.6110

	N = 5					N = 6					
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_6/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0128	1.0790	1.2247	1.3902	1.4810	1.0064	1.0454	1.1496	1.3048	1.4349	1.4905
2.0	1.0220	1.1391	1.4142	1.7558	1.9569	1.0110	1.0790	1.2693	1.5757	1.8536	1.9782
3.0	1.0354	1.2300	1.7321	2.4390	2.8974	1.0176	1.1288	1.4599	2.0549	2.6577	2.9481
4.0	1.0452	1.2995	2.0000	3.0781	3.8270	1.0225	1.1661	1.6129	2.4800	3.4302	3.9120
6.0	1.0596	1.4055	2.4495	4.2689	5.6625	1.0296	1.2219	1.8573	3.2305	4.9104	5.8275
8.0	1.0703	1.4870	2.8284	5.3800	7.4745	1.0349	1.2640	2.0539	3.8950	6.3291	7.7302
10.0	1.0789	1.5541	3.1623	6.4346	9.2687	1.0392	1.2982	2.2215	4.5015	7.7030	9.6228

Note:
The table only shows data for $Z_L > Z_0$ since the design can be reversed (load and source switched) for $Z_L < Z_0$.
(All of the Γ_n values become their negative, but this does not change the $|\Gamma|$ response.)

Binomial Multistage Transformer (cont.)

Example showing a microstrip line



A three-stage transformer is shown.

Binomial Multistage Transformer (cont.)

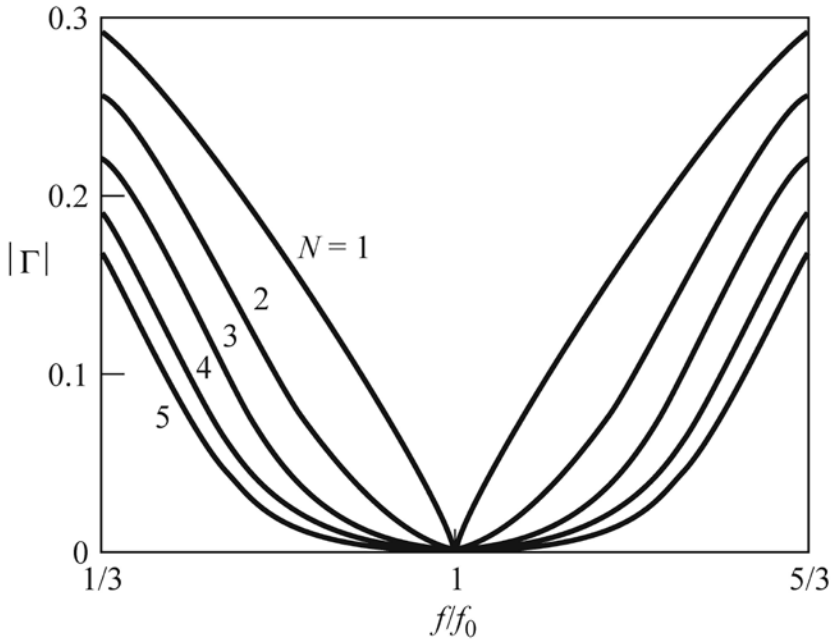


FIGURE 5.15 Reflection coefficient magnitude versus frequency for multisection binomial matching transformers of Example 5.6. $Z_L = 50\ \Omega$ and $Z_0 = 100\ \Omega$.

Note:
Increasing the number of lines increases the bandwidth.

Binomial Multistage Transformer (cont.)

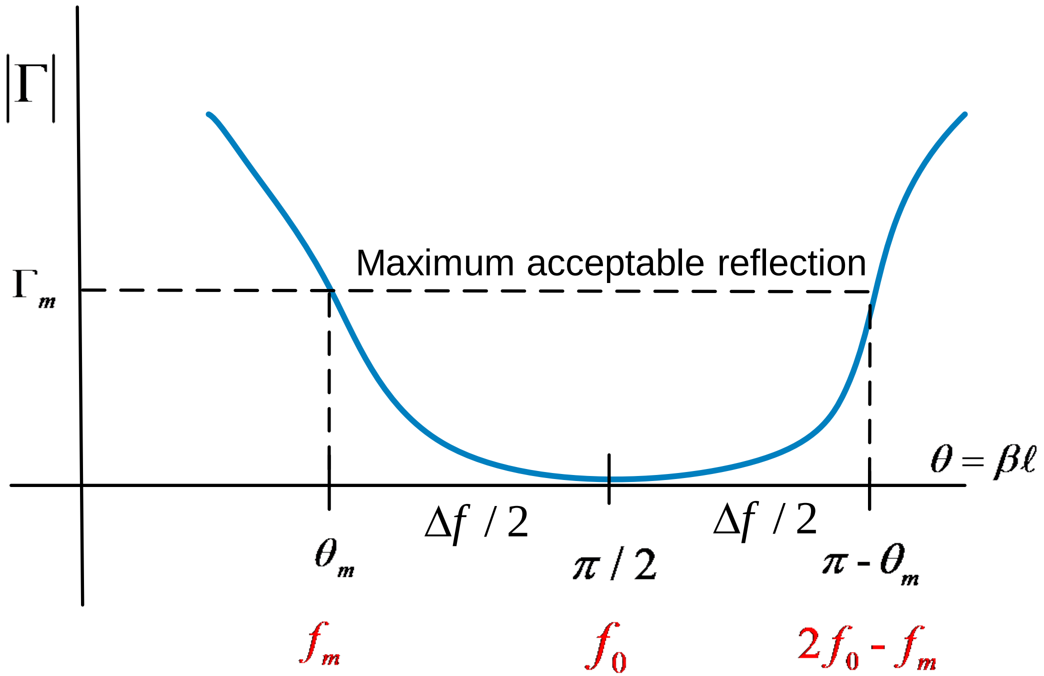
Bandwidth

First choose Γ_m

$$\Gamma(\theta) = A\, e^{-jN\theta}\, 2^N \cos^N \theta$$

$$\Rightarrow \Gamma_m = 2^N |A| \cos^N \theta_m$$

$$\Rightarrow \theta_m = \cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$$



The relative bandwidth is then:

$$\text{BW} = \frac{\Delta f}{f_0} = 2\frac{(f_0 - f_m)}{f_0} = 2 - 2\frac{f_m}{f_0} = 2 - 2\frac{\theta_m}{\pi/2} = 2 - \frac{4\theta_m}{\pi} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$$

Hence

$$\text{BW} = \frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$$

Binomial Multistage Transformer (cont.)

Summary of Design Formulas

$$\Gamma(\theta) = A\left(1+e^{-j2\theta}\right)^N = A\sum_{n=0}^{N} C_n^N \, e^{-j2n\theta}$$

Reflection coefficient response

$$\theta = \beta\ell = \left(\frac{f}{f_0}\right)\frac{\pi}{2} \qquad C_n^N = \frac{N!}{(N-n)!\; n!}$$

$$A = 2^{-N}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)$$

A coefficient

$$Z_{n+1} = Z_n\left(\frac{1+R_n}{1-R_n}\right) \qquad R_n = 2^{-N}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)C_n^N$$

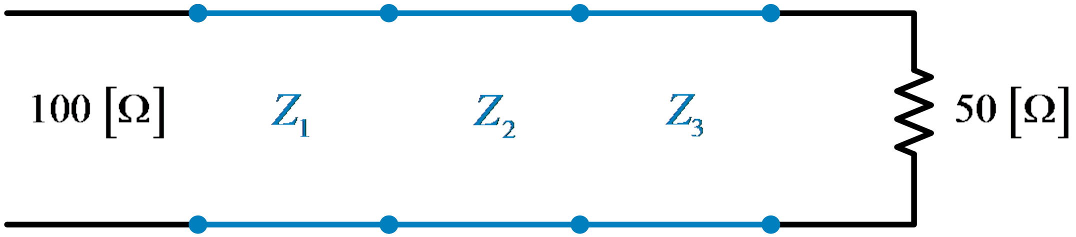
Design of line impedances

$$\mathrm{BW} = \frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$$

Bandwidth

Example

Example: Three-stage binomial transformer



Given:

$Z_L = 50$ [Ω]

$Z_0 = 100$ [Ω]

$\Gamma_m = 0.05$

↑

$\Gamma_m^{\text{dB}} = -26.0$ [dB]

$$A = 2^{-N}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)$$

$$N = 3 \quad \Rightarrow \quad A = 2^{-3}\left(\frac{50-100}{50+100}\right) = -0.0417$$

$$A = -0.0417$$

Example (cont.)

$$Z_{n+1} = Z_n \left(\frac{1+R_n}{1-R_n} \right) \qquad R_n = 2^{-N} \left(\frac{Z_L - Z_0}{Z_L + Z_0} \right) C_n^N$$

$$C_n^N = \frac{N!}{(N-n)!\ n!}$$

$$C_0^3 = 1$$

$$C_1^3 = 3$$

$$C_2^3 = 3$$

$$C_3^3 = 1$$

We then have:

$$Z_1 = Z_0 \left(\frac{1+R_0}{1-R_0} \right) \qquad R_0 = 2^{-3} \left(\frac{50-100}{50+100} \right) C_0^3$$

$$Z_2 = Z_1 \left(\frac{1+R_1}{1-R_1} \right) \qquad R_1 = 2^{-3} \left(\frac{50-100}{50+100} \right) C_1^3$$

$$Z_3 = Z_2 \left(\frac{1+R_2}{1-R_2} \right) \qquad R_2 = 2^{-3} \left(\frac{50-100}{50+100} \right) C_2^3$$

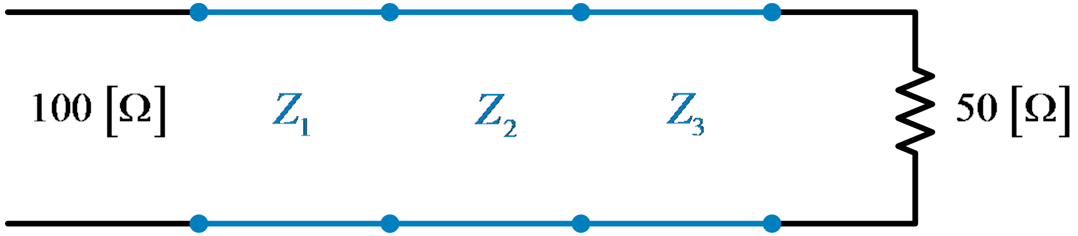
Example (cont.)

The results for the line impedances are:

$$Z_1 = 91.7\ [\Omega]$$

$$Z_2 = 70.7\ [\Omega]$$

$$Z_3 = 54.5\ [\Omega]$$



Example (cont.)

Using the table in Pozar we have:

$$Z_L / Z_0 = 2: \quad \left(Z_1, Z_2, Z_3\right) / Z_0 = 1.0907,\ 1.4142,\ 1.8337$$

(The above normalized load impedance is the reciprocal of what we actually have.)

Hence, switching the load and the source ends, we have

$$\left(Z_1, Z_2, Z_3\right) / Z_0 = 1.8337,\ 1.4142,\ 1.0907; \quad Z_0 = 50[\Omega]$$

Therefore, we have

$$Z_1 = 91.7\ [\Omega]$$
$$Z_2 = 70.7\ [\Omega]$$
$$Z_3 = 54.6\ [\Omega]$$

Example

Bandwidth:

$$\mathrm{BW} = \frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$$

$$A = -0.0417$$

$$\Gamma_m = 0.05$$

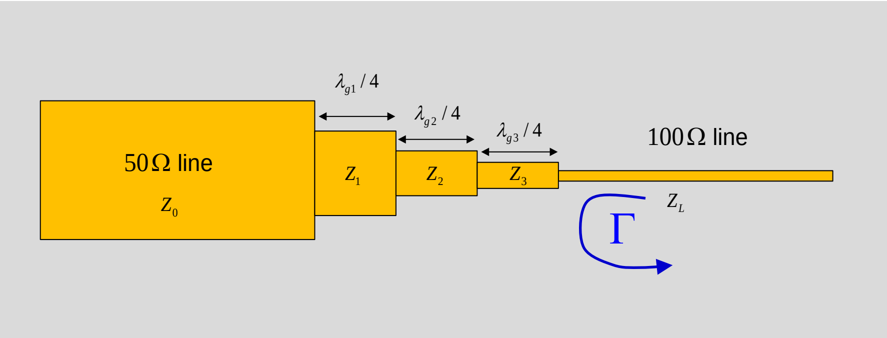
$$\Rightarrow \quad \mathrm{BW} = 0.713$$

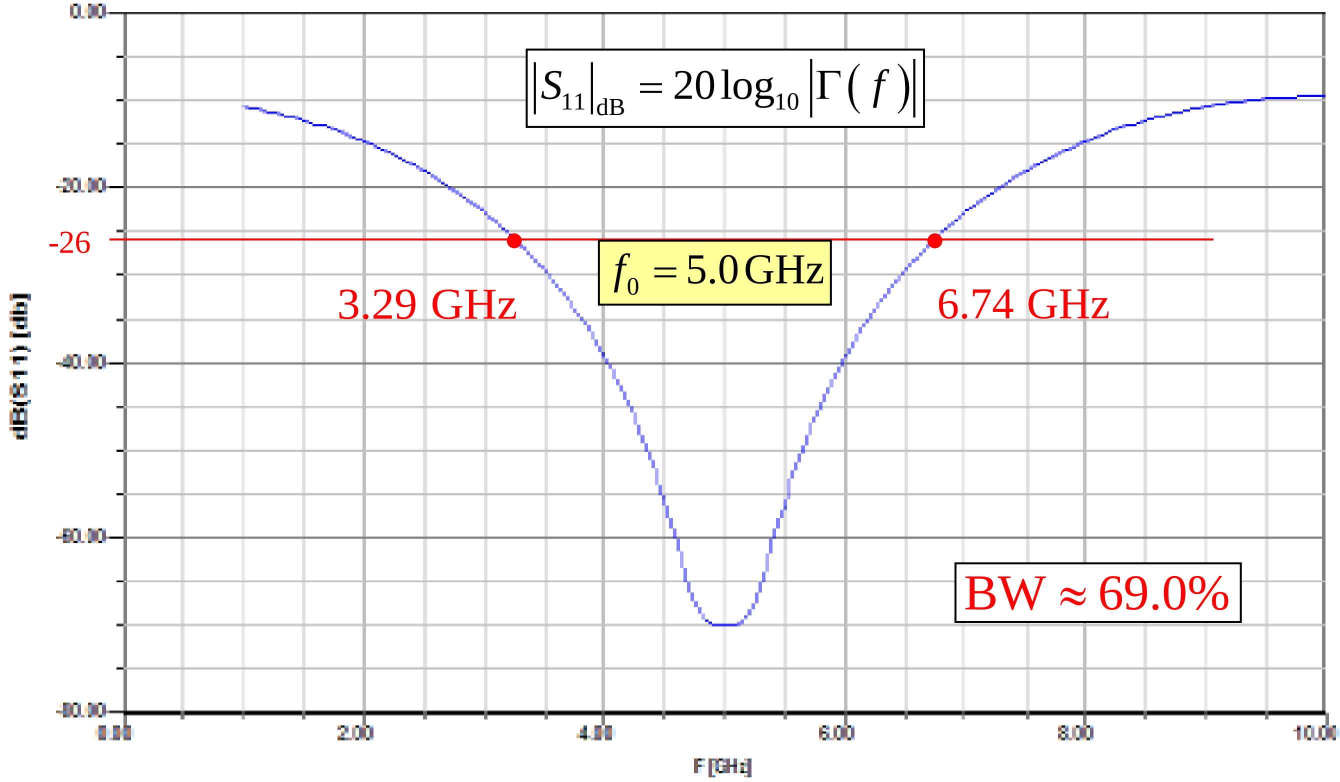
$$\mathrm{BW} = 71.3\%$$

Note:
A single quarter-wave transformer has a bandwidth of about 6% (for $\Gamma_m = 0.05$).

Example (cont.)

Microstrip





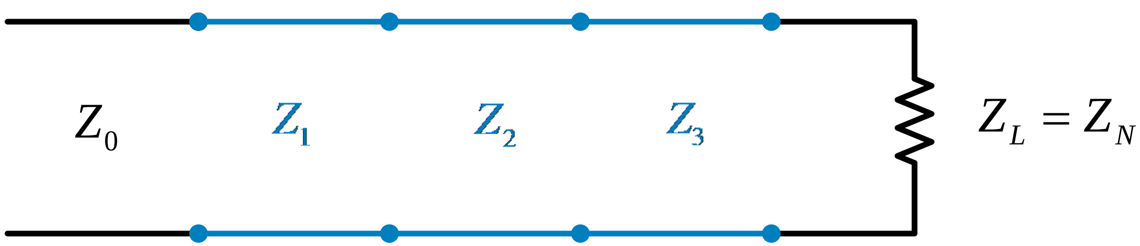
Response from Ansys Designer (5 GHz Design)

Load Check for Consistency

Consider an N stage transformer:

- ❖ We have N line impedances: $\left(Z_1, Z_2, \ldots Z_N\right)$
- ❖ We have N+1 reflection coefficients: $\left(\Gamma_0, \Gamma_1, \ldots \Gamma_N\right)$

If we start at the input end and work our way up, our Z_n values we will ensure that the first N reflection coefficients are correct. We then have no control over the final one, Γ_N. We can calculate Z_N to check the consistency (see how close it is to the specified load value).



Load Check for Consistency (cont.)

Example: Previous $N = 3$ Butterworth design

$$Z_L = 50\ [\Omega] \qquad Z_0 = 100\ [\Omega]$$

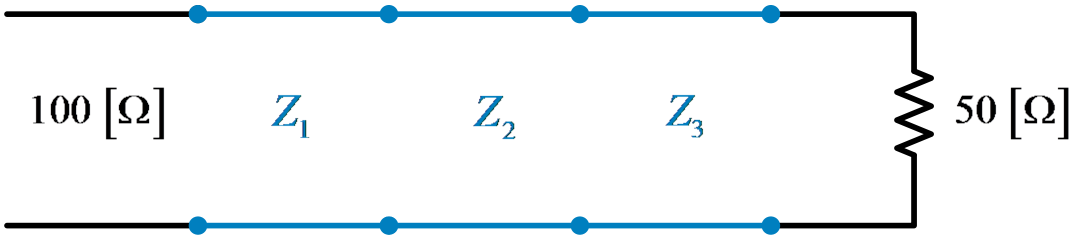
$$Z_1 = 91.7\ [\Omega] \qquad Z_2 = 70.7\ [\Omega] \qquad Z_3 = 54.6\ [\Omega]$$

For the load we have:

$$Z_4 = Z_3\left(\frac{1+R_3}{1-R_3}\right) \qquad R_3 = 2^{-3}\left(\frac{50-100}{50+100}\right)C_3^3$$

$$Z_4 = 50.14\ [\Omega]$$

(Not exact, but fairly consistent)



Chebyshev Multistage Matching Transformer

This is an example of Design #2.

Chebyshev polynomials of the first kind:

$$T_n(x) \equiv \begin{cases} \cos\left(n\cos^{-1} x\right), & |x| \le 1 \\ \cosh\left(n\cosh^{-1} x\right), & |x| \ge 1 \end{cases}$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$\vdots$$

$$T_n(x) = 2xT_{n\text{-}1}(x) - T_{n\text{-}2}(x)$$

For $-1 \le x \le 1$: $|T_n(x)| \le 1$

For $|x| > 1$: $|T_n(x)| > 1$

We choose the response to be in the form of a Chebyshev polynomial.

(This will lead to a finite Fourier cosine series in θ.)

Chebyshev Transformer (cont.)

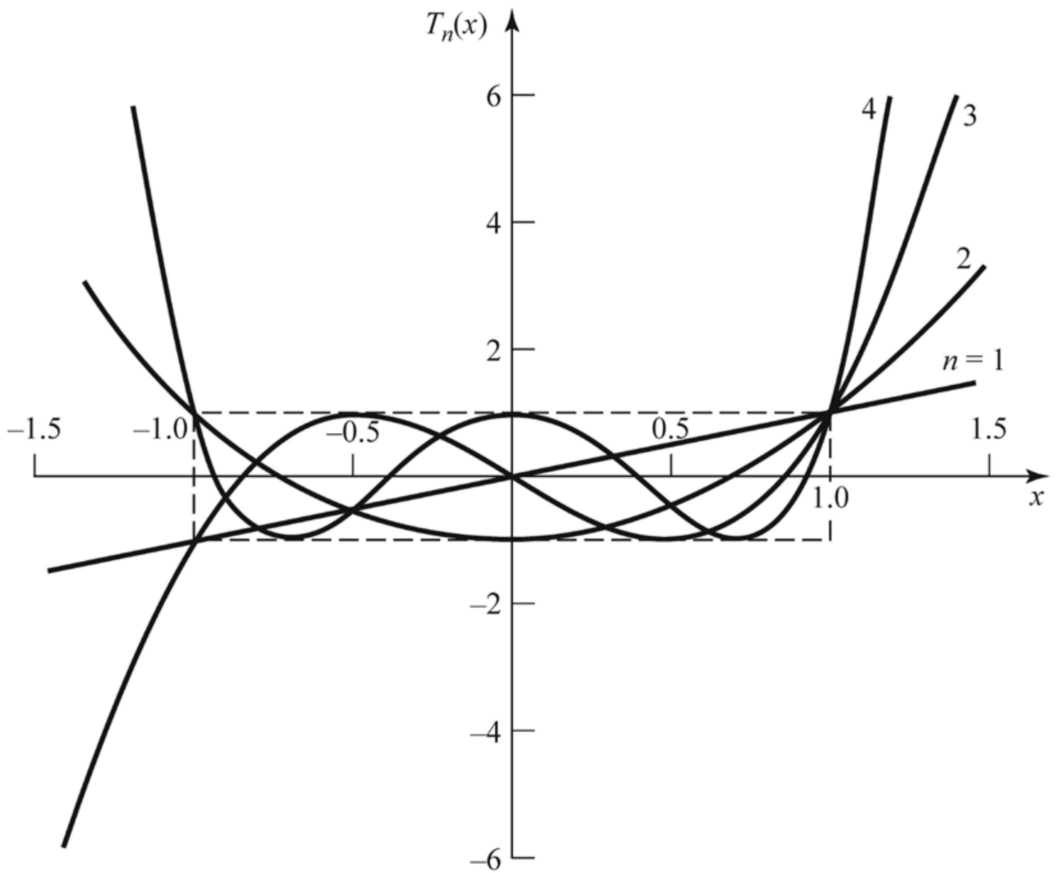


FIGURE 5.16 The first four Chebyshev polynomials, $T_n(x)$.

Chebyshev Transformer (cont.)

A Chebyshev response will have equal ripple within the bandwidth.

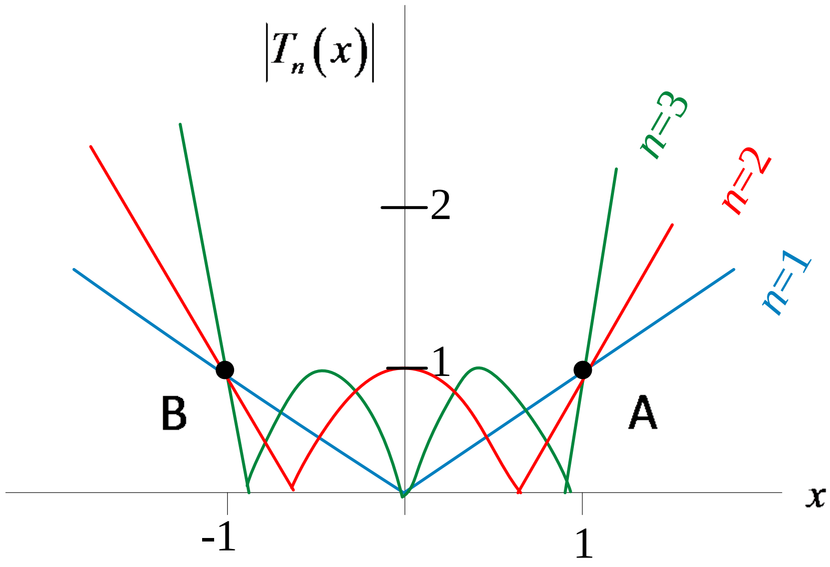
Choose:

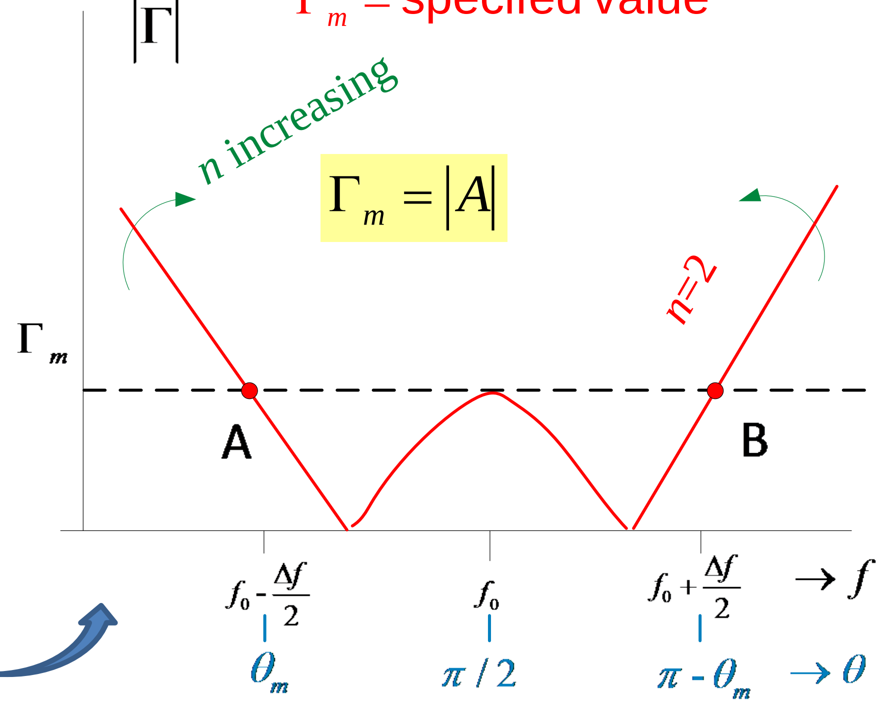
$$\Gamma(\theta) = A e^{-jN\theta} T_N\left(\sec\theta_m \cos\theta\right)$$

This can be put into a form involving the terms $\cos(n\theta)$ (i.e., a finite Fourier cosine series).

$$x = \sec\theta_m \cos\theta$$

$\Gamma_m =$ specifed value





Note: As frequency decreases, x increases.

Chebyshev Transformer (cont.)

We have that, after some algebra,

Note : $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

$$T_1(\sec\theta_m \cos\theta) = \sec\theta_m \cos\theta$$

$$T_2(\sec\theta_m \cos\theta) = 2(\sec\theta_m \cos\theta)^2 - 1 = \sec^2\theta_m (1+\cos 2\theta) - 1$$

$$T_3(\sec\theta_m \cos\theta) = \sec^3\theta_m (\cos 3\theta + 3\cos\theta) - 3\sec\theta_m \cos\theta$$

$$\vdots$$

$$T_n(\sec\theta_m \cos\theta) = 2(\sec\theta_m \cos\theta) T_{n-1}(\sec\theta_m \cos\theta) - T_{n-2}(\sec\theta_m \cos\theta)$$

$$x = \sec\theta_m \cos\theta$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$\vdots$$

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x)$$

Hence, the term $T_N(\sec\theta_m \cos\theta)$ can be cast into a finite cosine Fourier series expansion.

There is no general formula though!

Chebyshev Transformer (cont.)

Transformer design

Choice of response:

$$\Gamma(\theta) = A e^{-jN\theta} T_N(\sec\theta_m \cos\theta)$$

General form of response (Design #2):

$$\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + ... + \Gamma_n \cos(N-2n)\theta + ...\right]$$

From the above formulas we can extract the coefficients Γ_n
(no general formula is available).

Chebyshev Transformer (cont.)

$$\Gamma(\theta) = A e^{-jN\theta} T_N(\sec\theta_m \cos\theta)$$

Solve for A:

As $f \to 0 \;\; (\beta\ell = \theta \to 0)$:

$$\Gamma(0) = A T_N(\sec\theta_m) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Rightarrow \quad A = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)\frac{1}{T_N(\sec\theta_m)}$$

Chebyshev Transformer (cont.)

Alternative formula for A: Recall: $\Gamma(\theta) = A e^{-jN\theta} T_N(\sec\theta_m \cos\theta)$

At $\theta = \theta_m$

$$|\Gamma(\theta_m)| = \Gamma_m = |A| \, |T_N(\sec\theta_m \cos\theta_m)| = |A| \, |T_N(1)| = |A|$$

$$\Rightarrow A = \pm\Gamma_m$$

Which sign is correct?

At $\theta = 0$:

- $\Gamma(0) = A T_N(\sec\theta_m) = \dfrac{Z_L - Z_0}{Z_L + Z_0}$
- $T_N(\sec\theta_m) > 1 > 0 \quad (\sec\theta_m > 1, \text{ since } 0 < \theta_m < \pi/2)$

$\Rightarrow A$ has the same sign as $(Z_L - Z_0)$

Hence $$A = \operatorname{sgn}(Z_L - Z_0)\,\Gamma_m$$

Chebyshev Transformer (cont.)

Calculation of θ_m

At $f = 0: \;\Gamma(0) = A T_N(\sec\theta_m) = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$$\Rightarrow T_N(\sec\theta_m) = \frac{1}{A}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) = \frac{1}{\operatorname{sgn}(Z_L - Z_0)\Gamma_m}\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|$$

Also, we have from the definition of the Chebyshev polynomial:

$$T_N(\sec\theta_m) = \cosh\left(N\cosh^{-1}(\sec\theta_m)\right)$$

Hence, we have

$$\cosh\left(N\cosh^{-1}(\sec\theta_m)\right) = \frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|$$

Chebyshev Transformer (cont.)

From the last slide we have

$$\cosh\left(N\cosh^{-1}\left(\sec\theta_m\right)\right)=\frac{1}{\Gamma_m}\left|\frac{Z_L-Z_0}{Z_L+Z_0}\right|$$

We then have

$$\sec\theta_m=\cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L-Z_0}{Z_L+Z_0}\right|\right)\right]$$

Chebyshev Transformer (cont.)

Summary of Design Formulas

$$\Gamma(\theta) = A e^{-jN\theta} T_N(\sec\theta_m \cos\theta)$$

Reflection coefficient response

$$\theta = \beta\ell = \left(\frac{f}{f_0}\right)\frac{\pi}{2}$$

$$\sec\theta_m = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\Gamma_m}\left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|\right)\right]$$

θ_m term

$$A = \operatorname{sgn}(Z_L - Z_0)\,\Gamma_m$$

A coefficient

No formula is given for the line impedances. Use the Table from Pozar or generate ("by hand") the solution by expanding $\Gamma(\theta)$ into a polynomial with terms $\cos(n\theta)$.

Design of line impedances

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\theta_m$$

Bandwidth

Chebyshev Transformer (cont.)

TABLE 5.2 Chebyshev Transformer Design

	$N = 2$				$N = 3$					
	$\Gamma_m = 0.05$		$\Gamma_m = 0.20$		$\Gamma_m = 0.05$			$\Gamma_m = 0.20$		
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1347	1.3219	1.2247	1.2247	1.1029	1.2247	1.3601	1.2247	1.2247	1.2247
2.0	1.2193	1.6402	1.3161	1.5197	1.1475	1.4142	1.7429	1.2855	1.4142	1.5558
3.0	1.3494	2.2232	1.4565	2.0598	1.2171	1.7321	2.4649	1.3743	1.7321	2.1829
4.0	1.4500	2.7585	1.5651	2.5558	1.2662	2.0000	3.1591	1.4333	2.0000	2.7908
6.0	1.6047	3.7389	1.7321	3.4641	1.3383	2.4495	4.4833	1.5193	2.4495	3.9492
8.0	1.7244	4.6393	1.8612	4.2983	1.3944	2.8284	5.7372	1.5766	2.8284	5.0742
10.0	1.8233	5.4845	1.9680	5.0813	1.4385	3.1623	6.9517	1.6415	3.1623	6.0920

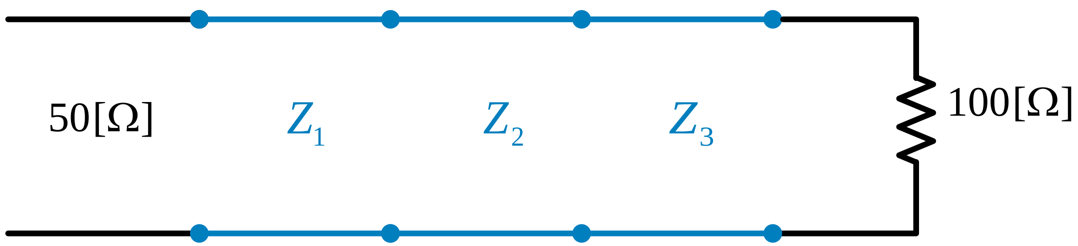
$N = 4$

	$\Gamma_m = 0.05$				$\Gamma_m = 0.20$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0892	1.1742	1.2775	1.3772	1.2247	1.2247	1.2247	1.2247
2.0	1.1201	1.2979	1.5409	1.7855	1.2727	1.3634	1.4669	1.5715
3.0	1.1586	1.4876	2.0167	2.5893	1.4879	1.5819	1.8965	2.0163
4.0	1.1906	1.6414	2.4369	3.3597	1.3692	1.7490	2.2870	2.9214
6.0	1.2290	1.8773	3.1961	4.8820	1.4415	2.0231	2.9657	4.1623
8.0	1.2583	2.0657	3.8728	6.3578	1.4914	2.2428	3.5670	5.3641
10.0	1.2832	2.2268	4.4907	7.7930	1.5163	2.4210	4.1305	6.5950

Note: The table only shows data for $Z_L > Z_0$ since the design can be reversed (load and source switched) for $Z_L < Z_0$.

Example

Example: Three-stage Chebyshev transformer



Given:

$Z_L = 100[\Omega]$

$Z_0 = 50[\Omega]$

$\Gamma_m = 0.05$

Assumed symmetry : $\Gamma_3 = \Gamma_0 , \Gamma_2 = \Gamma_1$

$$N = 3 \;\Rightarrow\; \Gamma(\theta) = A e^{-j3\theta} T_3(\sec\theta_m \cos\theta)$$

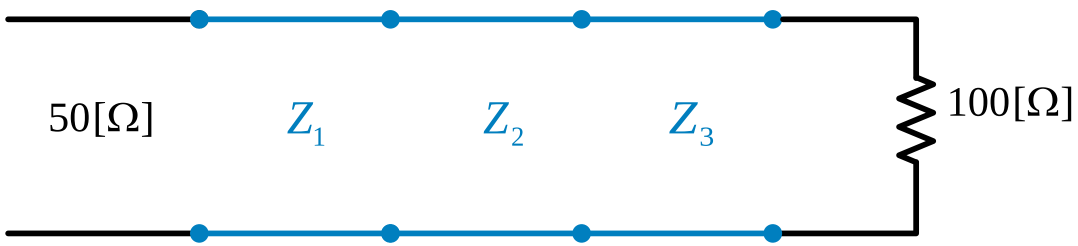
$$A = \operatorname{sgn}(Z_L - Z_0)\Gamma_m \Rightarrow A = 0.05$$

$$\sec\theta_m = \cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{1}{(0.05)}\left|\frac{100-50}{100+50}\right|\right)\right) = 1.408$$

$$\Rightarrow \theta_m = 0.78 \;\; [\text{rad}]$$

Example

Example: Three-stage Chebyshev transformer



Given:

$$Z_L = 100[\Omega]$$
$$Z_0 = 50[\Omega]$$
$$\Gamma_m = 0.05$$

$$\Gamma(\theta) = Ae^{-j3\theta} T_3(\sec\theta_m \cos\theta)$$

Hence,

$$\Gamma(\theta) = Ae^{-j3\theta}\left[\sec^3\theta_m(\cos 3\theta + 3\cos\theta) - 3\sec\theta_m \cos\theta\right]$$

Also,

$$\Gamma(\theta) = 2e^{-j3\theta}\left[\Gamma_0 \cos 3\theta + \Gamma_1 \cos\theta\right]$$

(finite Fourier cosine series form)

Equate

Recall: $\Gamma(\theta) = 2e^{-jN\theta}\left[\Gamma_0 \cos(N\theta) + \Gamma_1 \cos((N-2)\theta) + \ldots + \Gamma_n \cos((N-2n)\theta) + \ldots + \Gamma_{\frac{N-1}{2}} \cos\theta\right]$; N odd

Example

Equating coefficients from the previous equations on the last slide, we have:

$$\Gamma(\theta) = A e^{-j3\theta}\left[\sec^3\theta_m\left(\cos 3\theta + 3\cos\theta\right) - 3\sec\theta_m\cos\theta\right]$$

$$\Gamma(\theta) = 2e^{-j3\theta}\left[\Gamma_0\cos 3\theta + \Gamma_1\cos\theta\right]$$

⇓

$$\cos(3\theta):\quad 2\Gamma_0 = A\sec^3\theta_m$$

$$\cos(\theta):\quad 2\Gamma_1 = 3A\sec^3\theta_m - 3A\sec\theta_m$$

Example (cont.)

We now evaluate the reflection coefficients:

$$2\Gamma_0 = A\sec^3\theta_m$$

$$2\Gamma_1 = 3A\sec^3\theta_m - 3A\sec\theta_m$$

Recall:

$$A = 0.05$$

$$\theta_m = 0.78 \ \text{[rad]}$$

$$\sec\theta_m = 1.408$$

We then have:

$$\Gamma_0 = \frac{1}{2}(0.05)(1.408)^3 = 0.0698$$

$$\Rightarrow \ \Gamma_3 = \Gamma_0 = 0.0698$$

$$\Gamma_1 = \frac{1}{2}\left(3(0.05)(1.408)^3 - 3(0.05)(1.408)\right) = 0.1037$$

$$\Rightarrow \ \Gamma_2 = \Gamma_1 = 0.1037$$

Example (cont.)

Next, use $\dfrac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n}=\Gamma_n \quad \Rightarrow \quad Z_{n+1}=Z_n\left(\dfrac{1+\Gamma_n}{1-\Gamma_n}\right)$

so

$$Z_1=50\frac{(1+0.0698)}{(1-0.0698)}=57.5\ [\Omega]$$

$$Z_2=57.5\frac{(1+0.1037)}{(1-0.1037)}=70.8\ [\Omega]$$

$$Z_3=70.8\frac{(1+0.1037)}{(1-0.1037)}=87.2\ [\Omega]$$

Recall:

$$\Gamma_0=\Gamma_3=0.0698$$
$$\Gamma_1=\Gamma_2=0.1037$$

Results:

$$Z_1=57.5\ [\Omega]$$
$$Z_2=70.8\ [\Omega]$$
$$Z_3=87.2\ [\Omega]$$

Load Check :

$$Z_4=Z_L=87.2\frac{(1+0.0698)}{(1-0.0698)}=100.3\ [\Omega]$$

(The load is 100 [Ω]; this is pretty close.)

Example (cont.)

From Pozar Table $\left(\Gamma_m = 0.05,\ N = 3,\ Z_L / Z_0 = 2;\ Z_0 = 50\ [\Omega]\right)$:

$$Z_1 = (1.1475)50 = 57.4\ [\Omega]$$
$$Z_2 = (1.4142)50 = 70.7\ [\Omega]$$
$$Z_3 = (1.7429)50 = 87.1\ [\Omega]$$

$$Z_1 = 57.4\ [\Omega]$$
$$Z_2 = 70.7\ [\Omega]$$
$$Z_3 = 87.1\ [\Omega]$$

Example (cont.)

Bandwidth:

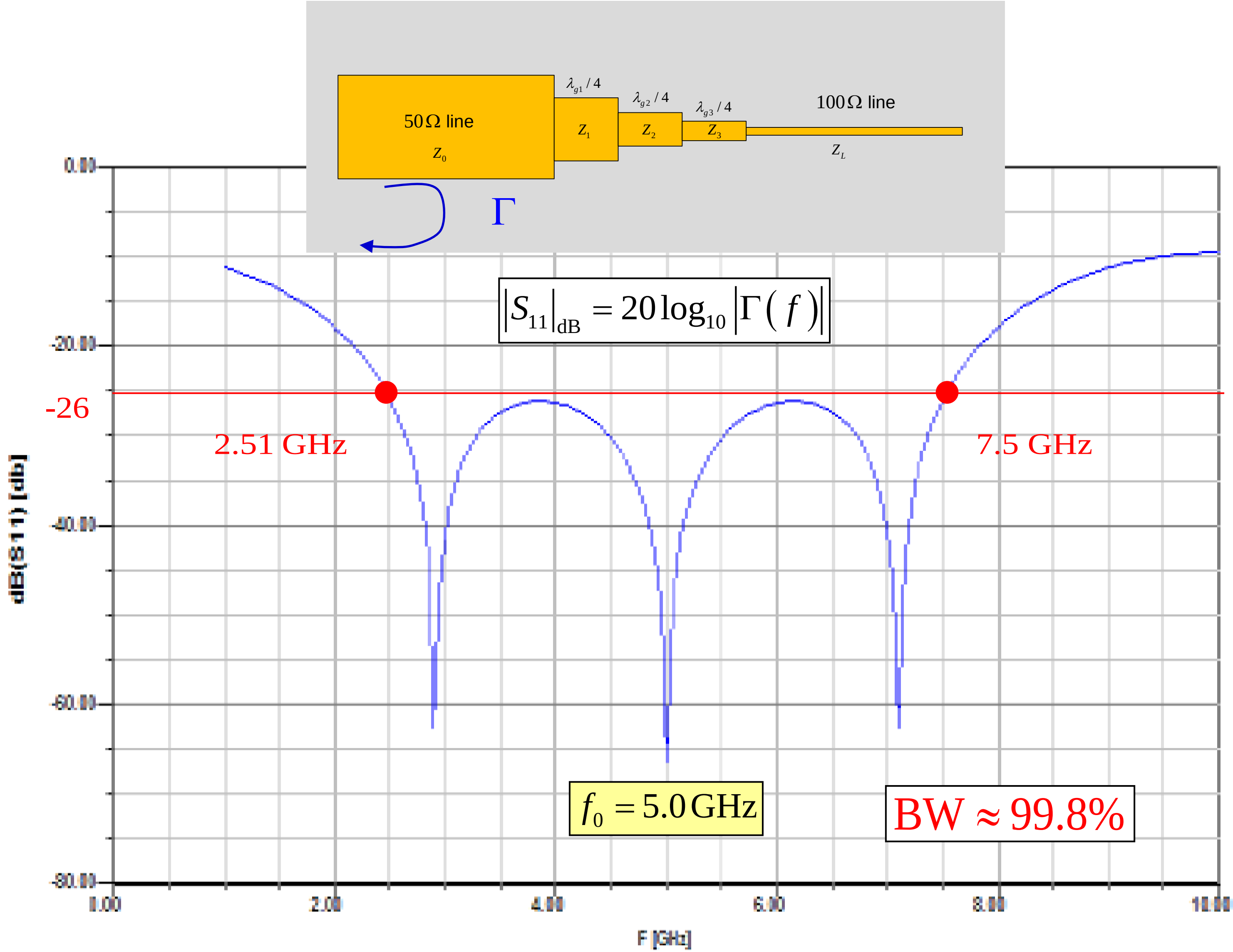
$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\theta_m$$

$$\theta_m = 0.78 \ \text{[rad]}$$

$$\Rightarrow \quad \frac{\Delta f}{f_0} = 1.01$$

$$\text{BW} = 101\%$$

Example (cont.)



Response from Ansys Designer (5 GHz Design)

Comparison of Designs

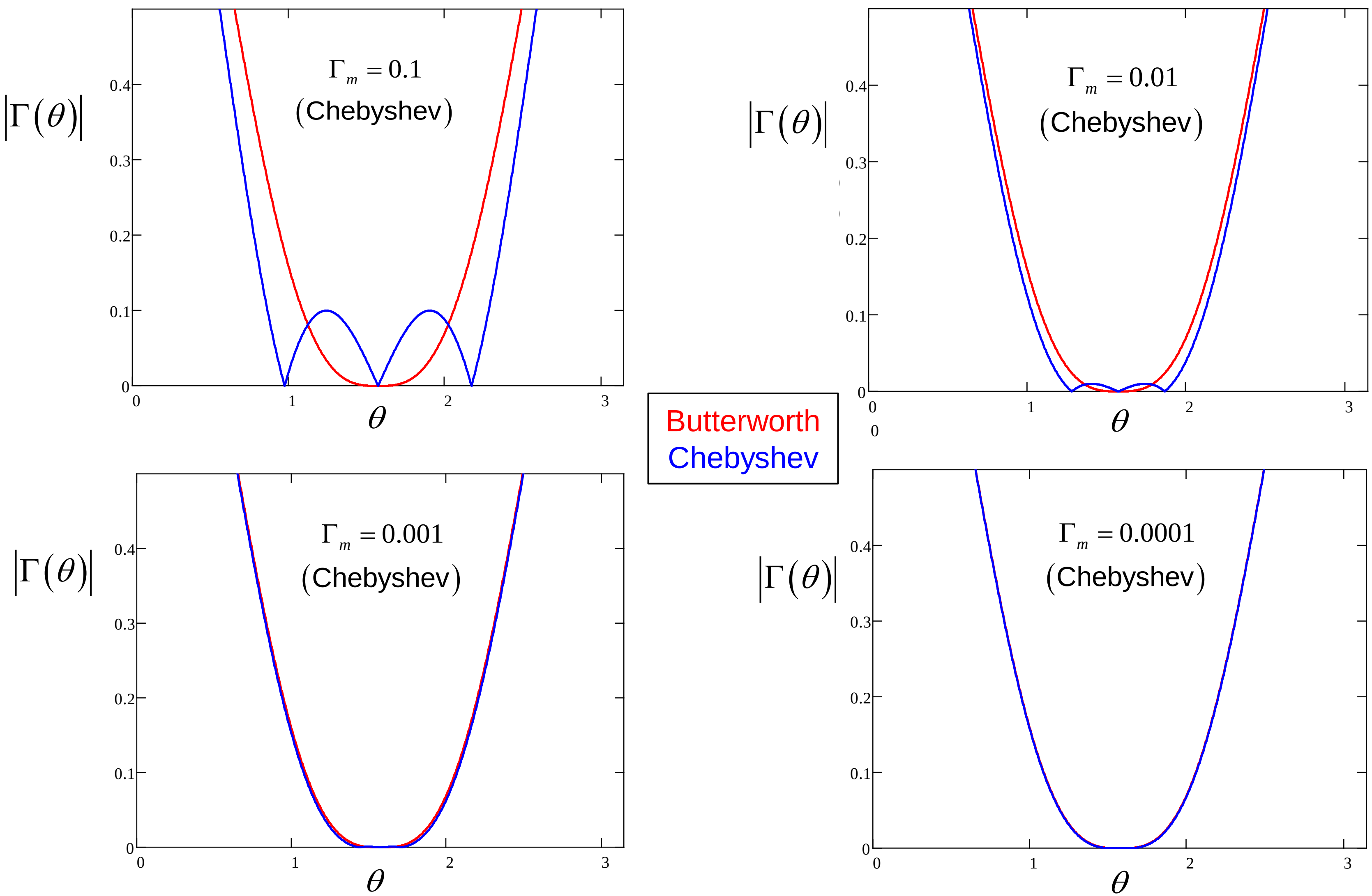
Comparison of Binomial (Butterworth) and Chebyshev

- The Chebyshev design has a higher bandwidth (100% vs. 69%).
- The increased bandwidth comes with a price: ripple in the passband.
- As the ripple level goes to zero, the Chebyshev response approaches that of the Butterworth.

Note:
It can be shown that the Chebyshev design gives the highest possible bandwidth for a given N and Γ_m.

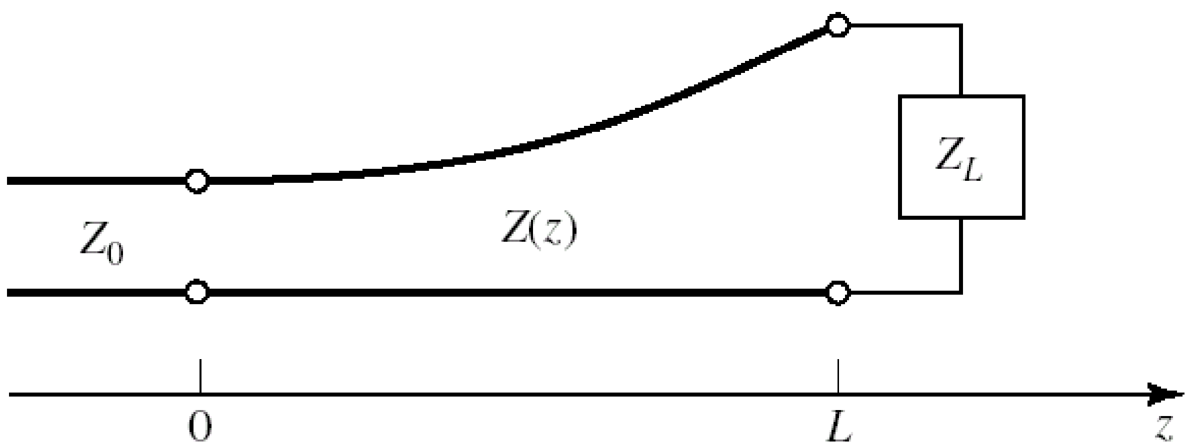
Comparison of Designs (cont.)

Comparison of Butterworth and Chebyshev Designs ($N = 3$)



Tapered Transformer

The Pozar book also talks about using continuously tapered lines to match between an input line Z_0 and an output load Z_L. (pp. 255-261). Please read this.



The "Klopfenstein taper" gives the $Z(z)$ that is best (lowest reflection coefficient) for a given length L.