

# ECE 5317-6351

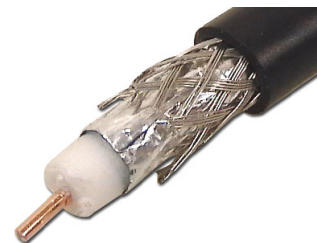
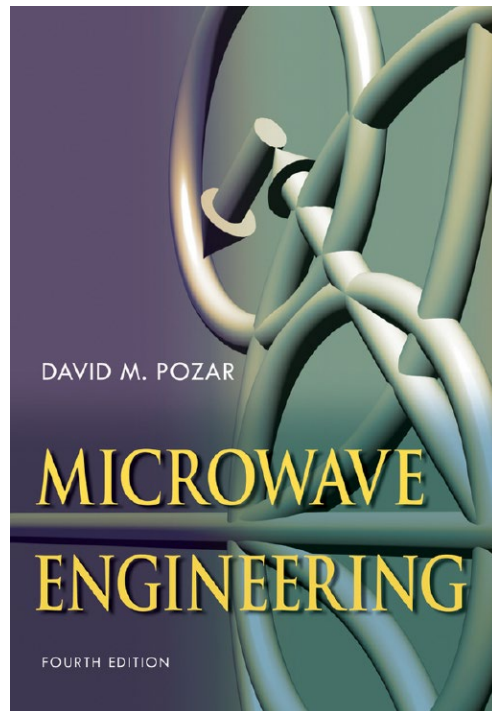
# Microwave Engineering

**Fall 2019**

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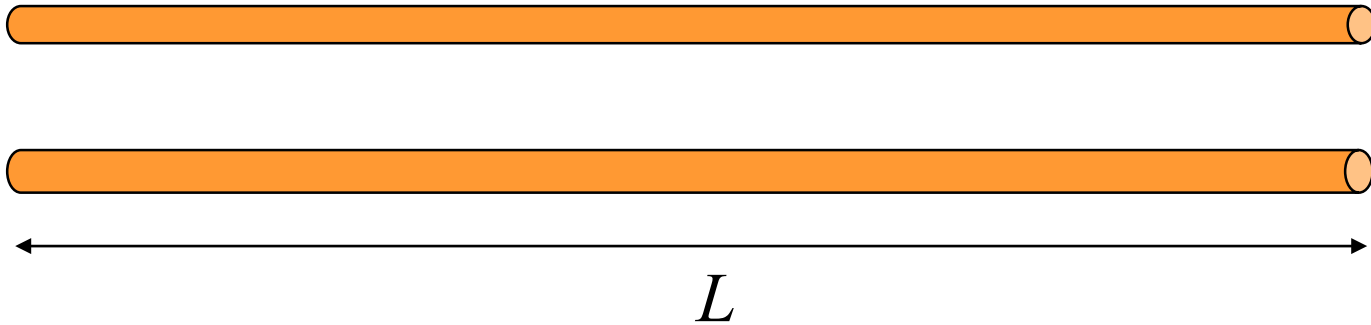
## Notes 2

### Transmission Lines Part 1: TL Theory



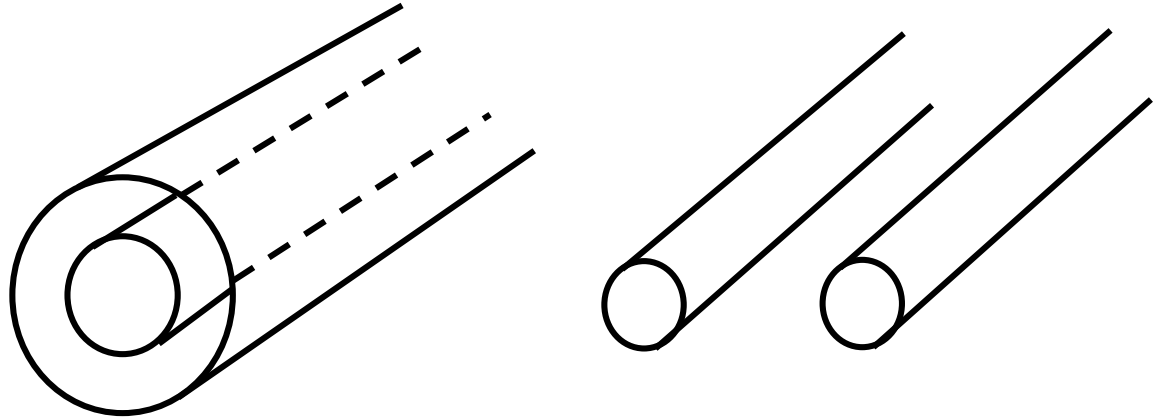
# Transmission-Line Theory

We need transmission-line theory whenever the length of a line is significant compared to a wavelength.



# Transmission Line

2 conductors



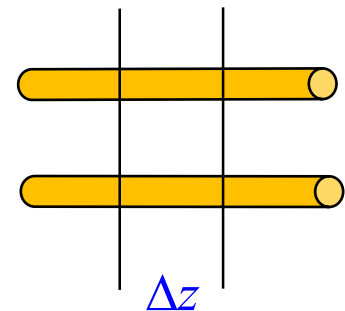
4 per-unit-length parameters:

$C$  = capacitance/length [F/m]

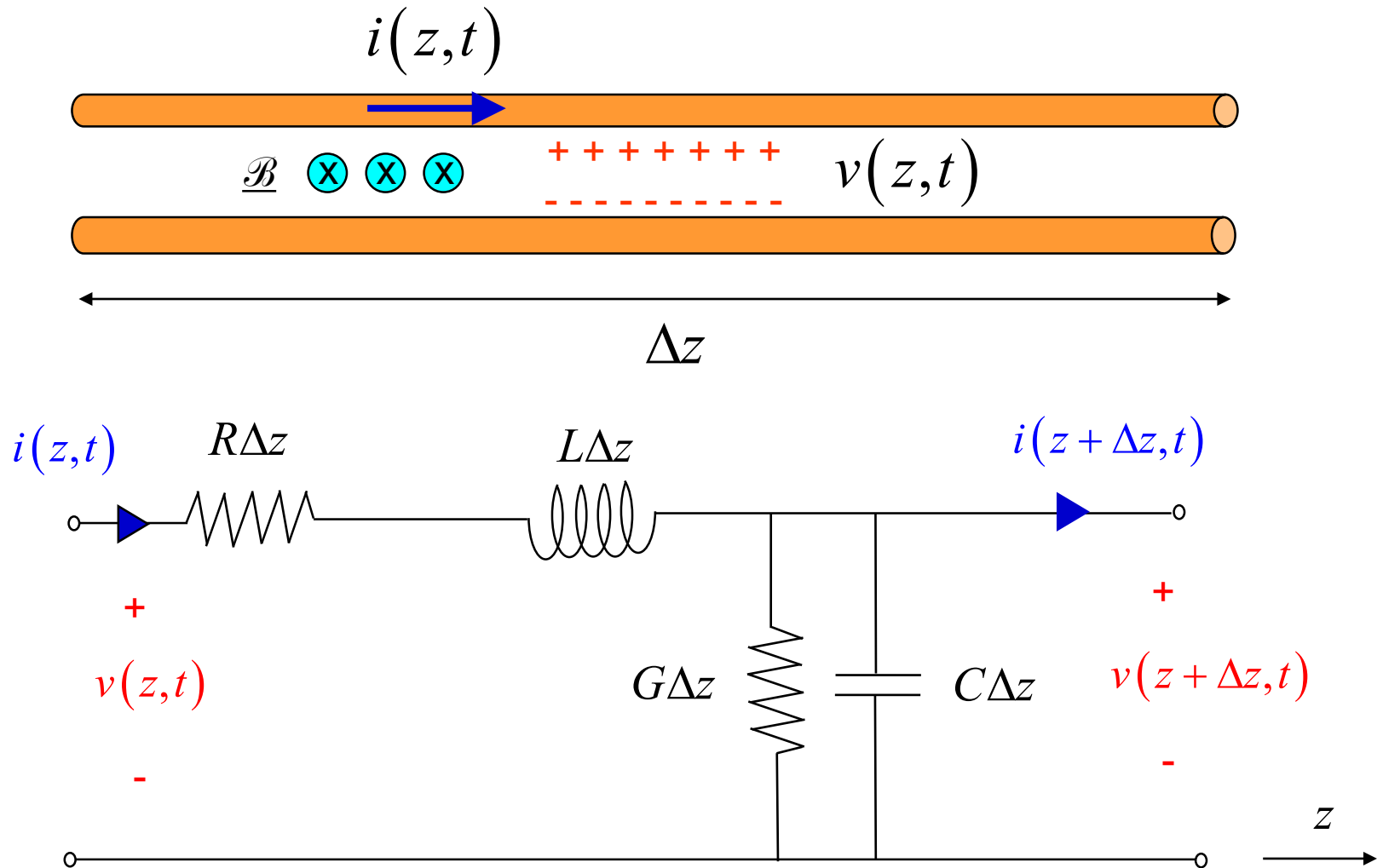
$L$  = inductance/length [H/m]

$R$  = resistance/length [ $\Omega$ /m]

$G$  = conductance/length [ $\text{U}/\text{m}$  or S/m]

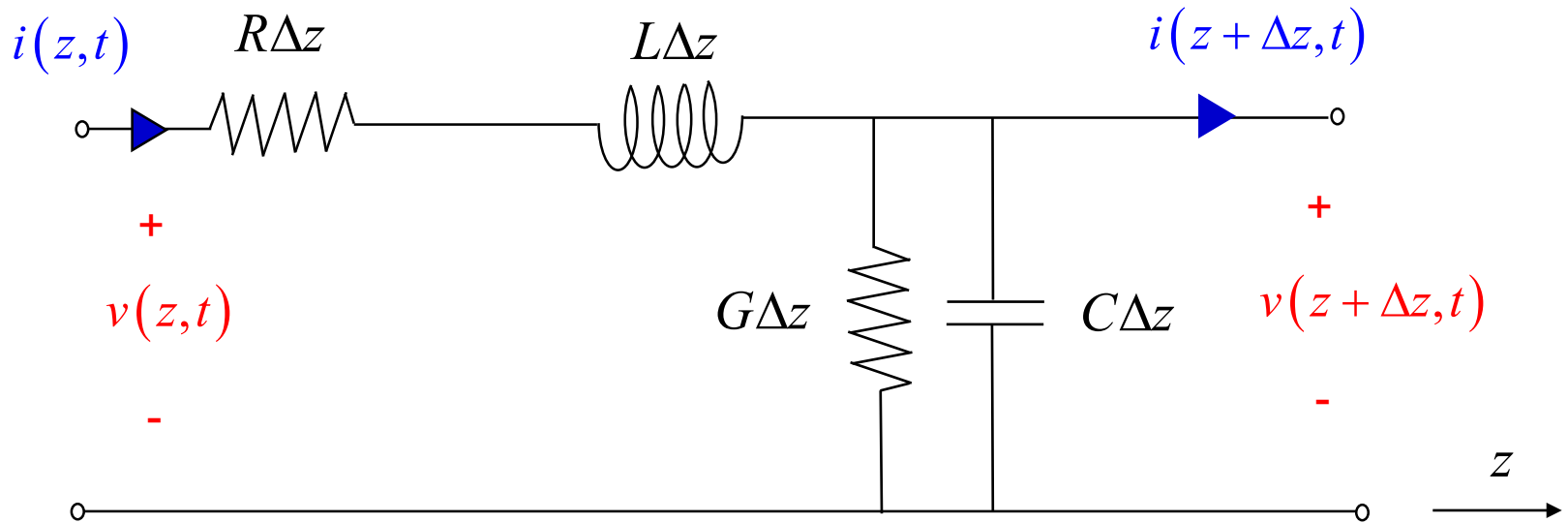


# Transmission Line (cont.)



**Note:** There are equal and opposite currents on the two conductors.  
 (We only need to work with the current on the top conductor, since we have chosen to put all of the series elements there.)

# Transmission Line (cont.)



$$v(z, t) = v(z + \Delta z, t) + i(z, t)R\Delta z + L\Delta z \frac{\partial i(z, t)}{\partial t}$$

$$i(z, t) = i(z + \Delta z, t) + v(z + \Delta z, t)G\Delta z + C\Delta z \frac{\partial v(z + \Delta z, t)}{\partial t}$$

# TEM Transmission Line (cont.)

Hence

$$\frac{v(z + \Delta z, t) - v(z, t)}{\Delta z} = -Ri(z, t) - L \frac{\partial i(z, t)}{\partial t}$$

$$\frac{i(z + \Delta z, t) - i(z, t)}{\Delta z} = -Gv(z + \Delta z, t) - C \frac{\partial v(z + \Delta z, t)}{\partial t}$$

Now let  $\Delta z \rightarrow 0$ :

$$\frac{\partial v}{\partial z} = -Ri - L \frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -Gv - C \frac{\partial v}{\partial t}$$

“Telegrapher’s  
Equations”

# TEM Transmission Line (cont.)

To combine these, take the derivative of the first one with respect to  $z$ :

$$\begin{aligned}\frac{\partial^2 v}{\partial z^2} &= -R \frac{\partial i}{\partial z} - L \frac{\partial}{\partial z} \left( \frac{\partial i}{\partial t} \right) \\ &= -R \frac{\partial i}{\partial z} - L \frac{\partial}{\partial t} \left( \frac{\partial i}{\partial z} \right) \quad \leftarrow \text{Switch the order of the derivatives.} \\ &= -R \left[ -Gv - C \frac{\partial v}{\partial t} \right] - L \left[ -G \frac{\partial v}{\partial t} - C \frac{\partial^2 v}{\partial t^2} \right] \\ &\quad \leftarrow \frac{\partial i}{\partial z} = -Gv - C \frac{\partial v}{\partial t}\end{aligned}$$

# TEM Transmission Line (cont.)

$$\frac{\partial^2 v}{\partial z^2} = -R \left[ -Gv - C \frac{\partial v}{\partial t} \right] - L \left[ -G \frac{\partial v}{\partial t} - C \frac{\partial^2 v}{\partial t^2} \right]$$

Hence, we have:

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG) \frac{\partial v}{\partial t} - LC \left( \frac{\partial^2 v}{\partial t^2} \right) = 0$$

The same differential equation also holds for  $i$ .

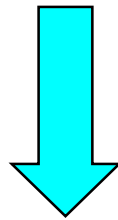
**Note:** There is no exact solution in the time domain, in the lossy case.



# TEM Transmission Line (cont.)

Time-Harmonic Waves:  $\frac{\partial}{\partial t} \rightarrow j\omega$

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$



$$\frac{d^2 V}{dz^2} - (RG)V - (RC + LG)j\omega V - LC(-\omega^2)V = 0$$

# TEM Transmission Line (cont.)

$$\frac{d^2V}{dz^2} = (RG)V + j\omega(RC + LG)V - (\omega^2 LC)V$$

Note that

$$RG + j\omega(RC + LG) - \omega^2 LC = (R + j\omega L)(G + j\omega C)$$

$Z = R + j\omega L$  = series impedance / unit length

$Y = G + j\omega C$  = parallel admittance / unit length

Then we can write:

$$\frac{d^2V}{dz^2} = (ZY)V$$

# TEM Transmission Line (cont.)

Define  $\gamma^2 \equiv ZY$       Then  $\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$

Solution:  $V(z) = Ae^{-\gamma z} + Be^{+\gamma z}$

$\gamma$  is called the “propagation constant”.

We have:  $\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$

**Question: Which sign of the square root is correct?**

# TEM Transmission Line (cont.)

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

We choose the principal square root.

Principal square root:  $z = re^{j\theta}$

$$\sqrt{z} = \sqrt{r} e^{j\theta/2} \quad -\pi < \theta \leq \pi \quad (\text{Note the square-root ("radical") symbol here.})$$

$$\Rightarrow \operatorname{Re}(\sqrt{z}) \geq 0$$

Examples :

$$4^{1/2} = \pm 2, \quad \sqrt{4} = 2$$

$$j^{1/2} = \pm \left( \frac{1+j}{\sqrt{2}} \right), \quad \sqrt{j} = \frac{1+j}{\sqrt{2}}$$

Hence

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \operatorname{Re} \gamma \geq 0$$

# TEM Transmission Line (cont.)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Denote:

$$\gamma = \alpha + j\beta$$

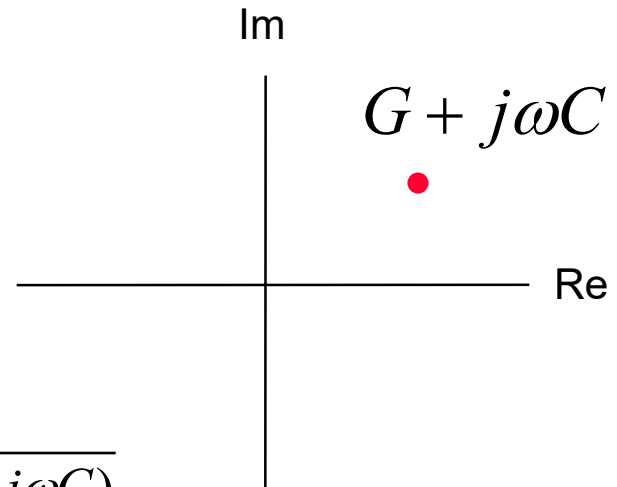
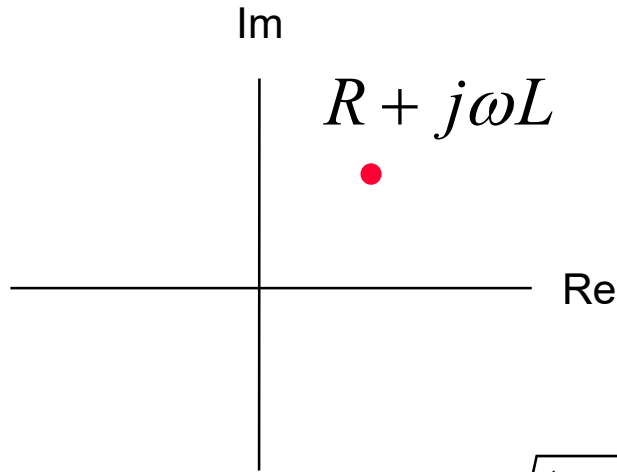
$\gamma$  = propagation constant [1/m]

$\beta$  = phase constant [rad/m]

$\alpha$  = attenuation constant [np/m]

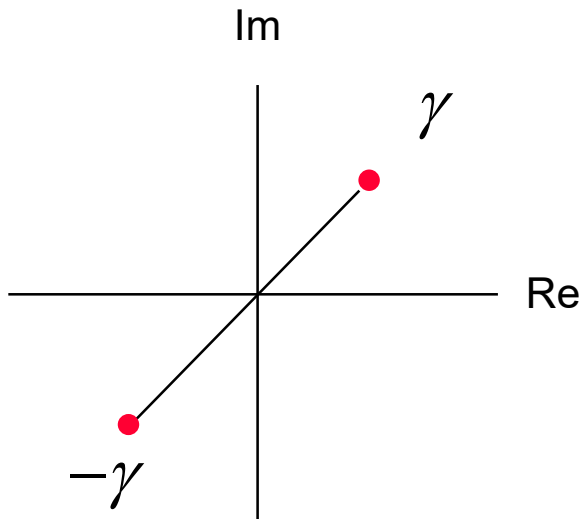
$$\alpha = \operatorname{Re} \gamma \geq 0$$

# TEM Transmission Line (cont.)



$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

There are two possible locations for the complex square root:



The principal square root must be in the first quadrant.

$$\alpha = \text{Re } \gamma$$

$$\beta = \text{Im } \gamma$$

Hence:

$$\alpha \geq 0, \beta \geq 0$$

# TEM Transmission Line (cont.)

Wave traveling in  $+z$  direction:

$$V(z) = Ae^{-\gamma z} = Ae^{-\alpha z} e^{-j\beta z} \quad (\gamma = \alpha + j\beta)$$



Wave is attenuating as it propagates.

Wave traveling in  $-z$  direction:

$$V(z) = Ae^{+\gamma z} = Ae^{+\alpha z} e^{+j\beta z} \quad (\gamma = \alpha + j\beta)$$



Wave is attenuating as it propagates.

# TEM Transmission Line (cont.)

Attenuation in dB/m:

$$V(z) = Ae^{-\alpha z} e^{-j\beta z}$$

$$\begin{aligned} \text{Attenuation in dB} &= -20 \log_{10} \left( \left| \frac{V_{out}}{V_{in}} \right| \right) \\ &= -20 \log_{10} \left( \frac{|A| e^{-\alpha z}}{|A|} \right) \\ &= -20(0.4343)(-\alpha z) \\ &= 8.686\alpha z \end{aligned}$$

Note:  $\log_{10}(x) = 0.4343 \ln(x)$

$$\text{Attenuation in dB/m} = 8.686\alpha$$



# Wavenumber Notation

$$V(z) = Ae^{-\gamma z} = Ae^{-\alpha z} e^{-j\beta z} \quad \gamma = \alpha + j\beta$$

$$V(z) = Ae^{-jk_z z} = Ae^{-\alpha z} e^{-j\beta z} \quad k_z = \beta - j\alpha$$

$$\gamma = jk_z$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \text{"propagation constant"}$$

$$k_z = -j\sqrt{(R + j\omega L)(G + j\omega C)} \quad \text{"propagation wavenumber"}$$

# TEM Transmission Line (cont.)

Forward travelling wave (a wave traveling in the positive  $z$  direction):

$$V^+(z) = V_0^+ e^{-\gamma z} = V_0^+ e^{-\alpha z} e^{-j\beta z}$$

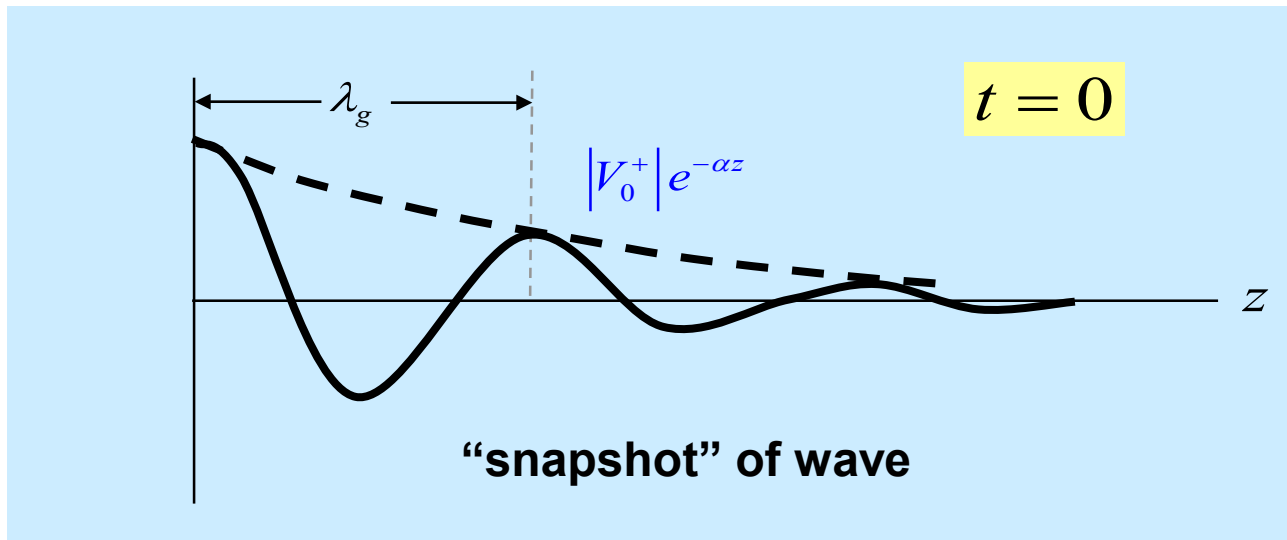
$$\begin{aligned} v^+(z, t) &= \text{Re} \left\{ \left( V_0^+ e^{-\alpha z} e^{-j\beta z} \right) e^{j\omega t} \right\} \\ &= \text{Re} \left\{ \left( \left( |V_0^+| e^{j\phi} \right) e^{-\alpha z} e^{-j\beta z} \right) e^{j\omega t} \right\} \\ &= |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi) \end{aligned}$$

The wave “repeats” when:

$$\beta \lambda_g = 2\pi$$

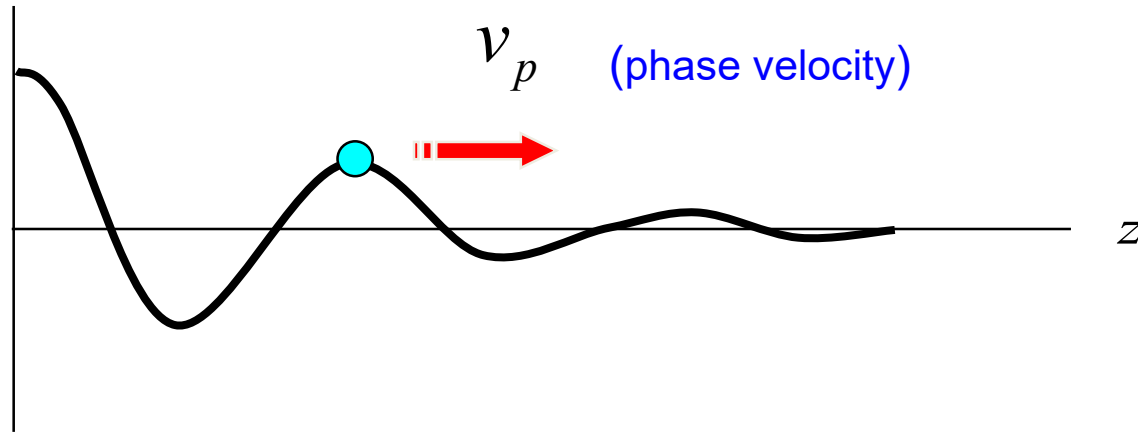
Hence:

$$\beta = \frac{2\pi}{\lambda_g}$$



# Phase Velocity

Let's track the velocity of a fixed point on the wave (a point of constant phase), e.g., the crest of the wave.



$$v^+(z, t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi)$$

# Phase Velocity (cont.)

Set  $\omega t - \beta z = \text{constant}$

$$\omega - \beta \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{\omega}{\beta}$$

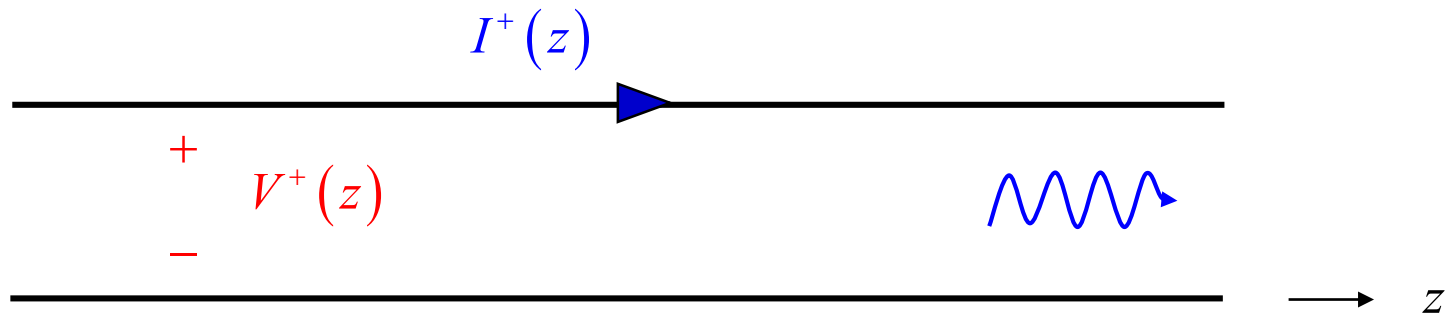
Hence

$$v_p = \frac{\omega}{\beta}$$

In expanded form:

$$v_p = \frac{\omega}{\text{Im}\left\{\sqrt{(R + j\omega L)(G + j\omega C)}\right\}}$$

# Characteristic Impedance $Z_0$



Assumption: A wave is traveling in the positive  $z$  direction.

$$Z_0 \equiv \frac{V^+(z)}{I^+(z)}$$

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$I^+(z) = I_0^+ e^{-\gamma z}$$

so 
$$Z_0 = \frac{V_0^+}{I_0^+}$$

(Note:  $Z_0$  is a number, not a function of  $z$ .)

# Characteristic Impedance $Z_0$ (cont.)

Use first Telegrapher's Equation:

$$\frac{\partial v}{\partial z} = -Ri - L \frac{\partial i}{\partial t}$$

so

$$\frac{dV}{dz} = -RI - j\omega LI = -ZI$$

Recall: 
$$\begin{cases} V^+(z) = V_0^+ e^{-\gamma z} \\ I^+(z) = I_0^+ e^{-\gamma z} \end{cases}$$

Hence 
$$-\gamma V_0^+ e^{-\gamma z} = -Z I_0^+ e^{-\gamma z}$$

# Characteristic Impedance $Z_0$ (cont.)

From this we have:

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}}$$

Use:

$$\left. \begin{aligned} Z &= R + j\omega L \\ Y &= G + j\omega C \end{aligned} \right\}$$

Both are in the first quadrant

$$\Rightarrow \gamma = \sqrt{ZY} \in \text{first quadrant}$$

$$\Rightarrow Z_0 = Z / \sqrt{ZY} \in \text{right - half plane}$$

$$\Rightarrow \text{Re } Z_0 \geq 0$$

$$\Rightarrow Z_0 = \sqrt{\frac{Z}{Y}} \quad (\text{principal square root})$$

# Characteristic Impedance $Z_0$ (cont.)

Hence, we have

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\operatorname{Re} Z_0 \geq 0$$



# General Case (Waves in Both Directions)

$$\begin{aligned} V(z) &= V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \\ &= |V_0^+| e^{j\phi^+} e^{-\alpha z} e^{-j\beta z} + |V_0^-| e^{j\phi^-} e^{+\alpha z} e^{+j\beta z} \end{aligned}$$

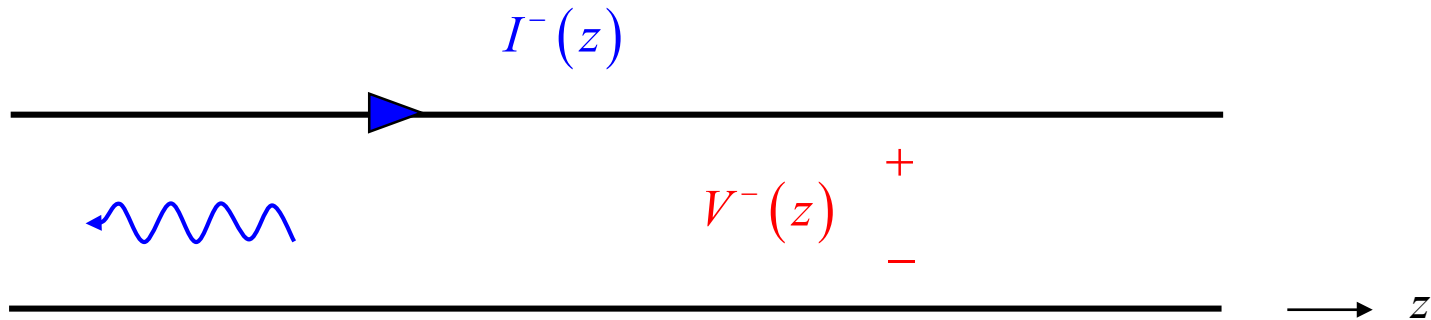
Wave in +z  
direction

Wave in -z  
direction

In the time domain:

$$\begin{aligned} v(z,t) &= \text{Re} \{ V(z) e^{j\omega t} \} \\ &= |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) \\ &\quad + |V_0^-| e^{+\alpha z} \cos(\omega t + \beta z + \phi^-) \end{aligned}$$

# Backward-Traveling Wave



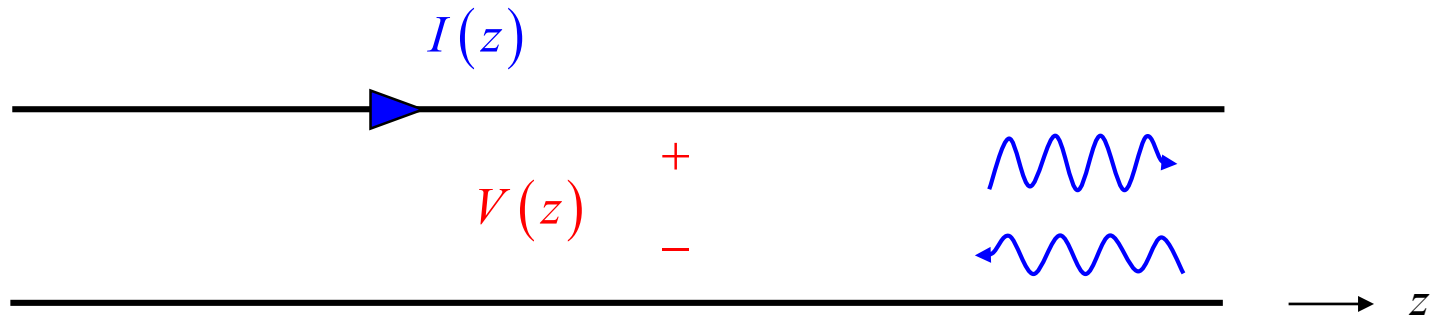
A wave is traveling in the negative  $z$  direction.

$$\frac{V^-(z)}{-I^-(z)} = Z_0 \quad \text{so} \quad \frac{V^-(z)}{I^-(z)} = -Z_0$$

**Note:**

The reference directions for voltage and current are chosen the same as for the forward wave.

# General Case

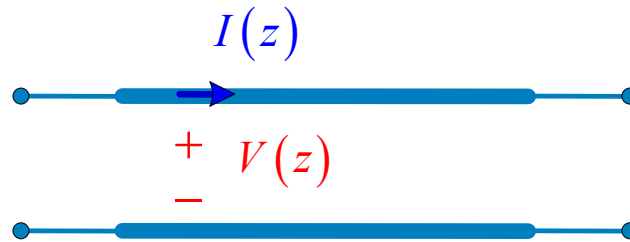


Most general case: A general superposition of forward and backward traveling waves:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{1}{Z_0} \left[ V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z} \right]$$

# Summary of Basic TL formulas



$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Guided wavelength:

$$\lambda_g = \frac{2\pi}{\beta} [\text{m}]$$

Phase velocity:

$$v_p = \frac{\omega}{\beta} [\text{m/s}]$$

Attenuation in dB/m =  $8.686\alpha$

# Lossless Case

$$R = 0, G = 0$$

$$\begin{aligned}\gamma = \alpha + j\beta &= \sqrt{(\cancel{R + j\omega L})(\cancel{G + j\omega C})} \\ &= j\omega\sqrt{LC}\end{aligned}$$

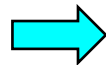
so

$$\begin{aligned}\alpha &= 0 \\ \beta &= \omega\sqrt{LC}\end{aligned}$$

$$v_p = \frac{\omega}{\beta}$$



$$Z_0 = \sqrt{\frac{\cancel{R + j\omega L}}{\cancel{G + j\omega C}}}$$



$$Z_0 = \sqrt{\frac{L}{C}}$$

(real and independent of freq.)

$$v_p = \frac{1}{\sqrt{LC}}$$

(independent of freq.)

# Lossless Case (cont.)

$$v_p = \frac{1}{\sqrt{LC}}$$

If the medium between the two conductors is lossless and homogeneous (uniform) and is characterized by  $(\epsilon, \mu)$ , then (proof given later):

$$LC = \mu\epsilon$$

The speed of light in a dielectric medium is  $c_d = \frac{1}{\sqrt{\mu\epsilon}}$

Hence, we have that:  $v_p = c_d$

and  $\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon} = k = k_0\sqrt{\epsilon_r\mu_r}$

In the lossless case the phase velocity does not depend on the frequency, and it is always equal to the speed of light (in the material).

# Terminated Transmission Line

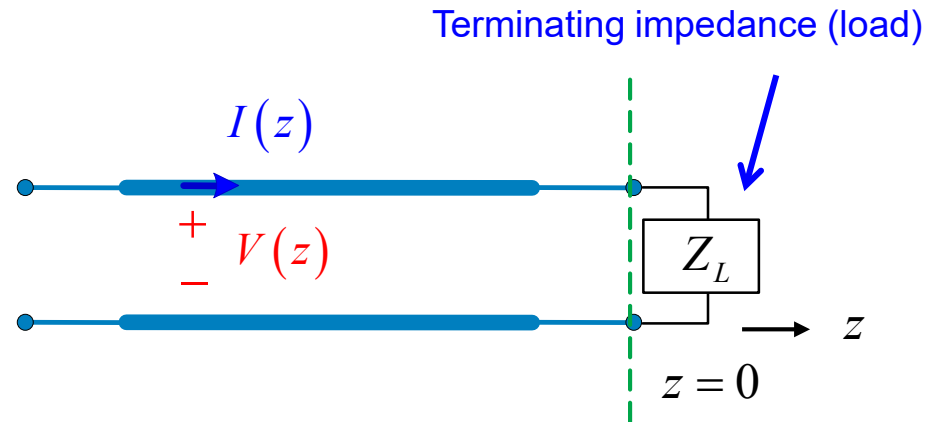
$$V(z) = \overbrace{V_0^+ e^{-\gamma z}}^{V^+(z)} + \overbrace{V_0^- e^{+\gamma z}}^{V^-(z)}$$

Amplitude of voltage wave propagating in positive  $z$  direction at  $z = 0$ .

$$V^+(0) = V_0^+$$

$$V^-(0) = V_0^-$$

Amplitude of voltage wave propagating in negative  $z$  direction at  $z = 0$ .

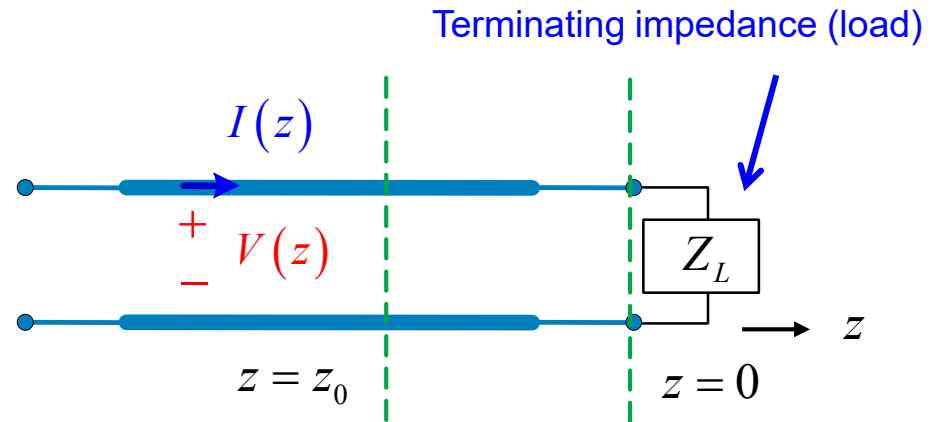


Where do we assign  $z = 0$  ?  
The usual choice is at the load.

# Terminated Transmission Line (cont.)

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

Can we use  $z = z_0$  as a reference plane?



$$V^+(z_0) = V_0^+ e^{-\gamma z_0}$$

$$\Rightarrow V_0^+ = V^+(z_0) e^{+\gamma z_0}$$

$$V^-(z_0) = V_0^- e^{+\gamma z_0}$$

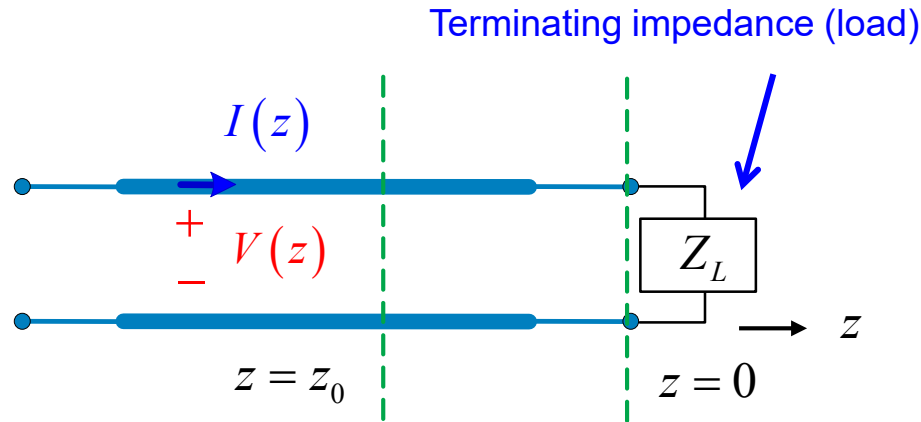
$$\Rightarrow V_0^- = V^-(z_0) e^{-\gamma z_0}$$

Hence

$$V(z) = V^+(z_0) e^{-\gamma(z-z_0)} + V^-(z_0) e^{\gamma(z-z_0)}$$



# Terminated Transmission Line (cont.)



Compare:

$$V(z) = V^+(0)e^{-\gamma z} + V^-(0)e^{+\gamma z}$$

$$V(z) = V^+(z_0)e^{-\gamma(z-z_0)} + V^-(z_0)e^{\gamma(z-z_0)}$$

This is simply a change of reference plane, from  $z=0$  to  $z=z_0$ .

# Terminated Transmission Line (cont.)

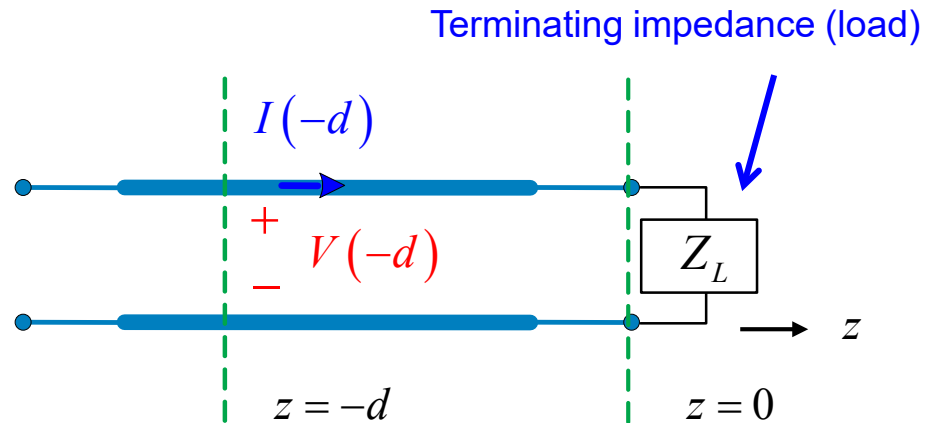
What is  $V(-d)$  ?

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$V(-d) = V_0^+ e^{\gamma d} + V_0^- e^{-\gamma d}$$

Propagating forwards

Propagating backwards



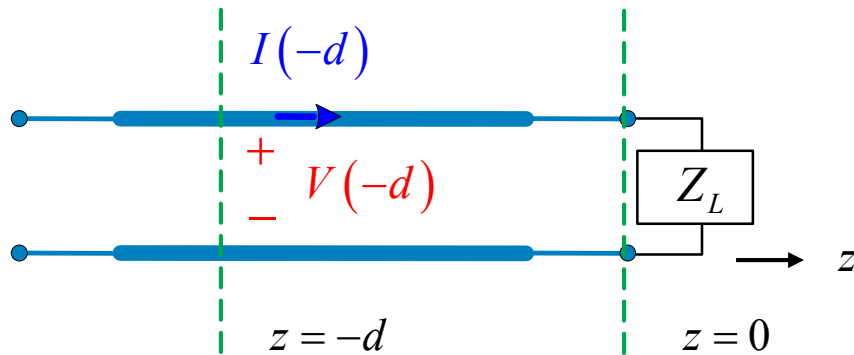
The current at  $z = -d$  is then:

$$I(-d) = \frac{V_0^+}{Z_0} e^{\gamma d} - \frac{V_0^-}{Z_0} e^{-\gamma d}$$

$d \equiv$  distance away from load

(This does not necessarily have to be the length of the entire line.)

# Terminated Transmission Line (cont.)



$\Gamma(-d)$  = reflection coefficient at  $z = -d$

$$V(-d) = V_0^+ e^{\gamma d} + V_0^- e^{-\gamma d} = V_0^+ e^{\gamma d} \left( 1 + \underbrace{\frac{V_0^-}{V_0^+}}_{\Gamma_L} e^{-2\gamma d} \right)$$

or

$$V(-d) = V_0^+ e^{\gamma d} (1 + \Gamma_L e^{-2\gamma d})$$

$\Gamma_L \equiv$  load reflection coefficient

Similarly,

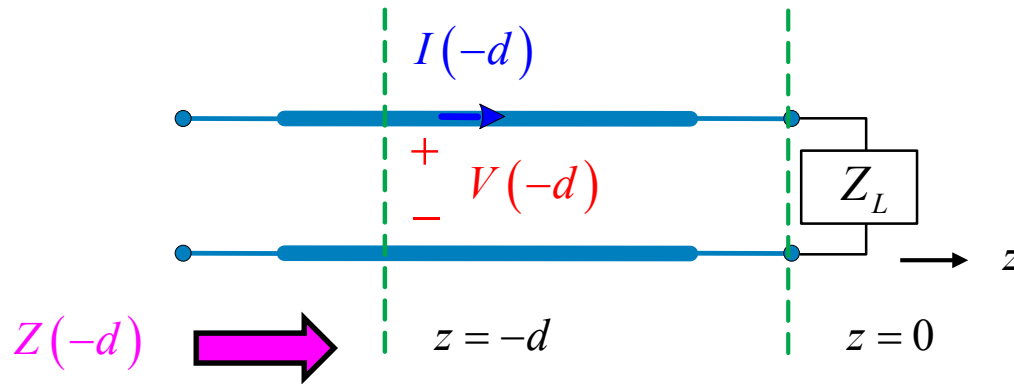
$$I(-d) = \frac{V_0^+}{Z_0} e^{\gamma d} (1 - \Gamma_L e^{-2\gamma d})$$

$$\Gamma_L \equiv \frac{V_0^-}{V_0^+}$$

$$\Gamma(-d) = \Gamma_L e^{-2\gamma d}$$

# Terminated Transmission Line (cont.)

$Z(-d)$  = impedance seen “looking” towards load at  $z = -d$ .



**Note:**  
If we are at the beginning of the line, we will call this the “input impedance”.

$$V(-d) = V_0^+ e^{\gamma d} (1 + \Gamma_L e^{-2\gamma d})$$

$$I(-d) = \frac{V_0^+}{Z_0} e^{\gamma d} (1 - \Gamma_L e^{-2\gamma d})$$

→ 
$$Z(-d) = \frac{V(-d)}{I(-d)} = Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right) = Z_0 \left( \frac{1 + \Gamma(-d)}{1 - \Gamma(-d)} \right)$$

# Terminated Transmission Line (cont.)

At the load ( $d = 0$ ):

$$Z(0) = Z_0 \left( \frac{1 + \Gamma_L}{1 - \Gamma_L} \right) \equiv Z_L \quad \Rightarrow \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

At any point on the line ( $d > 0$ ):

$$\Gamma(-d) = \Gamma_L e^{-2\gamma d}$$

# Terminated Transmission Line (cont.)

Recall

$$Z(-d) = Z_0 \left( \frac{1 + \Gamma(-d)}{1 - \Gamma(-d)} \right) = Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right)$$

Thus,

$$Z(-d) = Z_0 \left( \frac{1 + \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma d}}{1 - \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma d}} \right)$$

# Terminated Transmission Line (cont.)

Simplifying, we have:

$$\begin{aligned} Z(-d) &= Z_0 \left( \frac{1 + \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma d}}{1 - \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) e^{-2\gamma d}} \right) = Z_0 \left( \frac{(Z_L + Z_0) + (Z_L - Z_0) e^{-2\gamma d}}{(Z_L + Z_0) - (Z_L - Z_0) e^{-2\gamma d}} \right) \\ &= Z_0 \left( \frac{(Z_L + Z_0) e^{+\gamma d} + (Z_L - Z_0) e^{-\gamma d}}{(Z_L + Z_0) e^{+\gamma d} - (Z_L - Z_0) e^{-\gamma d}} \right) \\ &= Z_0 \left( \frac{Z_L \cosh(\gamma d) + Z_0 \sinh(\gamma d)}{Z_0 \cosh(\gamma d) + Z_L \sinh(\gamma d)} \right) \end{aligned}$$

Hence, we have

$$Z(-d) = Z_0 \left( \frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)} \right)$$

# Terminated Lossless Transmission Line

Lossless:  $\gamma = \cancel{\alpha} + j\beta = j\beta$

$$V(-d) = V_0^+ e^{j\beta d} (1 + \Gamma_L e^{-2j\beta d})$$

$$I(-d) = \frac{V_0^+}{Z_0} e^{j\beta d} (1 - \Gamma_L e^{-2j\beta d})$$

$$Z(-d) = Z_0 \left( \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}} \right)$$

$$Z(-d) = Z_0 \left( \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right)$$

**Note:**  $\tanh(\gamma d) = \tanh(j\beta d) = j \tan(\beta d)$

Impedance is periodic with period  $\lambda_g/2$ :

The tan function repeats when

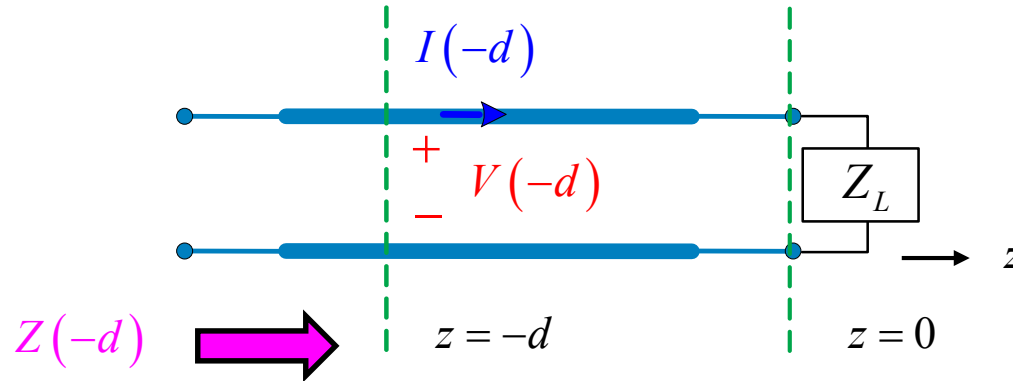
$$\beta(d_2 - d_1) = \pi$$

$$\frac{2\pi}{\lambda_g}(d_2 - d_1) = \pi$$

$$\Rightarrow d_2 - d_1 = \lambda_g / 2$$



# Summary for Lossy Transmission Line



$$Z(-d) = Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right)$$

$$= Z_0 \left( \frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)} \right)$$

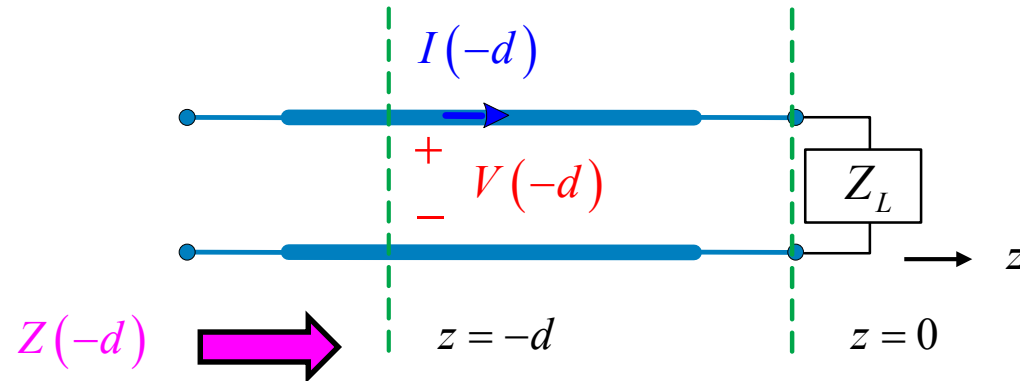
$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\lambda_g = \frac{2\pi}{\beta}$$

$$v_p = \frac{\omega}{\beta}$$

# Summary for Lossless Transmission Line



$$Z(-d) = Z_0 \left( \frac{1 + \Gamma_L e^{-2j\beta d}}{1 - \Gamma_L e^{-2j\beta d}} \right)$$

$$= Z_0 \left( \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right)$$

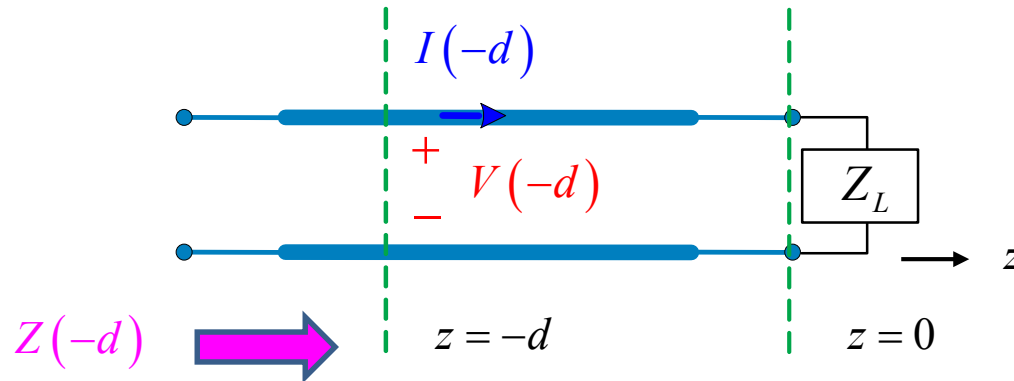
$$\beta = \omega \sqrt{LC} = \omega \sqrt{\mu \epsilon} = k$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k} = \lambda_d$$

$$v_p = \frac{\omega}{\beta} = c_d$$

# Matched Load ( $Z_L = Z_0$ )



$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$

No reflection from the load

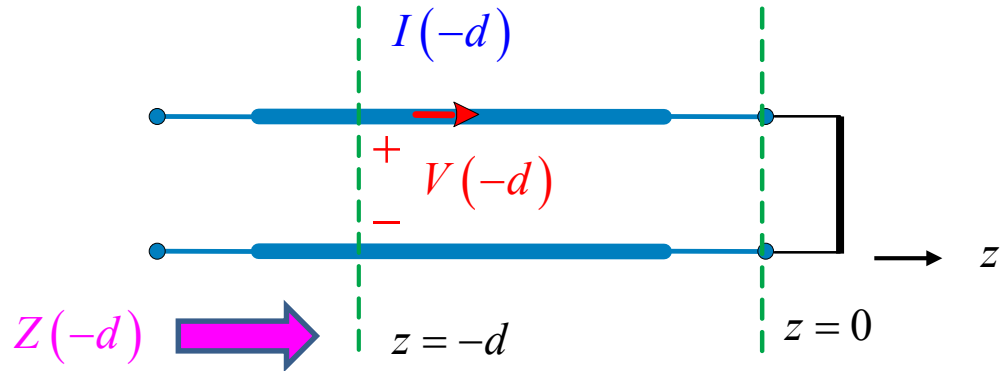
$$Z(-d) = Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}} \right) \Rightarrow Z(-d) = Z_0 \text{ for any } z$$

# Short-Circuit Load ( $Z_L=0$ )

## Lossless Case

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_L = -1$$



$$Z(-d) = Z_0 \left( \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right)$$

$$Z(-d) = jZ_0 \tan(\beta d) \quad \text{Always imaginary!}$$

# Short-Circuit Load ( $Z_L=0$ )

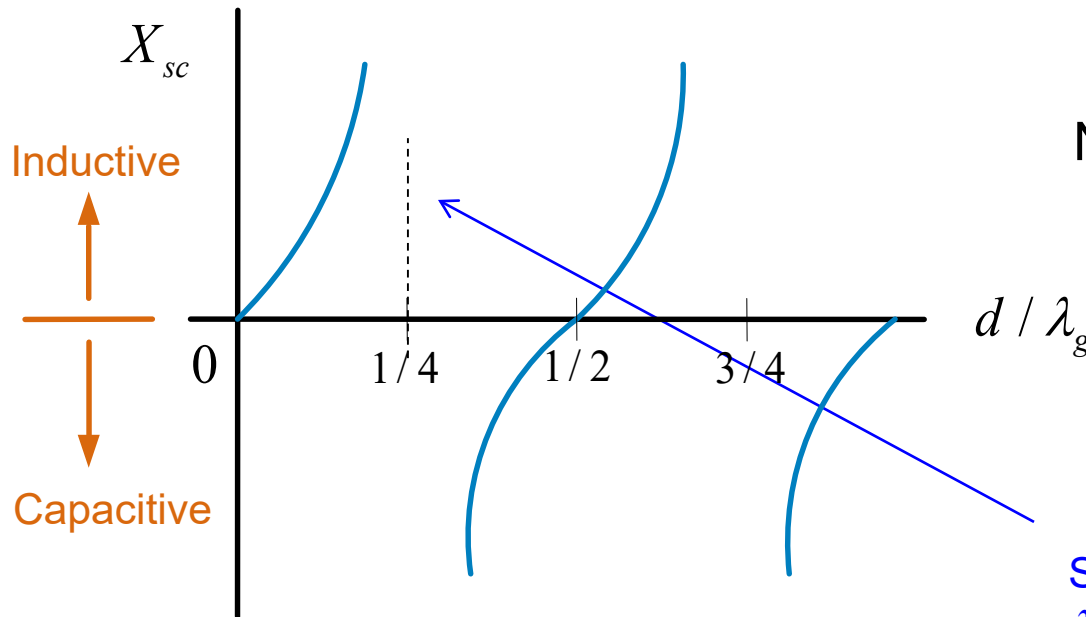
## Lossless Case

$$Z(-d) = jZ_0 \tan(\beta d)$$

$$Z(-d) = jX_{sc}$$

$$X_{sc} = Z_0 \tan(\beta d)$$

Note:  $\beta d = 2\pi \frac{d}{\lambda_g}$



$$\lambda_g = \lambda_d$$

(lossless)

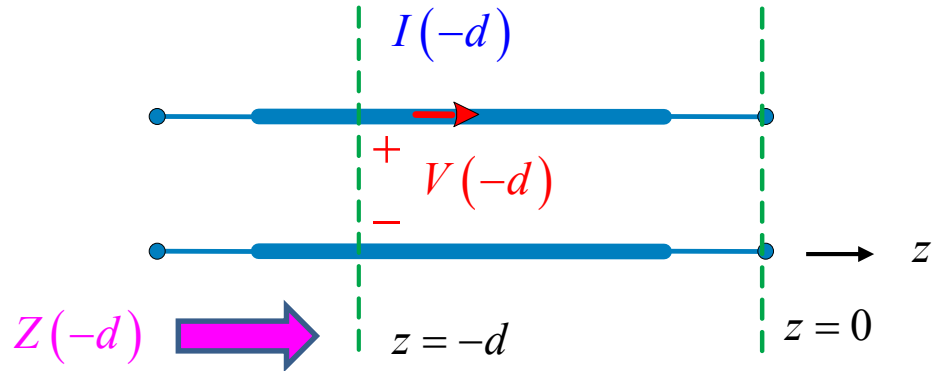
S.C. can become an O.C. with a  $\lambda_g/4$  transmission line.

# Open-Circuit Load ( $Z_L = \infty$ )

## Lossless Case

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_L = \frac{1 - Z_0 / Z_L}{1 + Z_0 / Z_L} \rightarrow 1$$



$$\Gamma_L = +1$$

$$Z(-d) = Z_0 \left( \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \right) \quad \text{or} \quad Z(-d) = Z_0 \left( \frac{1 + j(Z_0 / Z_L) \tan(\beta d)}{(Z_0 / Z_L) + j \tan(\beta d)} \right)$$

$$Z(-d) = -jZ_0 \cot(\beta d)$$

Always imaginary!

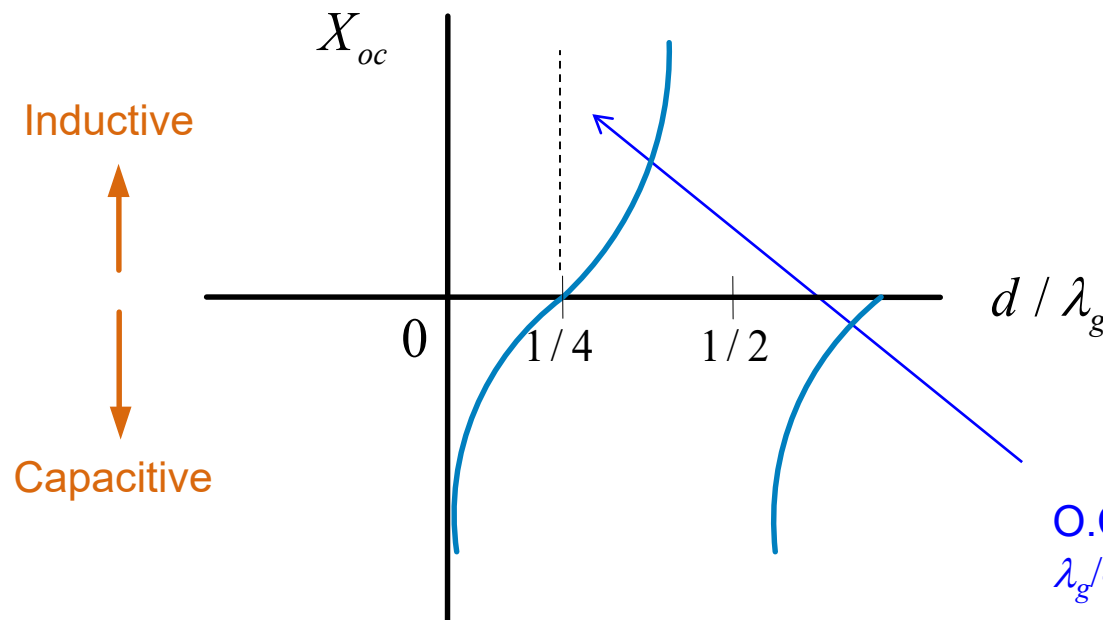
# Open-Circuit Load ( $Z_L = \infty$ )

## Lossless Case

$$Z(-d) = -jZ_0 \cot(\beta d)$$

$$Z(-d) = jX_{oc}$$

$$X_{oc} = -Z_0 \cot(\beta d)$$



Note:  $\beta d = 2\pi \frac{d}{\lambda_g}$

$$\lambda_g = \lambda_d$$

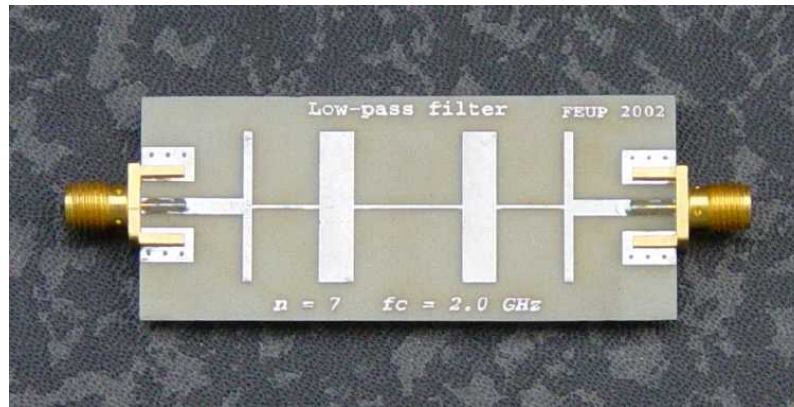
(lossless)

O.C. can become a S.C. with a  $\lambda_g/4$  transmission line.

# Using Transmission Lines to Synthesize Loads

We can obtain any reactance that we want from a short or open transmission line.

This is very useful in microwave engineering.

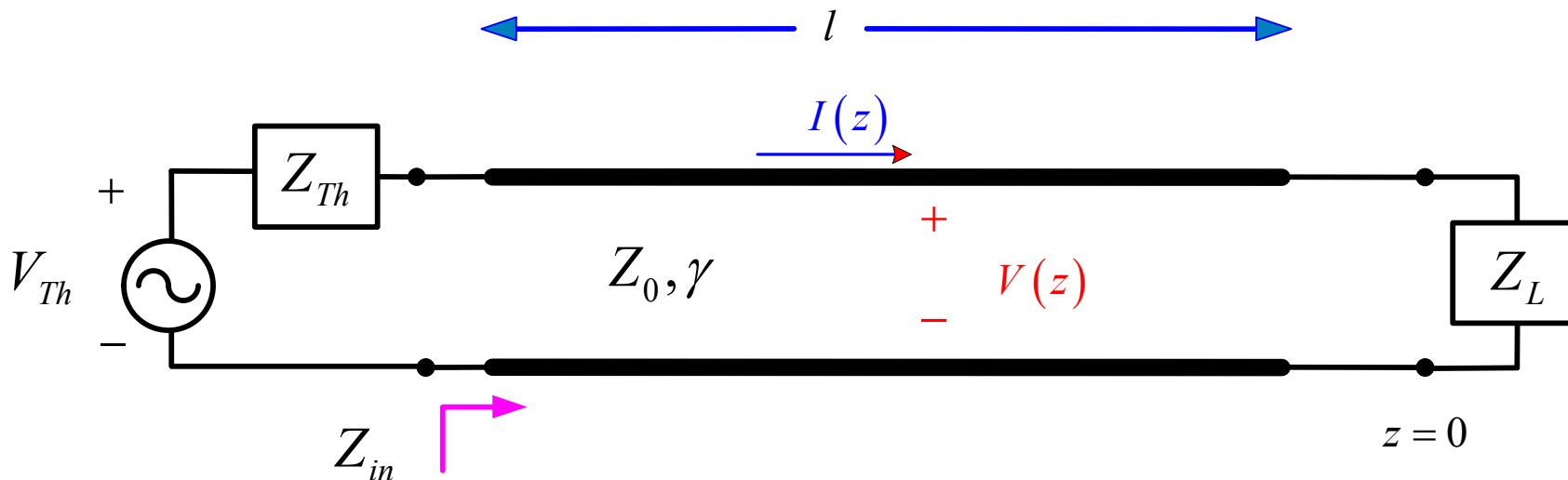


A microwave filter constructed from microstrip line.



# Voltage on a Transmission Line

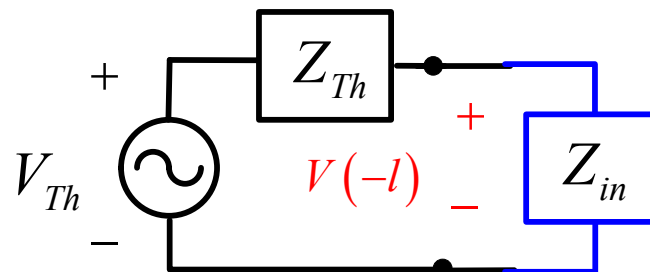
Find the voltage at any point on the line.



At the input:

$$V(-l) = V_{Th} \left( \frac{Z_{in}}{Z_{in} + Z_{Th}} \right)$$

$$Z_{in} = Z(-l) = Z_0 \left( \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \right)$$



# Voltage on a Transmission Line (cont.)

$$V(z) = \underbrace{V_0^+ e^{-\gamma z}}_{\uparrow} (1 + \Gamma_L e^{+2\gamma z}) \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Incident (forward) wave (not the same as the initial wave from the source!)

At  $z = -l$ :

$$V(-l) = V_0^+ e^{\gamma l} (1 + \Gamma_L e^{-2\gamma l}) = V_{Th} \left( \frac{Z_{in}}{Z_{in} + Z_{Th}} \right)$$

$$\Rightarrow V_0^+ = V_{Th} \left( \frac{Z_{in}}{Z_{in} + Z_{Th}} \right) e^{-\gamma l} \left( \frac{1}{1 + \Gamma_L e^{-2\gamma l}} \right)$$

Hence

$$V(z) = V_{Th} \left( \frac{Z_{in}}{Z_{in} + Z_{Th}} \right) e^{-\gamma(l+z)} \left( \frac{1 + \Gamma_L e^{+2\gamma z}}{1 + \Gamma_L e^{-2\gamma l}} \right)$$

# Voltage on a Transmission Line (cont.)

Let's derive an alternative form of the previous result.


$$\text{Start with: } Z_{in} = Z(-l) = Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}} \right)$$

$$\begin{aligned} \Rightarrow \frac{Z_{in}}{Z_{in} + Z_{Th}} &= \frac{Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}} \right)}{Z_0 \left( \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}} \right) + Z_{Th}} = \frac{Z_0 (1 + \Gamma_L e^{-2\gamma l})}{Z_0 (1 + \Gamma_L e^{-2\gamma l}) + Z_{Th} (1 - \Gamma_L e^{-2\gamma l})} \\ &= \frac{Z_0 (1 + \Gamma_L e^{-2\gamma l})}{(Z_{Th} + Z_0) + \Gamma_L e^{-2\gamma l} (Z_0 - Z_{Th})} \\ &= \left( \frac{Z_0}{Z_{Th} + Z_0} \right) \frac{(1 + \Gamma_L e^{-2\gamma l})}{1 + \Gamma_L e^{-2\gamma l} \left( \frac{Z_0 - Z_{Th}}{Z_{Th} + Z_0} \right)} \\ &= \left( \frac{Z_0}{Z_{Th} + Z_0} \right) \frac{(1 + \Gamma_L e^{-2\gamma l})}{1 - \Gamma_L e^{-2\gamma l} \left( \frac{Z_{Th} - Z_0}{Z_{Th} + Z_0} \right)} \end{aligned}$$

# Voltage on a Transmission Line (cont.)

Hence, we have

$$\frac{Z_{in}}{Z_{in} + Z_{Th}} = \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left( \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_s \Gamma_L e^{-2\gamma l}} \right)$$

Substitute  where  $\Gamma_s \equiv \frac{Z_{Th} - Z_0}{Z_{Th} + Z_0}$  (source reflection coefficient)

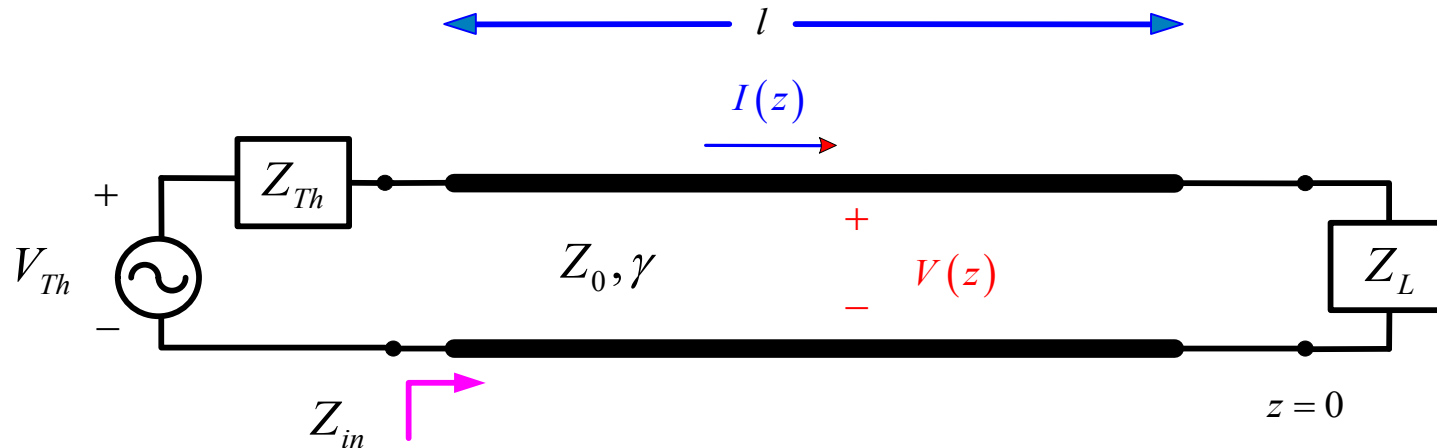
Recall:

$$V(z) = V_{Th} \left( \frac{Z_{in}}{Z_{in} + Z_{Th}} \right) e^{-\gamma(l+z)} \left( \frac{1 + \Gamma_L e^{+2\gamma z}}{1 + \Gamma_L e^{-2\gamma l}} \right)$$

Therefore, we have the following alternative form for the result:

$$V(z) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) e^{-\gamma(l+z)} \left( \frac{1 + \Gamma_L e^{+2\gamma z}}{1 - \Gamma_s \Gamma_L e^{-2\gamma l}} \right)$$

# Voltage on a Transmission Line (cont.)

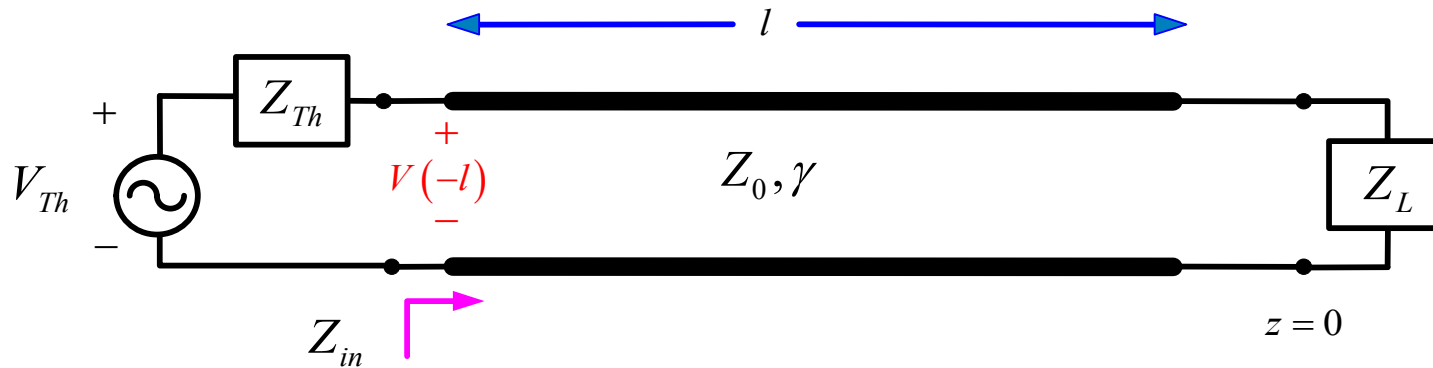


$$V(z) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) e^{-\gamma(z - (-l))} \left( \frac{1 + \Gamma_L e^{+2\gamma z}}{1 - \Gamma_s \Gamma_L e^{-2\gamma l}} \right)$$

This term accounts for the multiple (infinite) bounces.

The “initial” voltage wave that would exist if there were no reflections from the load (we have a semi-infinite transmission line or a matched load).

# Voltage on a Transmission Line (cont.)



Wave-bounce method (illustrated for  $z = -l$ ):

We add up all of the bouncing waves.

$$V(-l) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left[ \begin{aligned} &1 + \Gamma_L e^{-2\gamma l} + (\Gamma_L e^{-2\gamma l}) \Gamma_s \\ &+ [(\Gamma_L e^{-2\gamma l}) \Gamma_s] (\Gamma_L e^{-2\gamma l}) + [(\Gamma_L e^{-2\gamma l}) \Gamma_s (\Gamma_L e^{-2\gamma l})] \Gamma_s \\ &+ \dots \end{aligned} \right]$$

# Voltage on a Transmission Line (cont.)

$$V(-l) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left[ \begin{aligned} &1 + \Gamma_L e^{-2\gamma l} + (\Gamma_L e^{-2\gamma l}) \Gamma_s \\ &+ [(\Gamma_L e^{-2\gamma l}) \Gamma_s] (\Gamma_L e^{-2\gamma l}) + [(\Gamma_L e^{-2\gamma l}) \Gamma_s (\Gamma_L e^{-2\gamma l})] \Gamma_s \\ &+ \dots \end{aligned} \right]$$

Group together alternating terms:

$$V(-l) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left[ \begin{aligned} &1 + (\Gamma_L \Gamma_s e^{-2\gamma l}) + (\Gamma_L \Gamma_s e^{-2\gamma l})^2 + \dots \\ &+ \Gamma_L e^{-2\gamma l} \left[ 1 + (\Gamma_L \Gamma_s e^{-2\gamma l}) + (\Gamma_L \Gamma_s e^{-2\gamma l})^2 + \dots \right] \\ &+ \dots \end{aligned} \right]$$

Geometric series:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots = \frac{1}{1-z}, \quad |z| < 1 \quad z \equiv \Gamma_L \Gamma_s e^{-2\gamma l}$$

# Voltage on a Transmission Line (cont.)

Hence

$$V(-l) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left[ \frac{1}{1 - \Gamma_L \Gamma_s e^{-2\gamma l}} + \Gamma_L e^{-2\gamma d} \left( \frac{1}{1 - \Gamma_L \Gamma_s e^{-2\gamma l}} \right) \right]$$

or

$$V(-l) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) \left[ \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L \Gamma_s e^{-2\gamma l}} \right]$$

This agrees with the previous result (setting  $z = -l$ ).

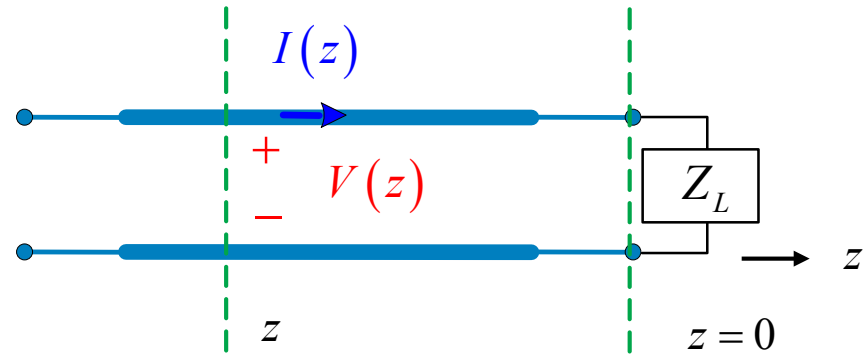
Previous (alternative) result:  $V(z) = V_{Th} \left( \frac{Z_0}{Z_0 + Z_{Th}} \right) e^{-\gamma(l+z)} \left( \frac{1 + \Gamma_L e^{+2\gamma z}}{1 - \Gamma_s \Gamma_L e^{-2\gamma l}} \right)$

The wave-bounce method is a *very tedious method* – not recommended.



# Time-Average Power Flow

$P(z)$  = power flowing in +  $z$  direction



At a distance  $d$  from the load:

$$P(z) = \frac{1}{2} \operatorname{Re} \{ V(z) I^*(z) \}$$

$$= \frac{1}{2} \operatorname{Re} \left[ \frac{|V_0^+|^2}{Z_0^*} e^{-2\alpha z} (1 + \Gamma_L e^{+2\gamma z}) (1 - \Gamma_L^* e^{+2\gamma^* z}) \right]$$

$$= \frac{1}{2} \operatorname{Re} \left[ \frac{|V_0^+|^2}{Z_0^*} e^{-2\alpha z} (1 - |\Gamma_L|^2 e^{+4\alpha z}) \right] + \frac{1}{2} \operatorname{Re} \left[ \frac{|V_0^+|^2}{Z_0^*} e^{-2\alpha z} (\Gamma_L e^{+2\gamma z} - \Gamma_L^* e^{+2\gamma^* z}) \right]$$

$$V(z) = V_0^+ e^{-\gamma z} (1 + \Gamma_L e^{+2\gamma z})$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} (1 - \Gamma_L e^{+2\gamma z})$$

$$\gamma = \alpha + j\beta$$

If  $Z_0 \approx$  real (low-loss transmission line):

$$P(z) \approx \frac{1}{2} \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z} (1 - |\Gamma_L|^2 e^{+4\alpha z}) \quad (\text{please see the note})$$

**Note:**

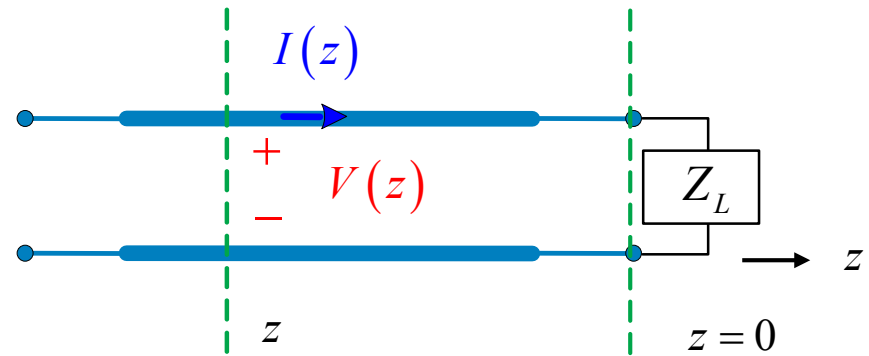
$$\Gamma_L e^{+2\gamma z} - \Gamma_L^* e^{+2\gamma^* z}$$

$$= \Gamma_L e^{+2\gamma z} - (\Gamma_L e^{+2\gamma z})^*$$

$$= \text{pure imaginary}$$

# Time-Average Power Flow (cont.)

Low-loss line



$$P(z) \approx \frac{1}{2} \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z} \left( 1 - |\Gamma_L|^2 e^{+4\alpha z} \right)$$

$$= \underbrace{\frac{1}{2} \frac{|V_0^+|^2}{Z_0} e^{-2\alpha z}}_{\text{Power in forward-traveling wave}} - \underbrace{\frac{1}{2} \frac{|V_0^+|^2}{Z_0} |\Gamma_L|^2 e^{+2\alpha z}}_{\text{Power in backward-traveling wave}}$$

Lossless line ( $\alpha = 0$ )

$$P(z) = \frac{1}{2} \frac{|V_0^+|^2}{Z_0} \left( 1 - |\Gamma_L|^2 \right)$$

**Note:**  
For a very lossy line, the total power is not the difference of the two individual powers.

# Quarter-Wave Transformer

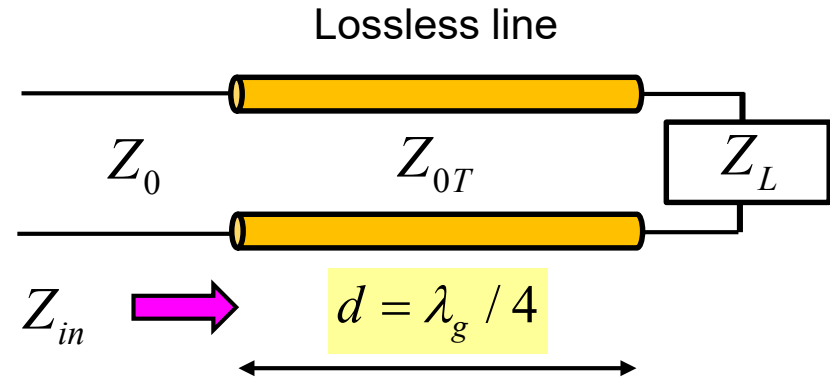
$$Z_{in} = Z_{0T} \left( \frac{Z_L + jZ_{0T} \tan \beta d}{Z_{0T} + jZ_L \tan \beta d} \right)$$

$$\beta d = \beta \frac{\lambda_g}{4} = \frac{2\pi}{\lambda_g} \frac{\lambda_g}{4} = \frac{\pi}{2}$$

$$\Rightarrow Z_{in} = Z_{0T} \left( \frac{jZ_{0T}}{jZ_L} \right)$$

so

$$Z_{in} = \frac{Z_{0T}^2}{Z_L}$$



Matching condition

$$\Gamma_{in} = 0 \Rightarrow Z_{in} = Z_0$$

$$\Rightarrow Z_0 = \frac{Z_{0T}^2}{Z_L}$$

(This requires  $Z_L$  to be real.)

Hence

$$Z_{0T} = \sqrt{Z_0 Z_L}$$

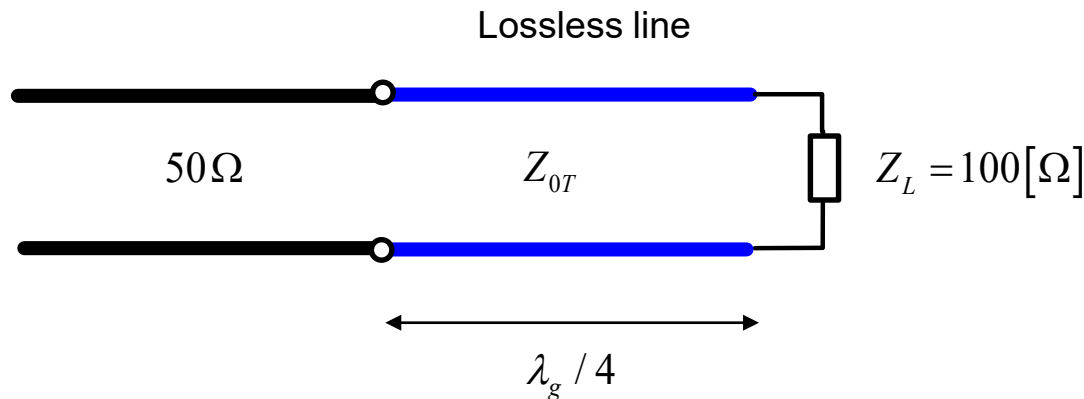
# Quarter-Wave Transformer (cont.)

## Example

Match a  $100\ \Omega$  load to a  $50\ \Omega$  transmission line at a given frequency.

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{k} = \frac{2\pi}{k_0\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \lambda_d$$

$$\lambda_0 = \frac{c}{f}$$



Note:  $\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon} = k$

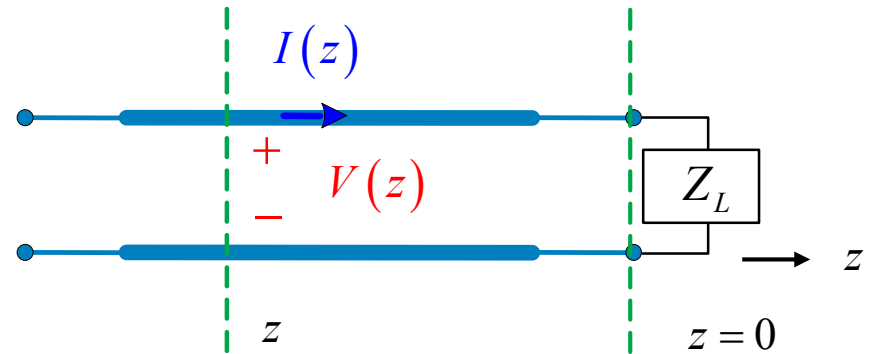
$$\begin{aligned} Z_{0T} &= \sqrt{100 \times 50} \\ &= 70.7 \end{aligned}$$

$$Z_{0T} = 70.7\ \Omega$$

# Voltage Standing Wave

## Lossless Case

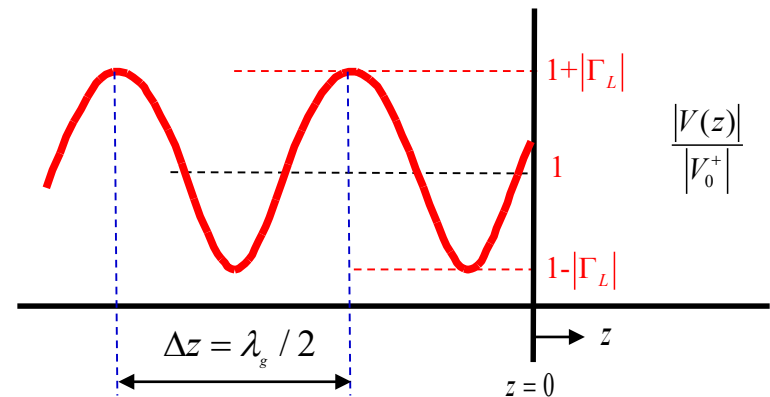
$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+2j\beta z}) \\ &= V_0^+ e^{-j\beta z} (1 + |\Gamma_L| e^{j\phi_L} e^{+2j\beta z}) \end{aligned}$$



$$|V(z)| = |V_0^+| \left| 1 + |\Gamma_L| e^{j\phi_L} e^{+j2\beta z} \right|$$

$$V_{\max} = |V_0^+| (1 + |\Gamma_L|)$$

$$V_{\min} = |V_0^+| (1 - |\Gamma_L|)$$



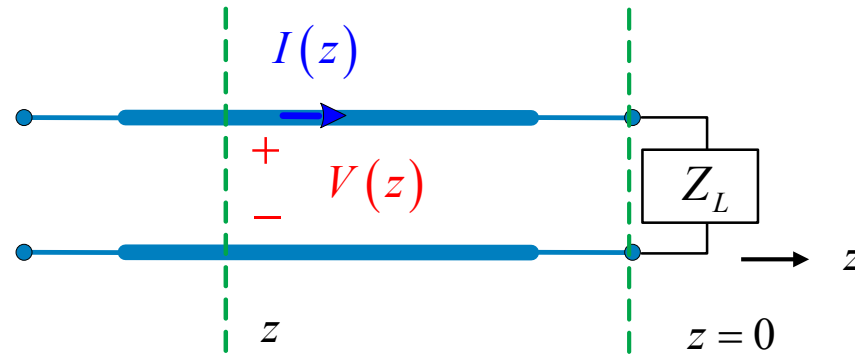
$$2\beta \Delta z = 2\pi \Rightarrow \Delta z = \lambda_g/2$$

**Note:** The voltage repeats every  $\lambda_g$ . The magnitude repeats every  $\lambda_g/2$ .

**Note:** The voltage changes by a minus sign after  $\lambda_g/2$ .

# Voltage Standing Wave Ratio

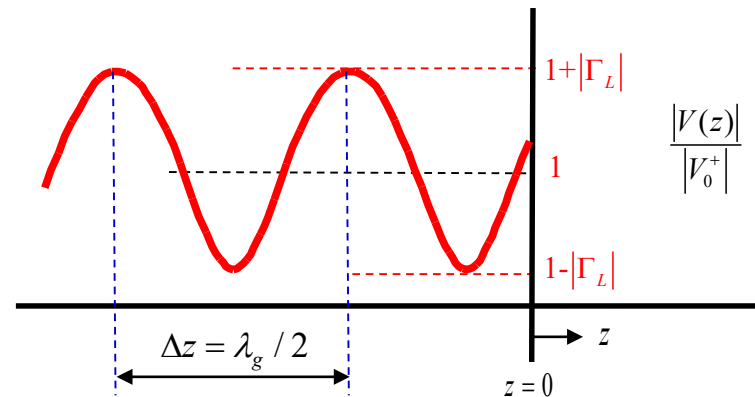
$$\text{Voltage Standing Wave Ratio (VSWR)} \equiv \frac{V_{\max}}{V_{\min}}$$



$$V_{\max} = |V_0^+| (1 + |\Gamma_L|)$$

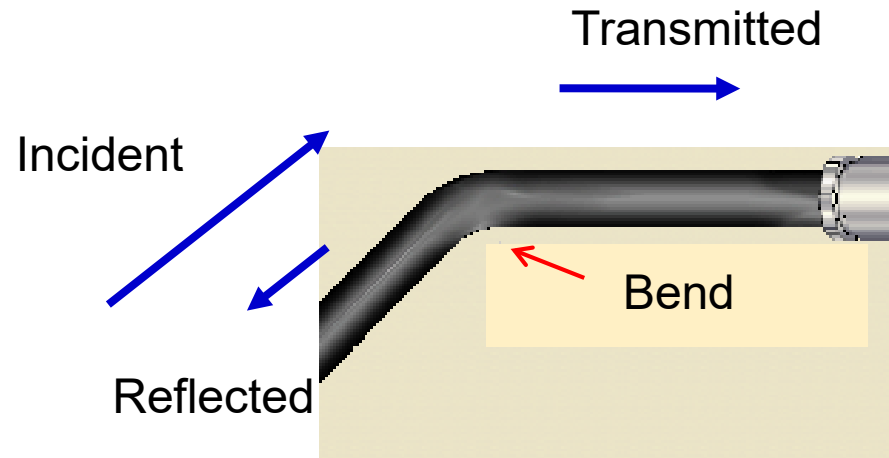
$$V_{\min} = |V_0^+| (1 - |\Gamma_L|)$$

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

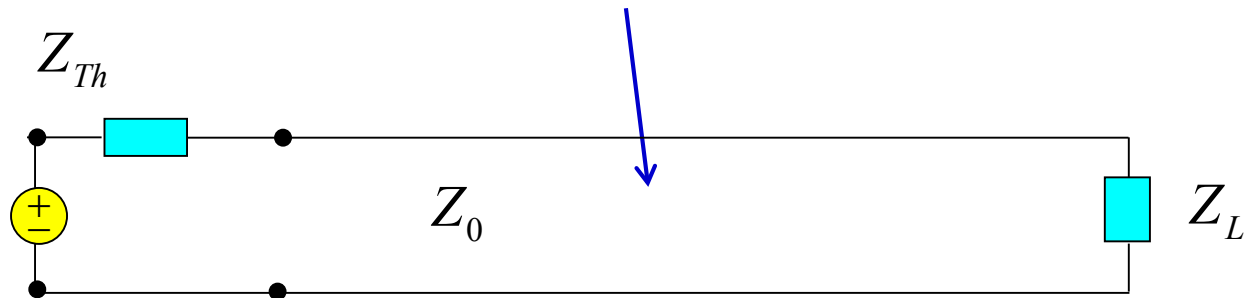


# Limitations of Transmission-Line Theory

At high frequency, **discontinuity effects** can become important.



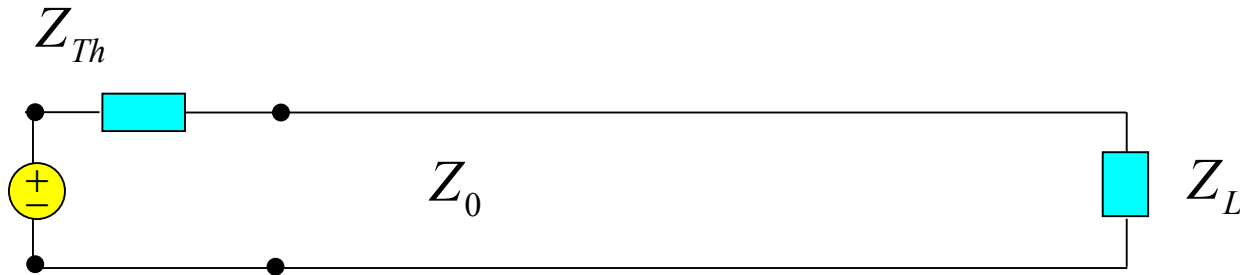
The simple TL model does not account for the bend.



# Limitations of Transmission-Line Theory (cont.)

At high frequency, radiation effects can also become important.

We want energy to travel from the generator to the load, without radiating.



When will radiation occur?

This is explored next.

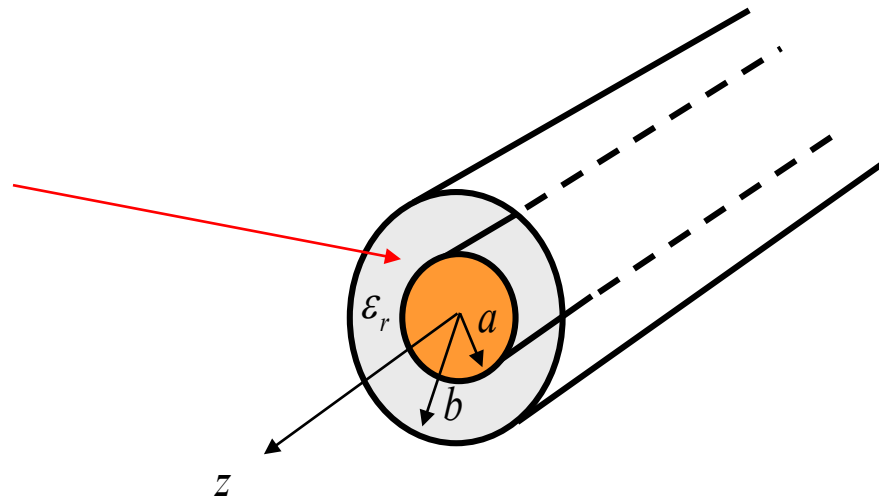


# Limitations of Transmission-Line Theory (cont.)

## Coaxial Cable

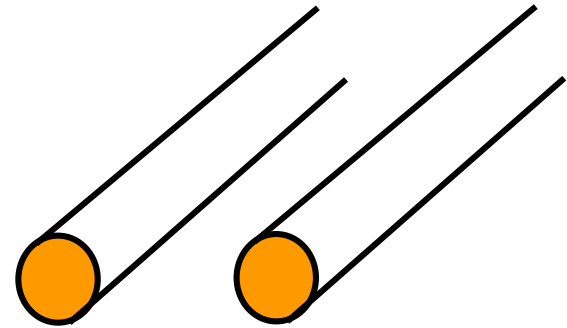
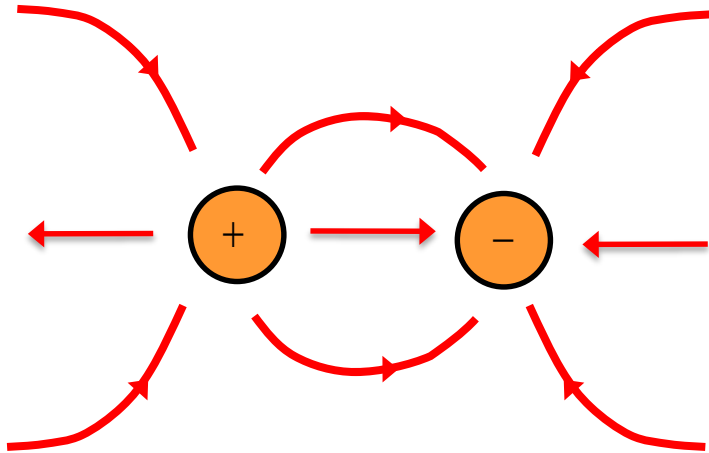
The coaxial cable is a perfectly shielded system – there is never any radiation at any frequency, as long as the metal thickness is large compared with a skin depth.

The fields are confined to the region between the two conductors.



# Limitations of Transmission-Line Theory (cont.)

The twin lead is an open type of transmission line – the fields extend out to infinity.

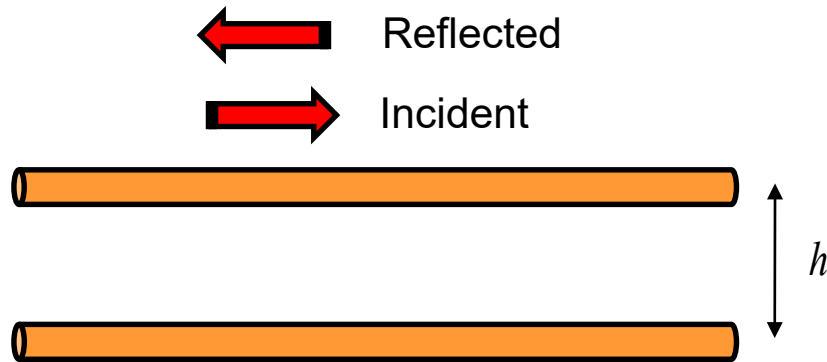


The extended fields may cause interference with nearby objects.  
(This may be improved by using “twisted pair”.)

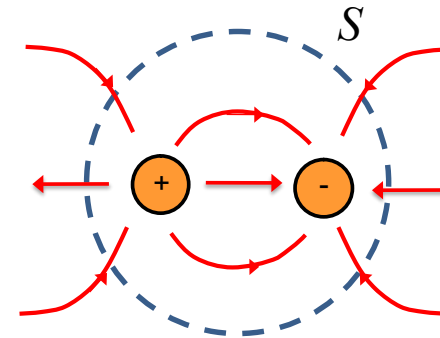
Having fields that extend to infinity is not the same thing as having radiation, however!

# Limitations of Transmission-Line Theory (cont.)

The infinite twin lead will not radiate by itself, regardless of how far apart the lines are (this is true for any transmission line).



$$P_t = \int_S \operatorname{Re} \left( \frac{1}{2} (\underline{E} \times \underline{H}^*) \right) \cdot \hat{\rho} dS = 0$$

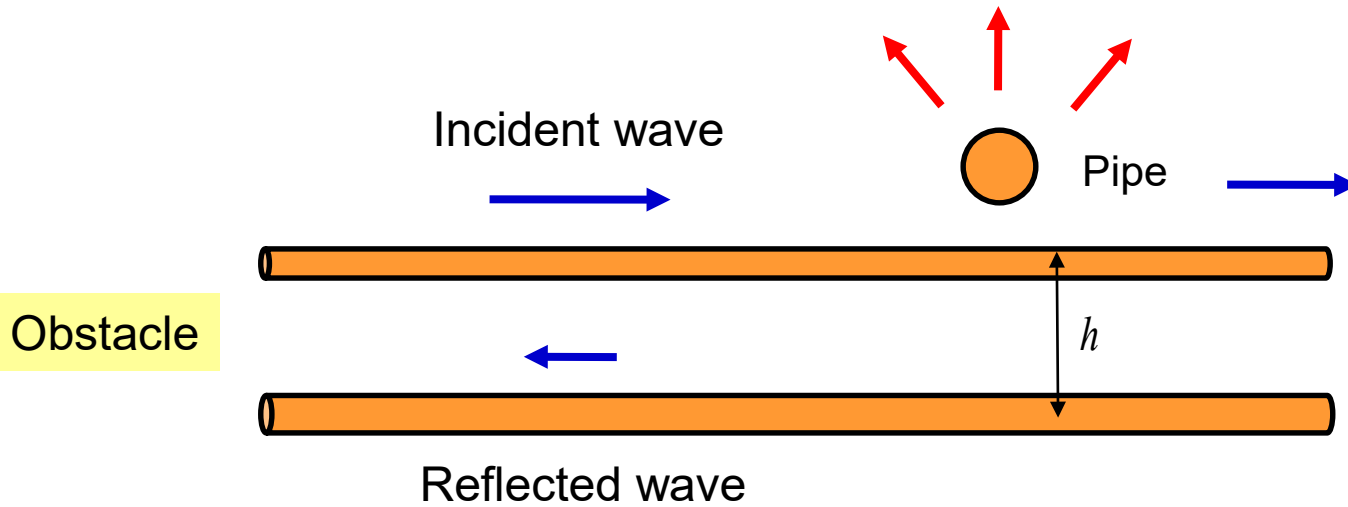


No attenuation on an infinite lossless line

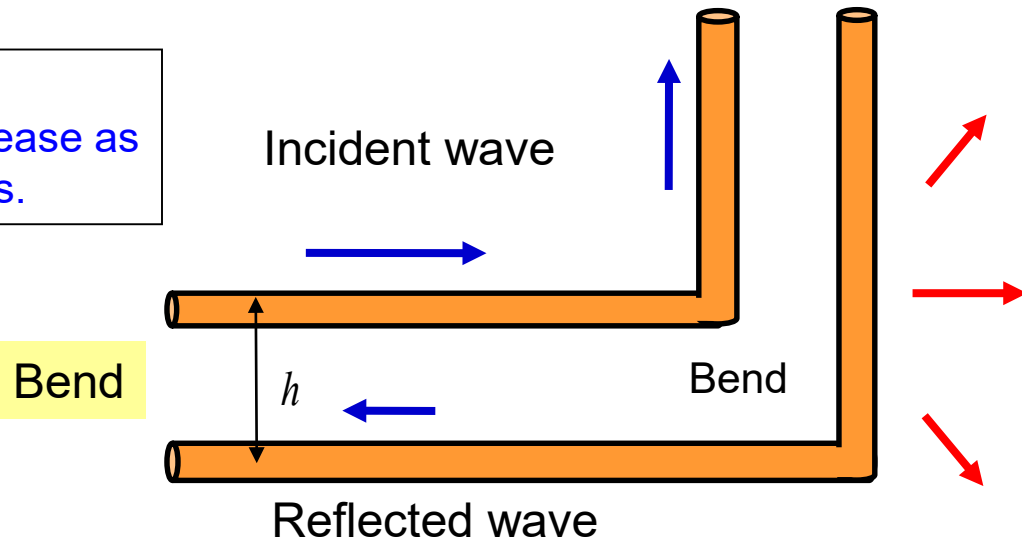
The incident and reflected waves represent an exact solution to Maxwell's equations on the infinite line, at any frequency.

# Limitations of Transmission-Line Theory (cont.)

A discontinuity on the twin lead will cause radiation to occur.



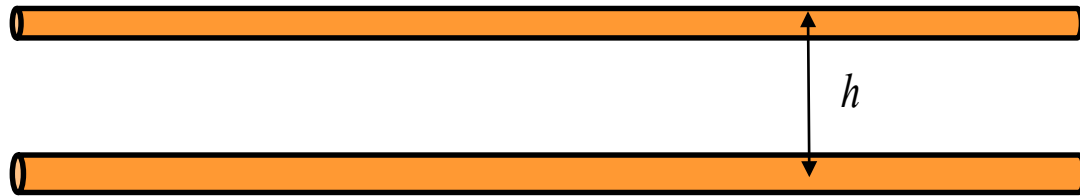
**Note:**  
Radiation effects usually increase as the frequency increases.



# Limitations of Transmission-Line Theory (cont.)

To reduce radiation effects of the twin lead at discontinuities:

- 1) Reduce the separation distance  $h$  (keep  $h \ll \lambda$ ).
- 2) Twist the lines (twisted pair).



CAT 5 cable  
(twisted pair)

