Adapted from notes by
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ECE 5317-6351
Microwave Engineering

Fall 2019

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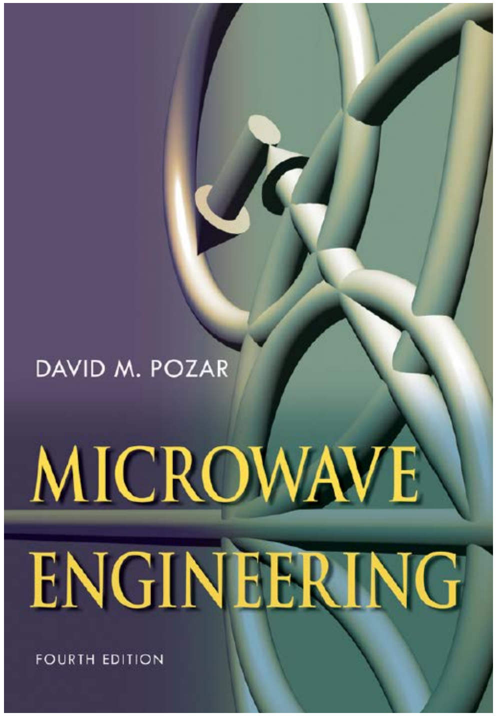
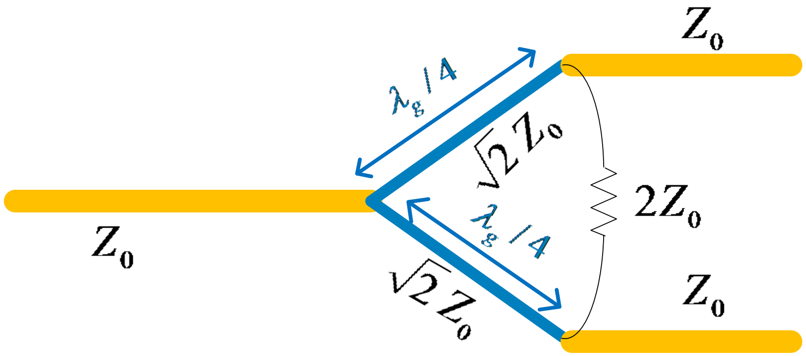


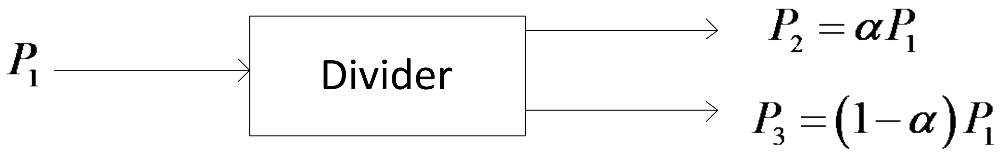
Notes 20

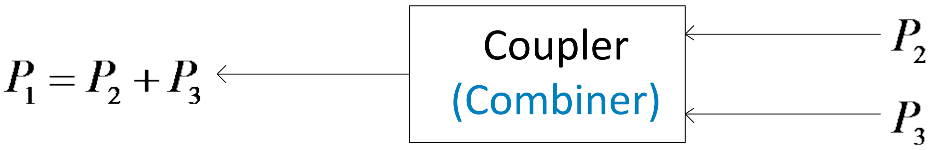
Power Dividers and Circulators



Power Dividers and Couplers

- A power divider is used to split a signal.
- A coupler is used to combine a signal.





These are examples of a <u>three-port</u> network.

Three Port Networks

General 3-port network:

$$[S] = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

Three Port Networks (cont.)

If all three ports are matched, and the device is reciprocal and lossless, we have:

$$[S] = \begin{pmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{pmatrix}$$

(The S matrix is unitary.)

(There are three distinct values.)

Such a device is not physically possible!

(Please see the next slide.)

Conclusion: If we want a match at all ports, we have to have a lossy device.

Power Dividers and Couplers (cont.)

$$[S]=\begin{pmatrix} 0 & S_{12} & S_{13} \\ S_{12} & 0 & S_{23} \\ S_{13} & S_{23} & 0 \end{pmatrix}$$

Lossless, reciprocal, and matched at all ports is not physically possible.

Lossless $\Rightarrow$ $[S]$ is unitary

Hence:

$$\left.\begin{aligned} |S_{12}|^2+|S_{13}|^2&=1 \\ |S_{12}|^2+|S_{23}|^2&=1 \\ |S_{13}|^2+|S_{23}|^2&=1 \end{aligned}\right\}$$

These cannot all be satisfied.

Only one of them can be nonzero.

$$\left.\begin{aligned} S_{13}\,S_{23}^*&=0 \\ S_{12}\,S_{23}^*&=0 \\ S_{12}\,S_{13}^*&=0 \end{aligned}\right\}$$

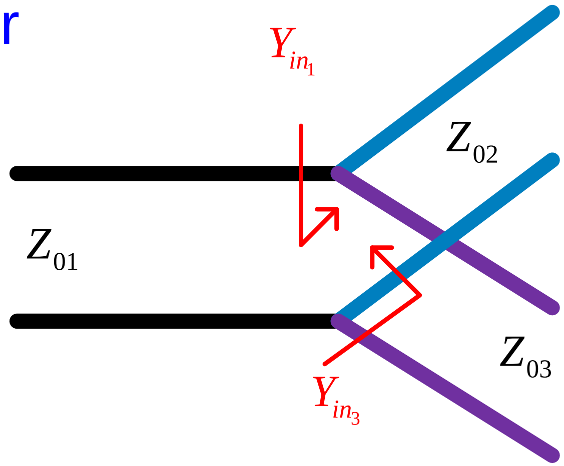
At least 2 of S_{13}, S_{12}, S_{23} must be zero.

(If only one is zero (or none is zero), we cannot satisfy all three.)

Power Dividers

T-Junction: A lossless divider

$$Y_{\text{in}_1} = \frac{1}{Z_{02}} + \frac{1}{Z_{03}}$$



To match: $Z_{01} = Z_{\text{in}_1} = \frac{1}{Y_{\text{in}_1}} = Z_{02} \parallel Z_{03} = \frac{Z_{02}Z_{03}}{Z_{02}+Z_{03}}$

We then have: $Y_{\text{in}_3} = \frac{1}{Z_{01}} + \frac{1}{Z_{02}} = \frac{Z_{02}+Z_{03}}{Z_{02}Z_{03}} + \frac{1}{Z_{02}} = \frac{Z_{02}+2Z_{03}}{Z_{02}Z_{03}} = \frac{1}{Z_{03}}\left(\frac{Z_{02}+2Z_{03}}{Z_{02}}\right)$

Thus, $Y_{\text{in}_3} \neq \frac{1}{Z_{03}}$ Similarly, $Y_{\text{in}_2} \neq \frac{1}{Z_{02}}$

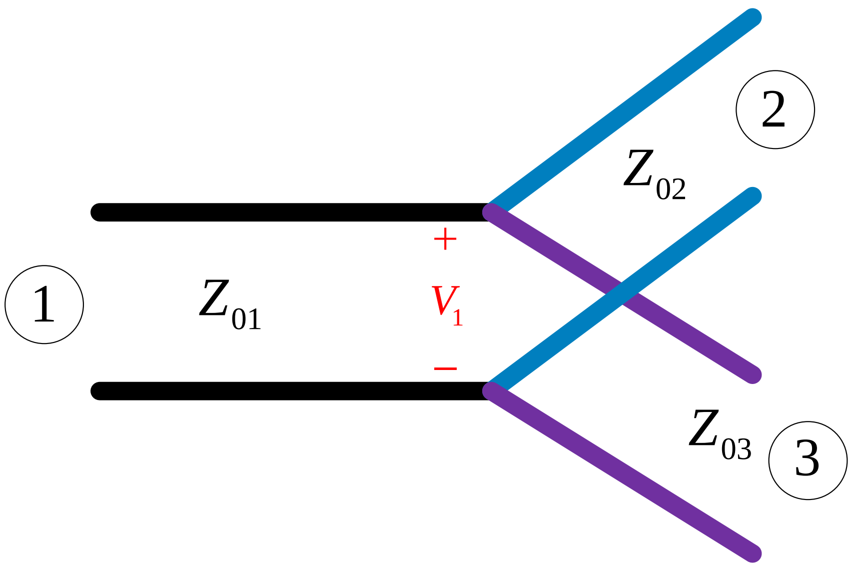
If we match at port 1, we cannot match at the other ports!

Note:
Matching at ports 2 and 3 would be helpful when output lines 2 and 3 are not matched to their loads.

Power Dividers (cont.)

Assuming port 1 matched:

$$Z_{01} = \frac{Z_{02} Z_{03}}{Z_{02} + Z_{03}}$$



$$P_{\text{in}_1} = \frac{1}{2} \frac{|V_1|^2}{Z_{01}}$$

$$P_{\text{out}_2} = \frac{1}{2} \frac{|V_1|^2}{Z_{02}} = \frac{Z_{01}}{Z_{02}} P_{\text{in}_1}$$

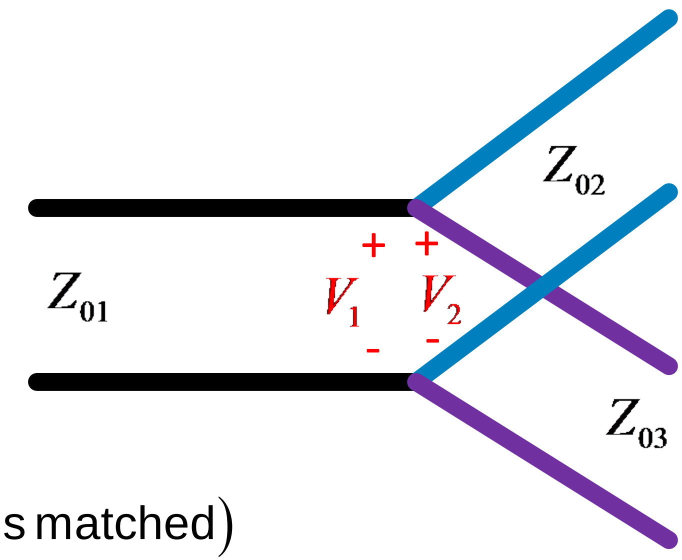
$$P_{\text{out}_3} = \frac{1}{2} \frac{|V_1|^2}{Z_{03}} = \frac{Z_{01}}{Z_{03}} P_{\text{in}_1}$$

$$\frac{P_{\text{out2}}}{P_{\text{out3}}} = \frac{Z_{03}}{Z_{02}}$$

We can design the splitter to control the powers going into the two output lines.

Power Dividers (cont.)

Examine the reflection at each port (S_{ii}):



$$S_{11} = \left.\frac{V_1^- / \sqrt{Z_{01}}}{V_1^+ / \sqrt{Z_{01}}}\right|_{a_2=a_3=0} = \left.\frac{V_1^-}{V_1^+}\right|_{a_2=a_3=0}$$

$$= \frac{Z_{in1} - Z_{01}}{Z_{in1} + Z_{01}} = \frac{Z_{02} \| Z_{03} - Z_{01}}{Z_{02} \| Z_{03} + Z_{01}} \quad \left(\text{zero if port 1 is matched}\right)$$

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{a_1=a_3=0} = \frac{Z_{01} \| Z_{03} - Z_{02}}{Z_{01} \| Z_{03} + Z_{02}}$$

$$S_{33} = \left.\frac{V_3^-}{V_3^+}\right|_{a_1=a_2=0} = \frac{Z_{01} \| Z_{02} - Z_{03}}{Z_{01} \| Z_{02} + Z_{03}}$$

Note: A match on port 1 requires:

$$Z_{01} < Z_{02}, \quad Z_{01} < Z_{03}$$

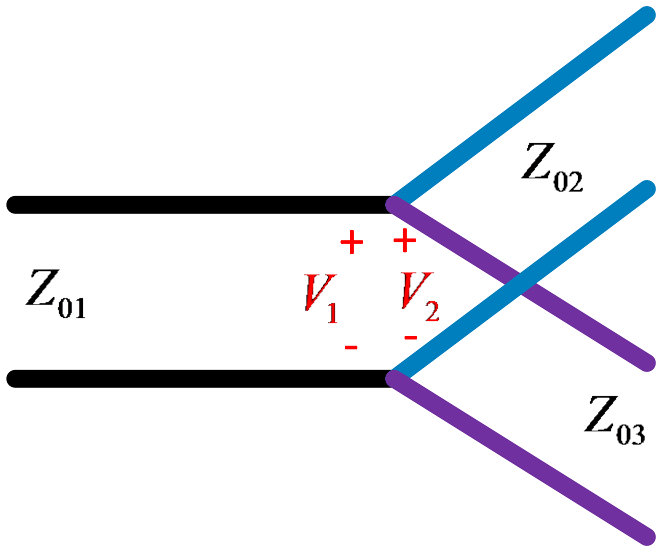
(The two output lines combine in parallel.)

The first term in the numerators must be less than the second term. $\Rightarrow S_{22} \neq 0, \quad S_{33} \neq 0$

Power Dividers (cont.)

Also, we have:

$$S_{21} = \left. \frac{\dfrac{V_2^-}{\sqrt{Z_{02}}}}{\dfrac{V_1^+}{\sqrt{Z_{01}}}} \right|_{a_2 = a_3 = 0} = \frac{V_2^-}{V_1^+} \sqrt{\frac{Z_{01}}{Z_{02}}}$$



$$V_2^- / V_1^+ = V_2 / V_1^+ = V_1 / V_1^+$$

Also

$$V_1 = V_1^+ \left(1 + S_{11}\right) \Rightarrow V_1 / V_1^+ = \left(1 + S_{11}\right)$$

Hence

$$S_{21} = \left(1 + S_{11}\right) \sqrt{\frac{Z_{01}}{Z_{02}}} = S_{12}$$

Similarly:

$$S_{31} = \left(1 + S_{11}\right) \sqrt{\frac{Z_{01}}{Z_{03}}} = S_{13}$$

$$S_{32} = S_{23} = \left(1 + S_{22}\right) \sqrt{\frac{Z_{02}}{Z_{03}}}$$

Power Dividers (cont.)

If port 1 is matched: $$Z_{01} = \frac{Z_{02} Z_{03}}{Z_{02} + Z_{03}}$$

$$\Rightarrow S_{11} = 0 \;;\quad S_{22} = \frac{Z_{01} \| Z_{03} - Z_{02}}{Z_{01} \| Z_{03} + Z_{02}} \;;\quad S_{33} = \frac{Z_{01} \| Z_{02} - Z_{03}}{Z_{01} \| Z_{02} + Z_{03}}$$

$$S_{21} = S_{12} = \left(1 + S_{11}\right)\sqrt{\frac{Z_{01}}{Z_{02}}} = \sqrt{\frac{Z_{01}}{Z_{02}}} = \sqrt{\frac{Z_{03}}{Z_{02} + Z_{03}}}$$

$$S_{31} = S_{13} = \left(1 + S_{11}\right)\sqrt{\frac{Z_{01}}{Z_{03}}} = \sqrt{\frac{Z_{01}}{Z_{03}}} = \sqrt{\frac{Z_{02}}{Z_{02} + Z_{03}}}$$

$$[S] = \begin{pmatrix} 0 & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix}$$

Only $S_{11} = 0$

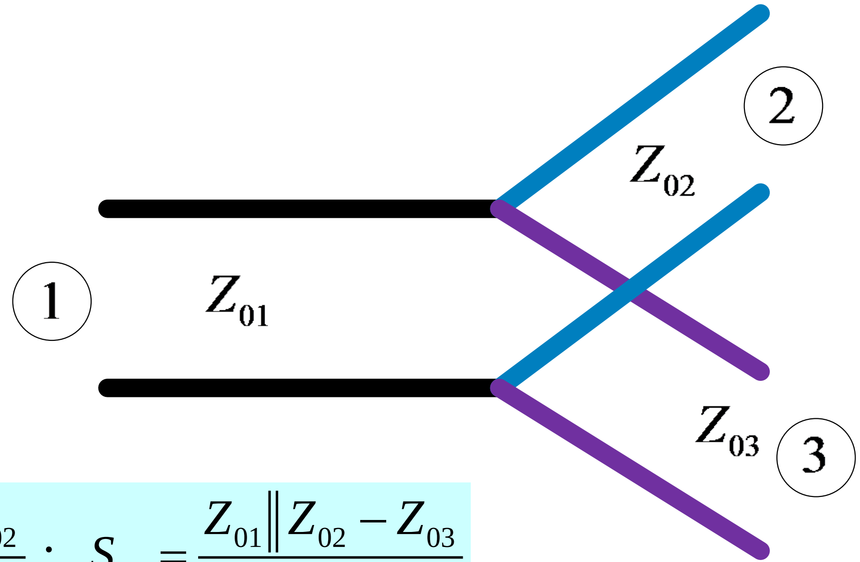
From last slide: $$S_{32} = S_{23} = \left(1 + S_{22}\right)\sqrt{\frac{Z_{02}}{Z_{03}}}$$

The output ports 2 and 3 are <u>not</u> isolated.

Power Dividers (cont.)

Summary

$$Z_{01} = \frac{Z_{02} Z_{03}}{Z_{02} + Z_{03}}$$



$$\frac{P_{\text{out}3}}{P_{\text{out}2}} = \frac{Z_{02}}{Z_{03}}$$

$$S_{11} = 0 \;;\quad S_{22} = \frac{Z_{01} \| Z_{03} - Z_{02}}{Z_{01} \| Z_{03} + Z_{02}} \;;\quad S_{33} = \frac{Z_{01} \| Z_{02} - Z_{03}}{Z_{01} \| Z_{02} + Z_{03}}$$

$$S_{21} = S_{12} = \sqrt{\frac{Z_{03}}{Z_{02} + Z_{03}}}$$

$$S_{31} = S_{13} = \sqrt{\frac{Z_{02}}{Z_{02} + Z_{03}}}$$

$$S_{32} = S_{23} = \left(1 + S_{22}\right) \sqrt{\frac{Z_{02}}{Z_{03}}}$$

- The input port is matched, but not the output ports.
- The output ports are not isolated.

Waves reflected from devices on ports 2 and 3 with cause <u>interference</u> with the other devices.

Power Dividers (cont.)

Example: Microstrip T-junction power divider

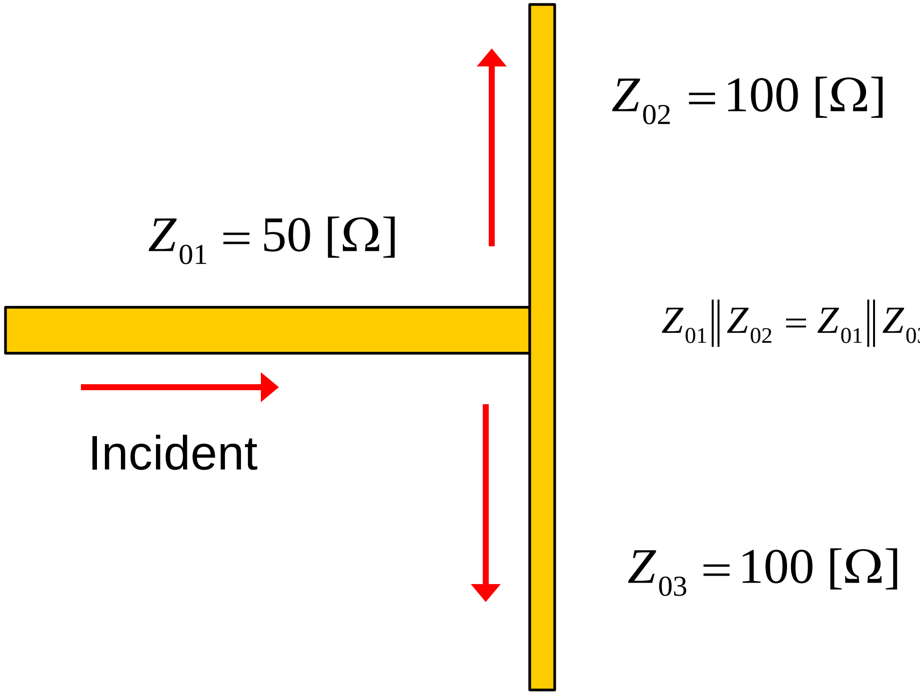
$$S_{11} = 0$$

$$S_{22} = S_{33} = -\frac{1}{2}$$

$$S_{21} = S_{12} = \sqrt{\frac{1}{2}}$$

$$S_{31} = S_{13} = \sqrt{\frac{1}{2}}$$

$$S_{32} = S_{23} = \frac{1}{2}$$

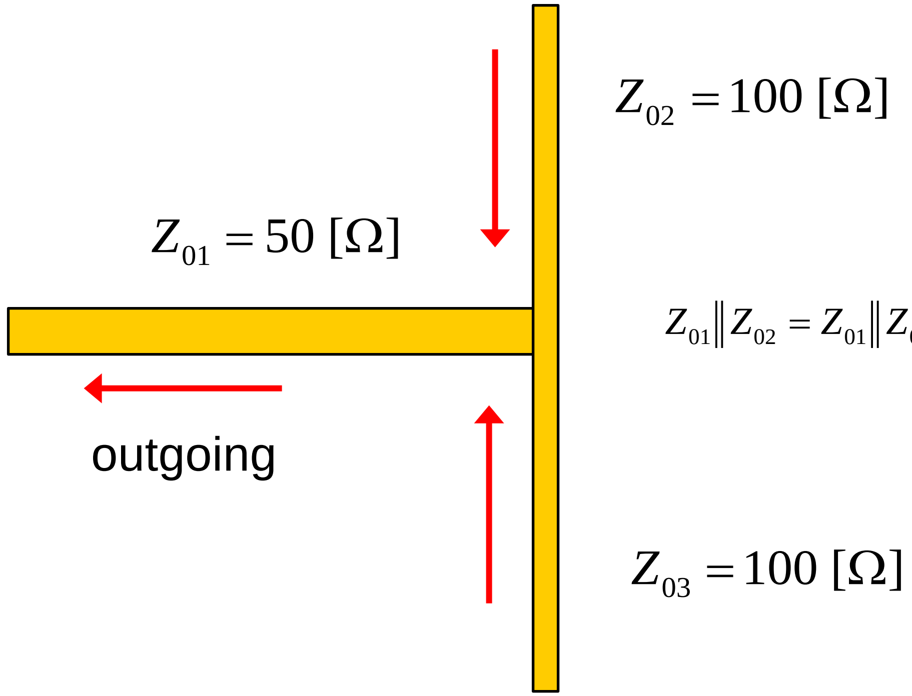


$$Z_{01} \| Z_{02} = Z_{01} \| Z_{03} = 50 \| 100 = 33.333\ [\Omega]$$

Note:
Quarter-wave transformers could be put on the output lines to bring the final output lines back to 50 [Ω].

Power Dividers (cont.)

The matched power divider also works as a matched power combiner.



$$Z_{01} \| Z_{02} = Z_{01} \| Z_{03} = 50 \| 100 = 33.333\ [\Omega]$$

$$S_{11} = 0$$

$$S_{22} = S_{33} = -\frac{1}{2}$$

$$S_{21} = S_{12} = \sqrt{\frac{1}{2}}$$

$$S_{31} = S_{13} = \sqrt{\frac{1}{2}}$$

$$S_{32} = S_{23} = \frac{1}{2}$$

$$\begin{aligned} b_3 &= \cancel{a_1} S_{31} + a_2 S_{32} + a_3 S_{33} \\ &= a_3 \left(S_{33} + S_{32} \right) \\ &= a_3 \left(-\frac{1}{2} + \frac{1}{2} \right) \\ &= 0 \end{aligned}$$

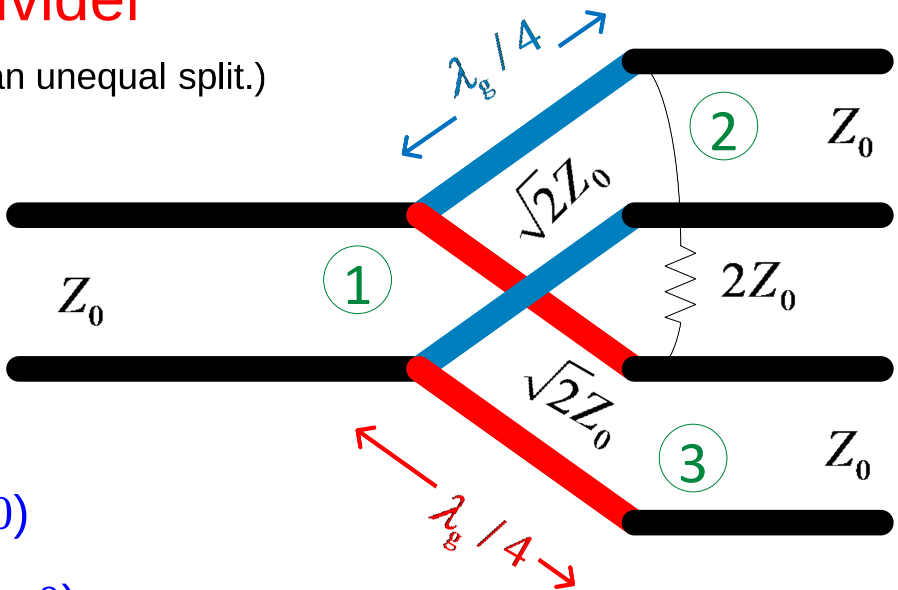
Equal waves are incident from ports 2 and 3 ($a_2 = a_3$).

There is no reflection if <u>equal</u> waves are incident on ports 2 and 3, and port 1 is matched.

Wilkinson Power Divider

Equal-split (3 dB) power divider

(The Wilkenson can also be designed to have an unequal split.)



- All ports matched ($S_{11} = S_{22} = S_{33} = 0$)
- Output ports are isolated ($S_{23} = S_{32} = 0$)

Note: No power is lost in going from port 1 to ports 2 and 3:

$$\left|S_{21}\right|^2 = \left|S_{31}\right|^2 = \frac{1}{2}$$

$$[S] = \frac{-j}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

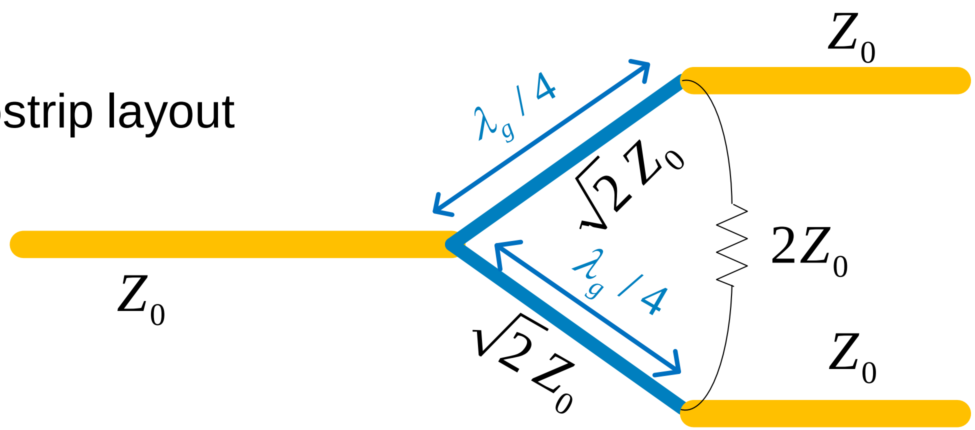
Obviously not unitary

The derivation is in the appendix.

Wilkinson Power Divider (cont.)

$$[S] = \frac{-j}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Microstrip layout



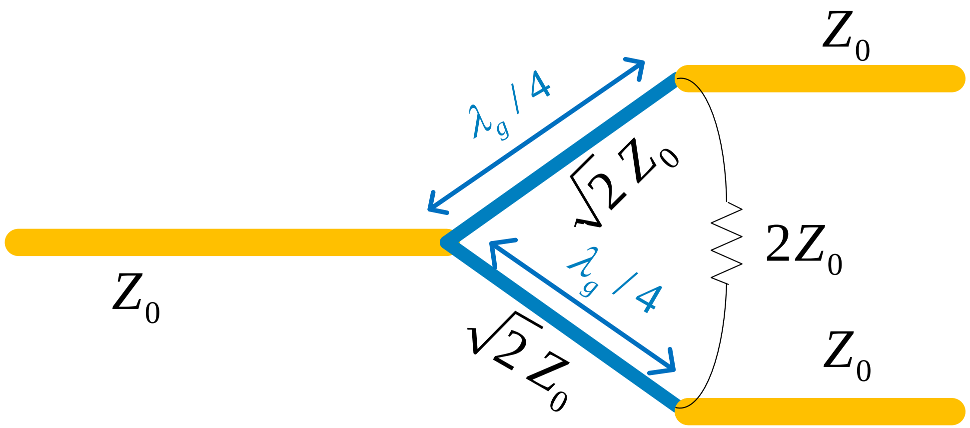
$$S_{11} = S_{22} = S_{33} = 0$$

$$S_{32} = S_{23} = 0$$

All three ports are matched, and the output ports are isolated.

Wilkinson Power Divider (cont.)

$$[S] = \frac{-j}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



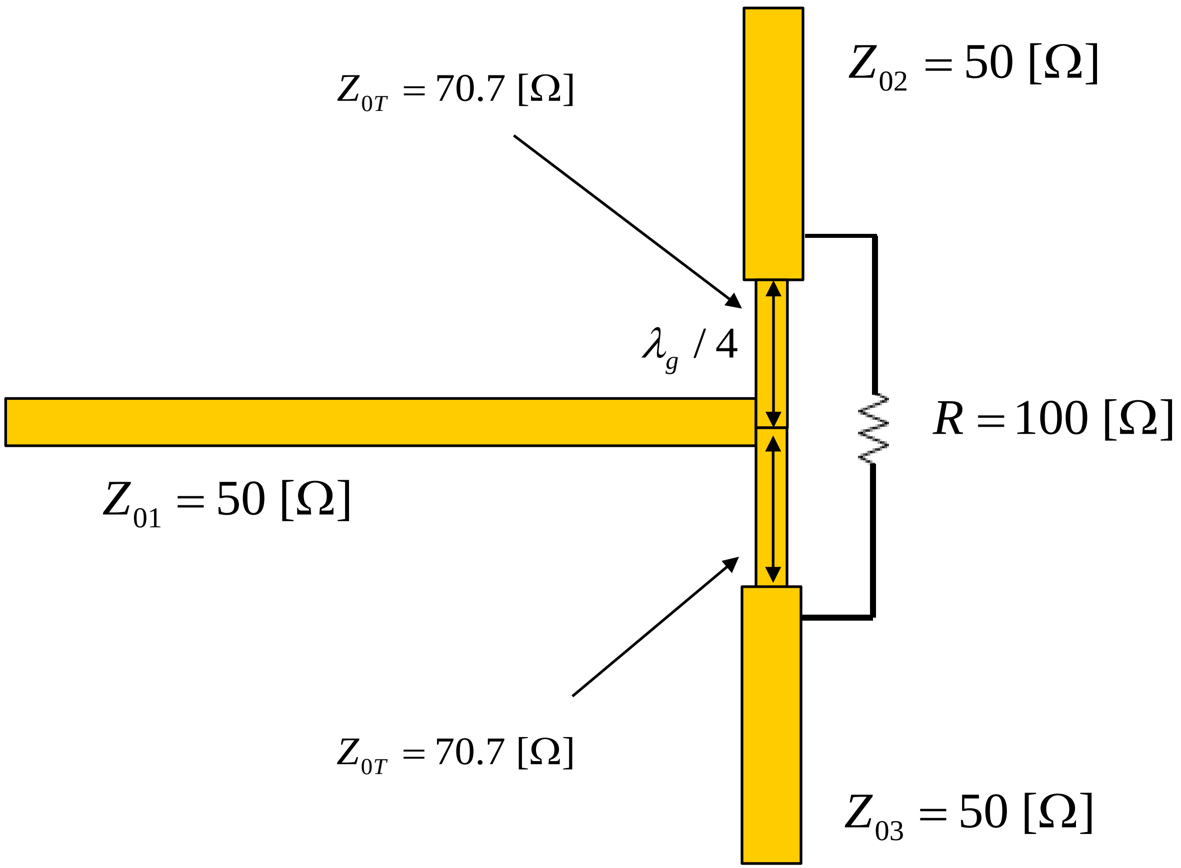
$$S_{21} = S_{31} = \frac{-j}{\sqrt{2}}$$

$$S_{12} = S_{13} = \frac{-j}{\sqrt{2}}$$

- When a wave is incident from port 1, half of the total incident power gets transmitted to each output port (no loss of power).
- When a wave is incident from port 2 or port 3, half of the power gets transmitted to port 1 and half gets absorbed by the resistor, but nothing gets through to the other output port (the two output ports are isolated from each other).

Wilkinson Power Divider (cont.)

Example: Microstrip Wilkinson power divider

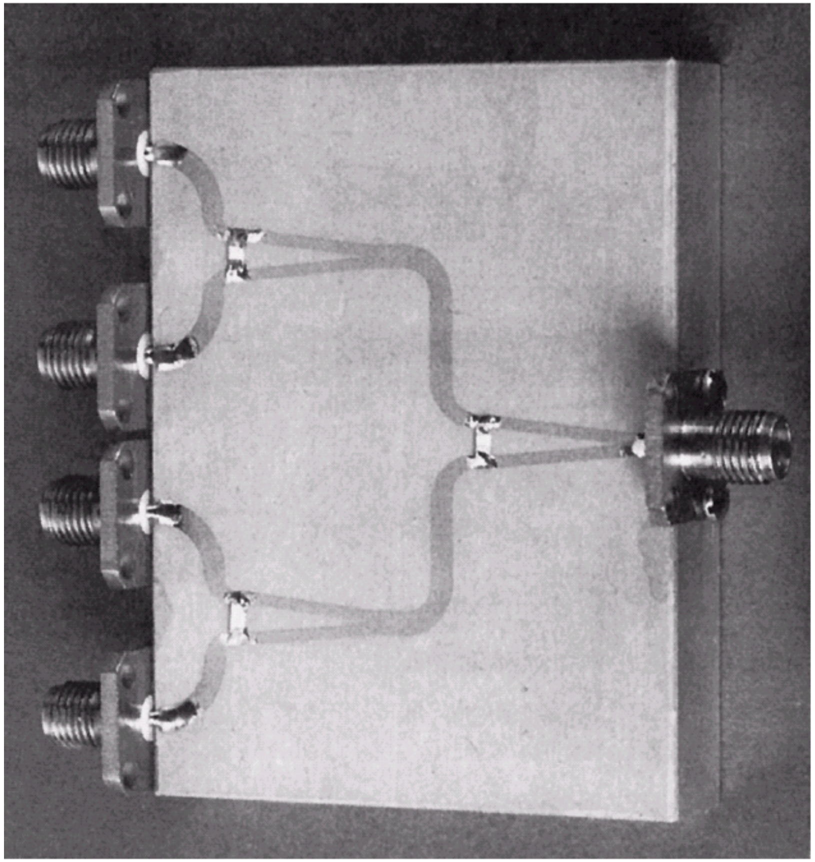


Wilkinson Power Divider (cont.)

Figure 7.15 of Pozar

Photograph of a four-way corporate power divider network using three microstrip Wilkinson power dividers. Note the isolation chip resistors.

Courtesy of M.D. Abouzahra, MIT Lincoln Laboratory.



Wilkinson Power Divider (cont.)

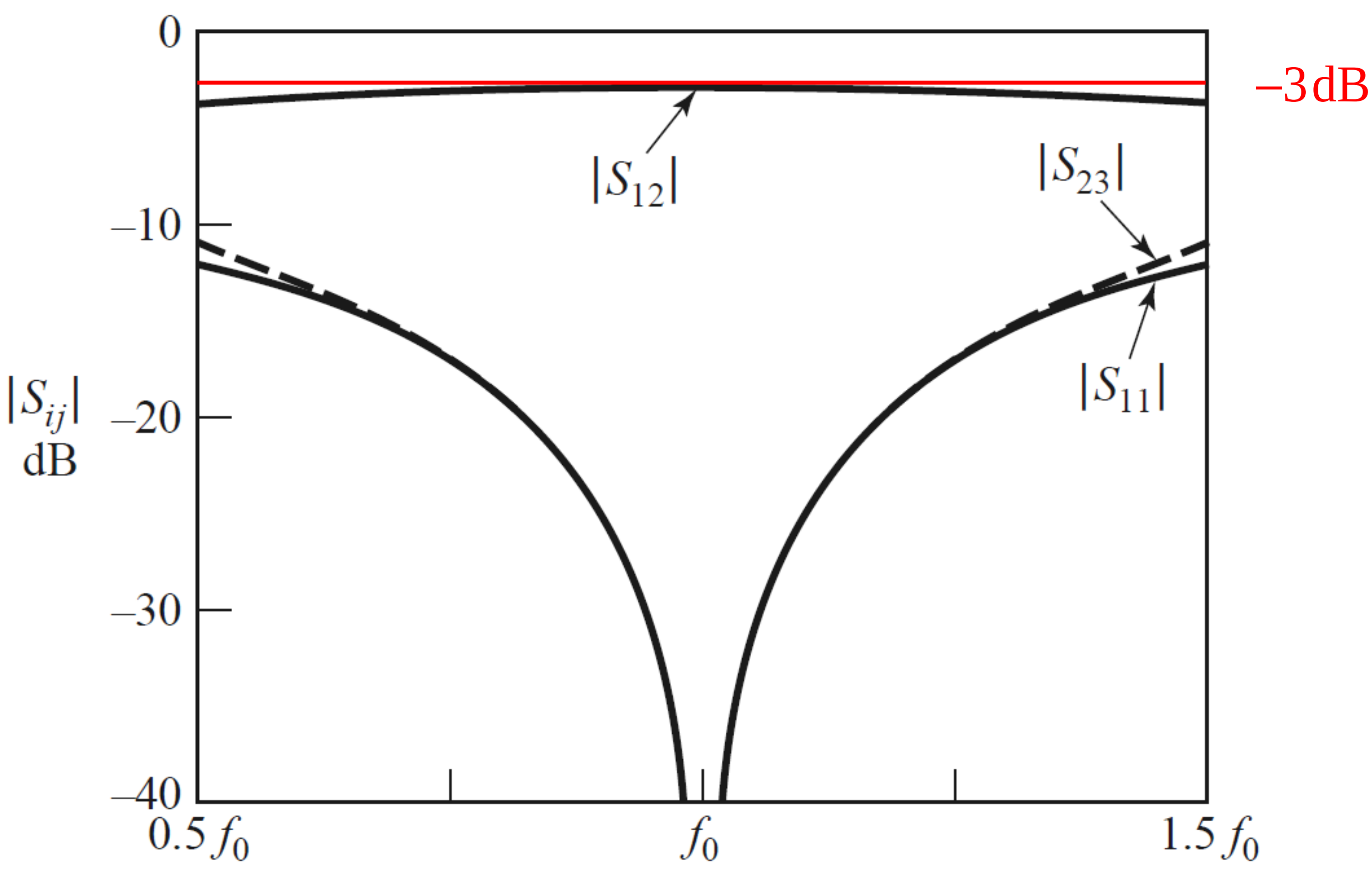


Figure 7.12 of Pozar

Frequency response of an equal-split Wilkinson power divider. Port 1 is the input port; ports 2 and 3 are the output ports.

Circulators

Now consider a 3-port network that is non-reciprocal, with all ports matched, and is lossless:

$$\Rightarrow [S] = \begin{pmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{pmatrix}$$

$$S_{ij} \neq S_{ji}$$

"Circulator"

(There are six distinct values.)

These equations will be satisfied if:

Lossless

(unitary)

$$\Rightarrow \left\{ \begin{aligned} &|S_{21}|^2 + |S_{31}|^2 = 1 \\ &|S_{12}|^2 + |S_{32}|^2 = 1 \\ &|S_{13}|^2 + |S_{23}|^2 = 1 \\ &S_{31}\,S_{32}^* = 0 \\ &S_{21}\,S_{23}^* = 0 \\ &S_{12}\,S_{13}^* = 0 \end{aligned} \right.$$

1

$$S_{12} = S_{23} = S_{31} = 0$$

$$|S_{21}| = |S_{32}| = |S_{13}| = 1$$

or

2

$$S_{21} = S_{32} = S_{13} = 0$$

$$|S_{12}| = |S_{23}| = |S_{31}| = 1$$

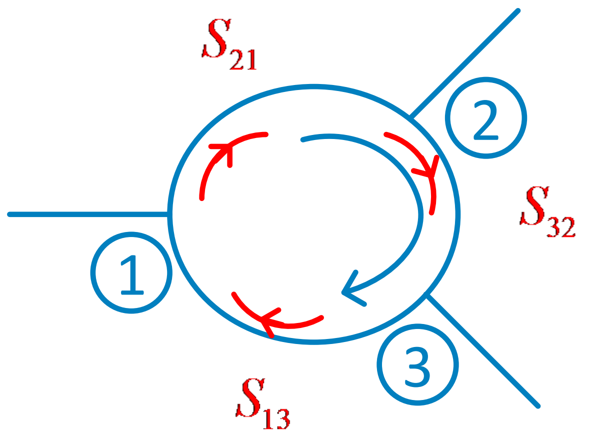
Circulators (cont.)

1

$$[S] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

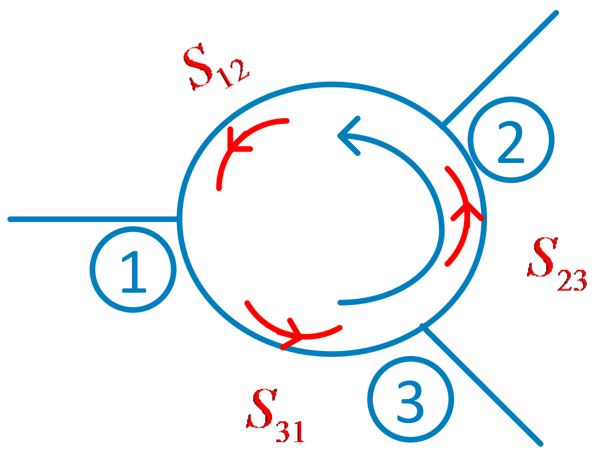
Note: We have assumed here that the phases of all the S parameters are zero.

Clockwise (LH) circulator



2

$$[S] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



Counter-clockwise (RH) circulator

A wave goes in one port and comes out from the adjacent port!

Circulators (cont.)

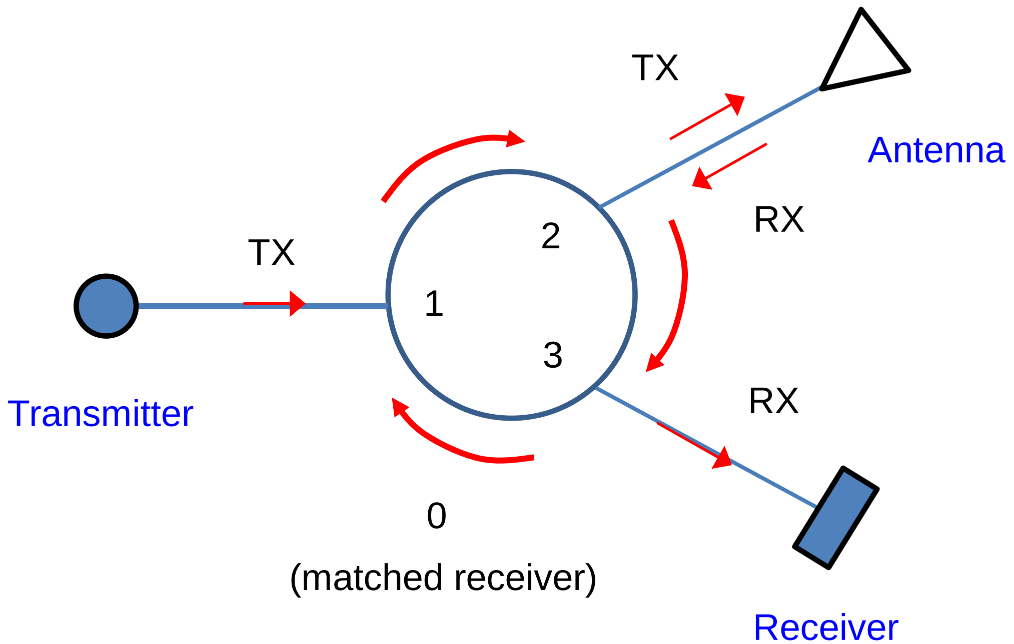
An example of a circulator:

Frequency: 0.698 – 0.96 GHz
VSWR: <1.3
Isolation : >18 dB
Power: 1000 W
Insertion loss: <0.35 dB
Length: 1.75 in
Width: 2 in
Height: 1.02 in
Temperature range: -20 – 70 deg C
Price: $439.94

Circulators can be made using biased ferrite materials.

Circulators (cont.)

Application: Wireless system



❖ The same antenna can be used for transmit and receive.

❖ The transmit and receive frequencies can even be the same.

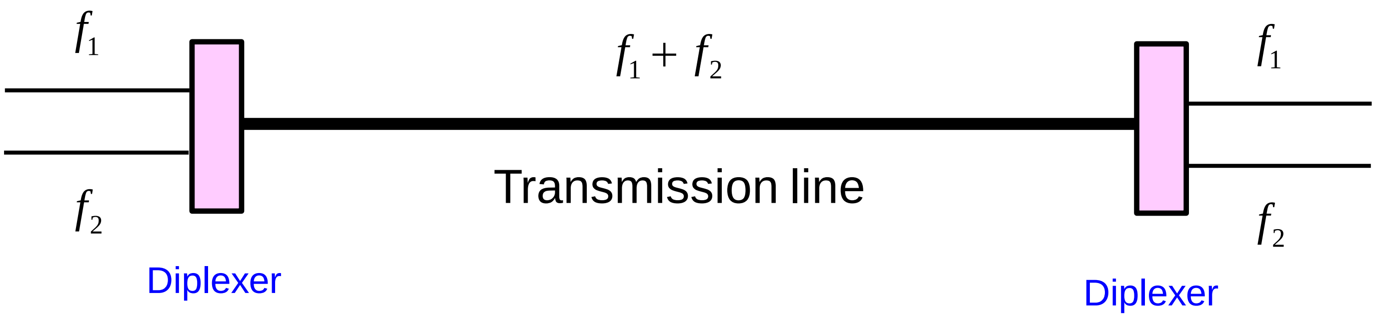
Note: A circulator used this way is often called a duplexer.

Duplexer

A duplexer:

Diplexer

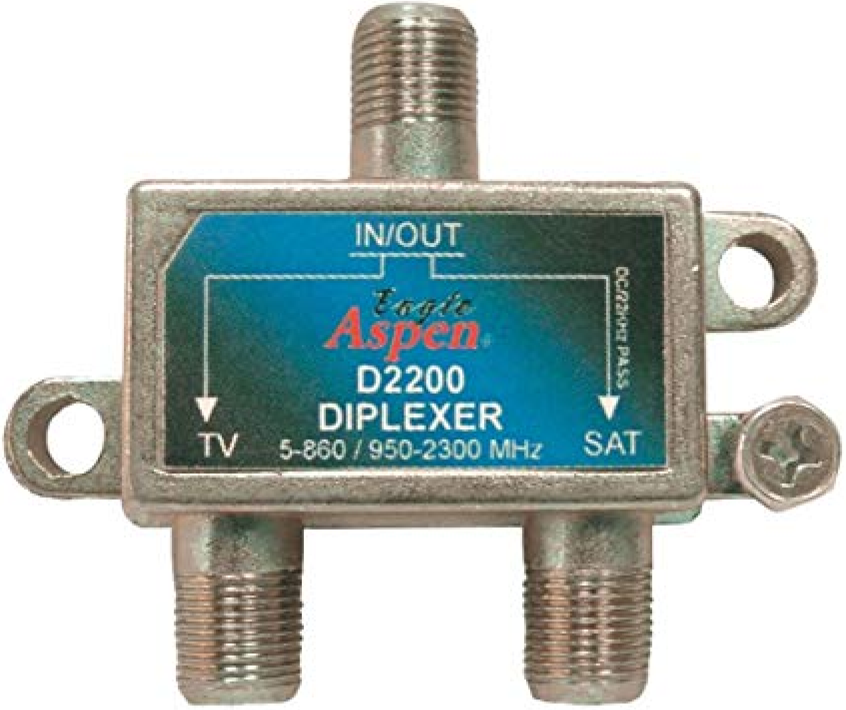
A diplexer) is a type of filter that combines or splits two different frequencies (f_1 and f_2).



Triplexer: three frequencies
Multiplexer: multiple frequencies

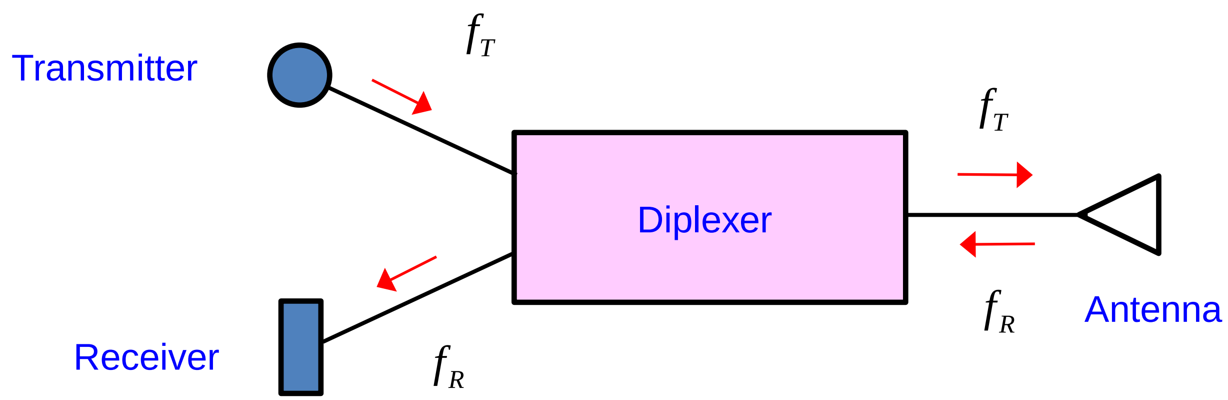
Diplexer (cont.)

An example of a diplexer:



Diplexer (cont.)

Note: A diplexer can be used to transmit and receive two different channels with the same antenna (as with a duplexer), if the frequencies are separated enough.



Note:
This requires a high isolation between the transmit and receive ports, to avoid interference.

Isolator

An isolator is a two-port device that is nonreciprocal:

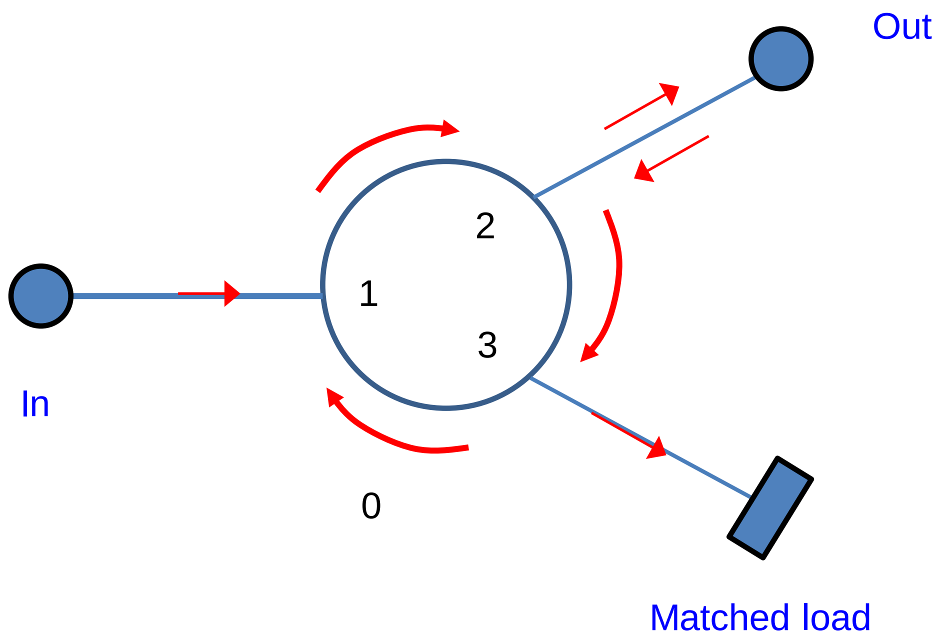
$$[S] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- ❖ A wave coming in on port 1 goes through to port 2.
- ❖ A wave coming in on port 2 gets absorbed (does not go to port 1).

Frequency: 0.8 - 1 GHz
VSWR: 1.25
Isolation : 20 dB
Power: 2 W
Insertion loss: <0.4 dB
Length: 1.25 in
Width: 2 in
Height: 0.75 in
Temperature range: -20 – 60 deg C
Price: $279.82

Circulator as Isolator

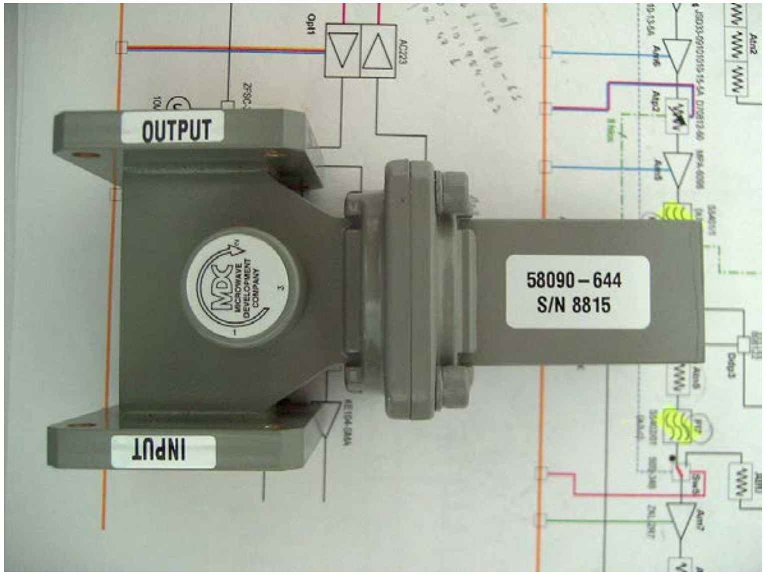
A circulator can also be used to make an isolator:



- A signal from the input port goes to the output port.
- A signal from the output port does not get to the input port.

Circulator as Isolator (cont.)

A waveguide-based circulator with a matched load at port 3, acting as an isolator



https://en.wikipedia.org/wiki/Circulator

Appendix

The analysis of the Wilkinson power divider is given here.

- ❖ Even/odd mode analysis is used to analyze the Wilkinson power divider.
- ❖ This requires two separate analyzes (even and odd).
- ❖ Each analysis involves only a <u>two-port</u> device instead of a three-port device.

Appendix (cont.)

- Even and odd analysis is used to analyze the structure when port 2 is excited.

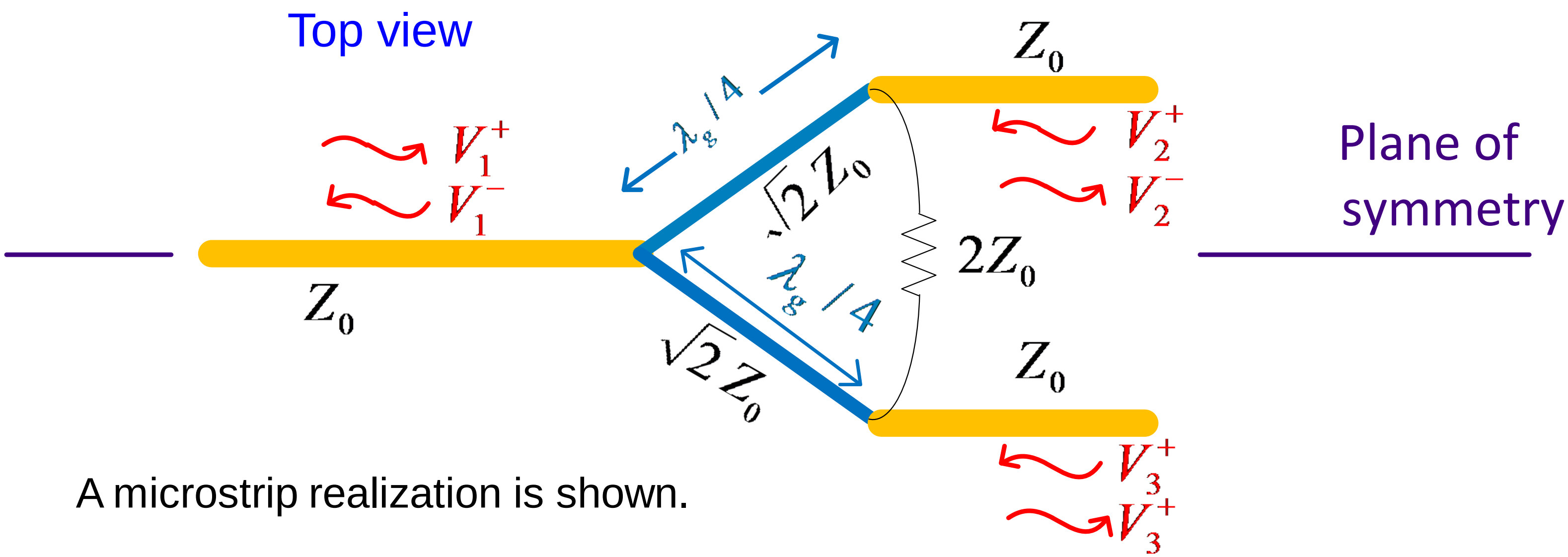
  $\Rightarrow$ To determine S_{22}, S_{32}

- Only even analysis is needed to analyze the structure when port 1 is excited.

  $\Rightarrow$ To determine S_{11}, S_{21}

The other five components of the S matrix can be found by using physical symmetry and reciprocity (the symmetry of the S matrix).

Appendix (cont.)



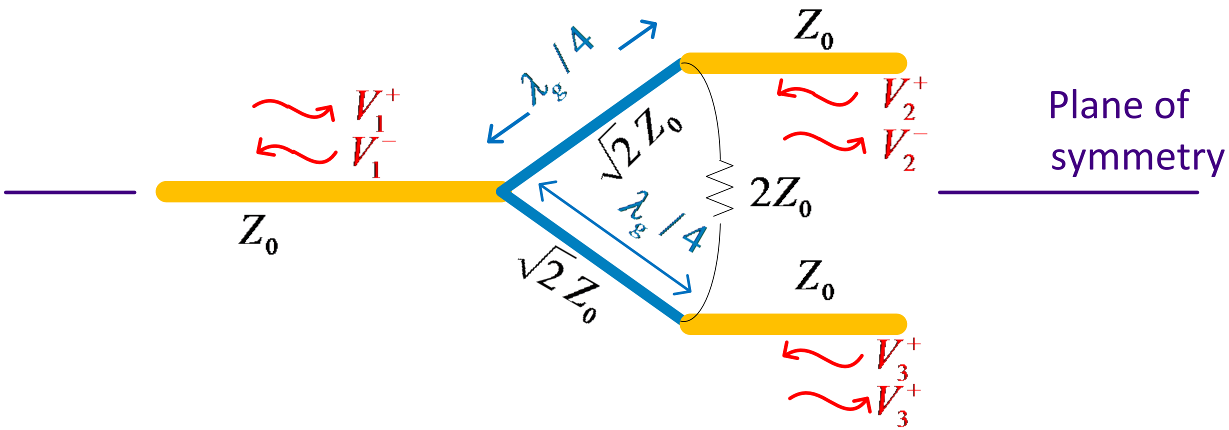
A microstrip realization is shown.

Split structure along plane of symmetry (POS)

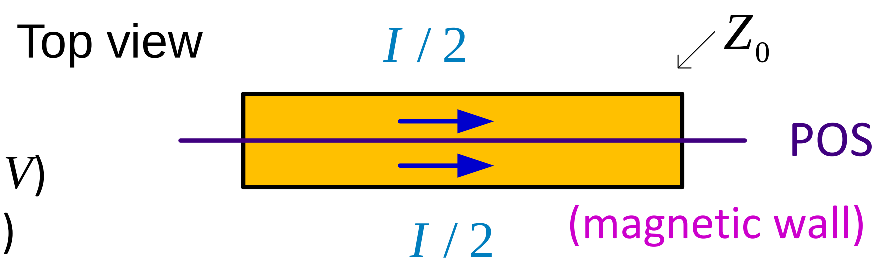
Even $\Rightarrow$ voltage even about POS $\Rightarrow$ place OC along POS

Odd $\Rightarrow$ voltage odd about POS $\Rightarrow$ place SC along POS

Appendix (cont.)



How do you split a transmission line? (This is needed for the even case.)



Z_0 microstrip line

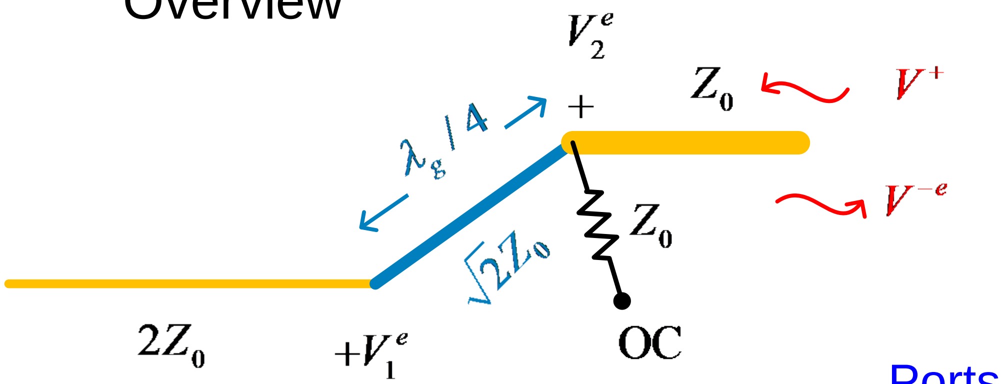
Voltage is the same for each half of line (V)
Current is halved for each half of line ($I/2$)

$$\Rightarrow Z_0^h = \frac{V}{I/2} = 2Z_0$$

For each half

Appendix (cont.)

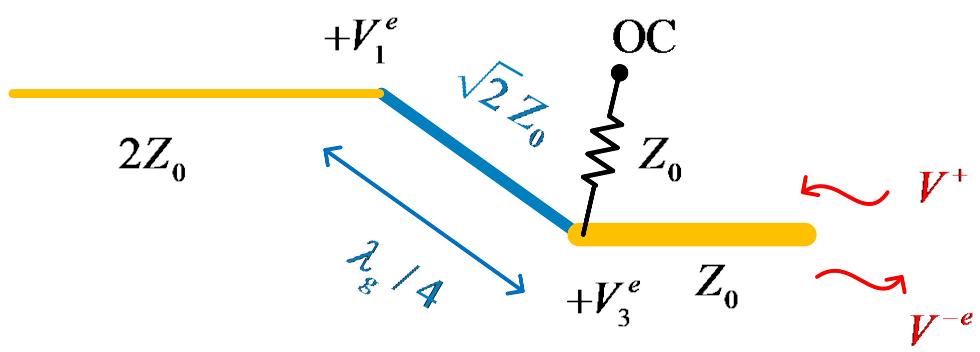
Port 2 Excitation
"even" problem
Overview



Note:
The $2Z_0$ resistor has been split into two Z_0 resistors in series.

Ports 2 and 3 are excited in phase.

$$\Rightarrow S_{22}^e,\ S_{32}^e$$



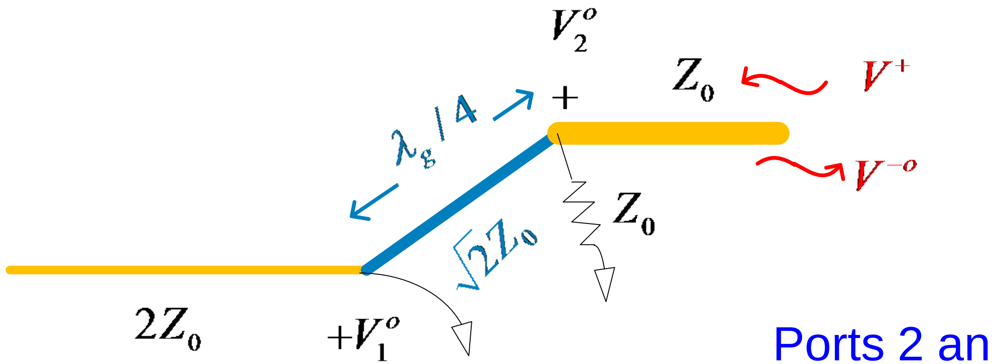
$$S_{22}^e \equiv \frac{V^{-e}}{V^+}$$

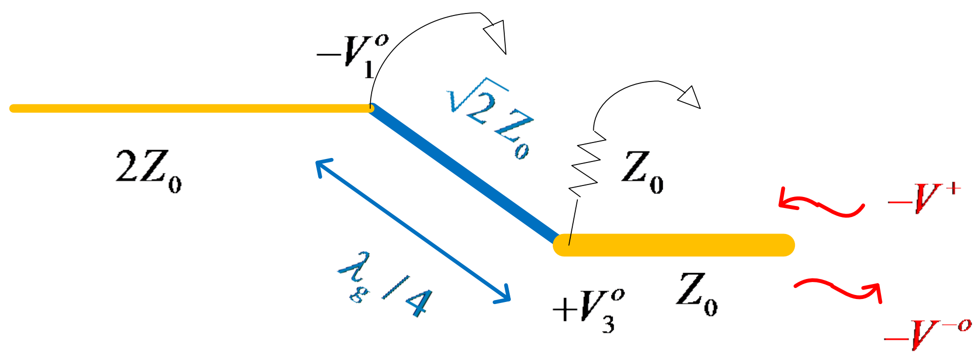
$$S_{32}^e \equiv \frac{V^{-e}}{V^+} = S_{22}^e$$

Note : $V_3^e = V_2^e$

Appendix (cont.)

Port 2 Excitation
"odd" problem
Overview





Note:
The $2Z_0$ resistor has been split into two Z_0 resistors in series.

Ports 2 and 3 are excited 180º out of phase.

$$\Rightarrow S_{22}^o,\ S_{32}^o$$

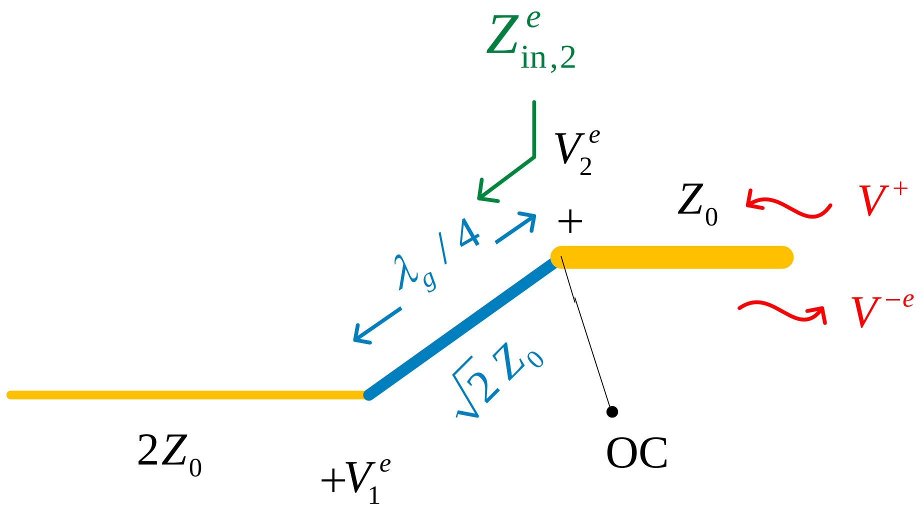
$$S_{22}^o \equiv \frac{V^{-o}}{V^+}$$

$$S_{32}^o \equiv \frac{-V^{-o}}{V^+} = -S_{22}^o$$

Note : $V_1^o = 0, \quad V_3^o = -V_2^o$

Appendix (cont.)

Port 2 Excitation
"even" problem
Analysis



$$Z^e_{\text{in}\,2} = \frac{\left(\sqrt{2}\,Z_0\right)^2}{2Z_0} = Z_0$$

$$\Rightarrow S^e_{22} = \frac{Z^e_{\text{in}\,2} - Z_0}{Z^e_{\text{in}\,2} + Z_0} = 0$$

Recall:

$$Z_{in} = \frac{Z_T^2}{Z_L}$$

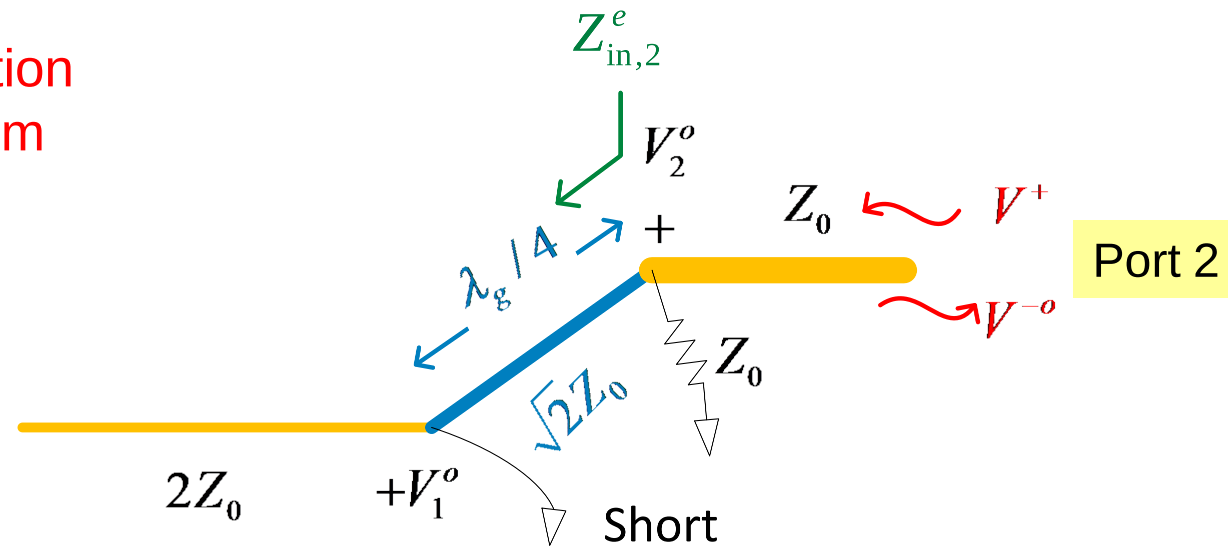
(quarter-wave transformer)

Also, by physical symmetry: $S^e_{33} = S^e_{22} = 0$

Also, in the even case: $S^e_{32} = S^e_{22} \quad \Rightarrow \quad S^e_{32} = 0$

Appendix (cont.)

Port 2 Excitation
"odd" problem
Analysis



$$V_1^o = 0$$

$$Z_{\text{in}\,2}^{o} = \infty \,\|\, Z_0 = Z_0$$

$$\Rightarrow S_{22}^{o} = \frac{Z_{\text{in}\,2}^{o} - Z_0}{Z_{\text{in}\,2}^{o} + Z_0} = 0$$

Also, by physical symmetry: $S_{33}^{o} = S_{22}^{o} = 0$

Also, in the odd case: $S_{32}^{o} = -S_{22}^{o} \;\Rightarrow\; S_{32}^{o} = 0$

Appendix (cont.)

We now add the results from the even and odd cases together:

$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{a_1=a_3=0} = \frac{V^{-e}+V^{-o}}{V^{+}+V^{+}} = \frac{V^{-e}+V^{-o}}{2V^{+}} = \frac{1}{2}\left(S_{22}^e + S_{22}^o\right) = \frac{1}{2}(0+0) = 0$$

$$\Rightarrow \quad S_{33} = 0 \quad \text{(by symmetry)}$$

$$S_{32} = \left.\frac{V_3^-}{V_2^+}\right|_{a_1=a_3=0} = \frac{V^{-e}-V^{-o}}{V^{+}+V^{+}} = \frac{V^{-e}-V^{-o}}{2V^{+}} = \frac{1}{2}\left(S_{32}^e - S_{32}^o\right) = \frac{1}{2}(0-0) = 0$$

$$\Rightarrow \quad S_{23} = 0 \quad \text{(by reciprocity)}$$

In summary, for port 2 excitation, we have:

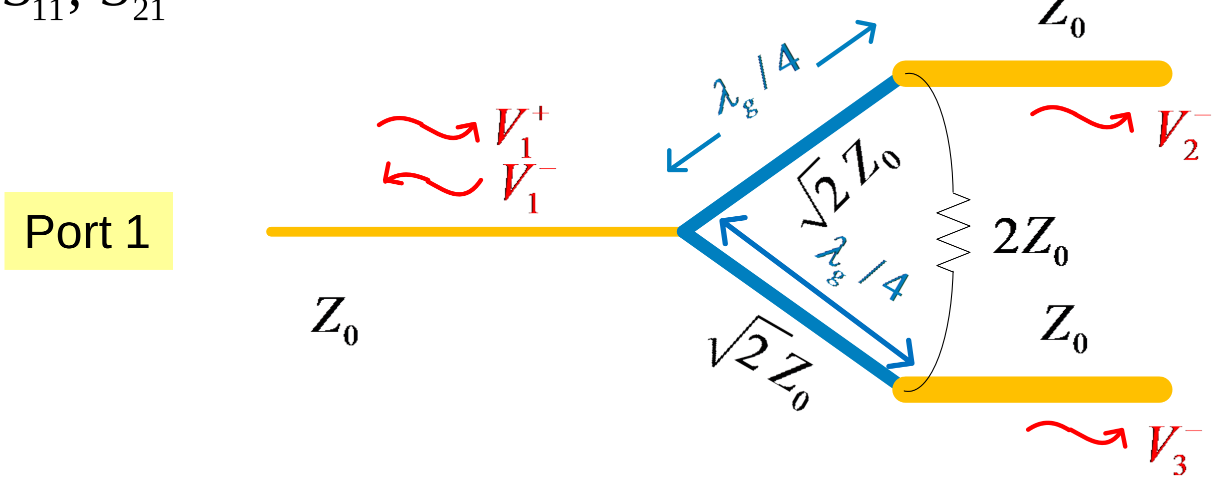
$$S_{22} = 0$$
$$S_{33} = 0$$
$$S_{32} = S_{23} = 0$$

Note:
Since all ports have the same Z_0, we ignore the normalizing factor $\sqrt{Z_0}$ in the S parameter definition.

Appendix (cont.)

Port 1 Excitation
"even" problem
Overview

$$\Rightarrow S_{11}^{e},\ S_{21}^{e}$$

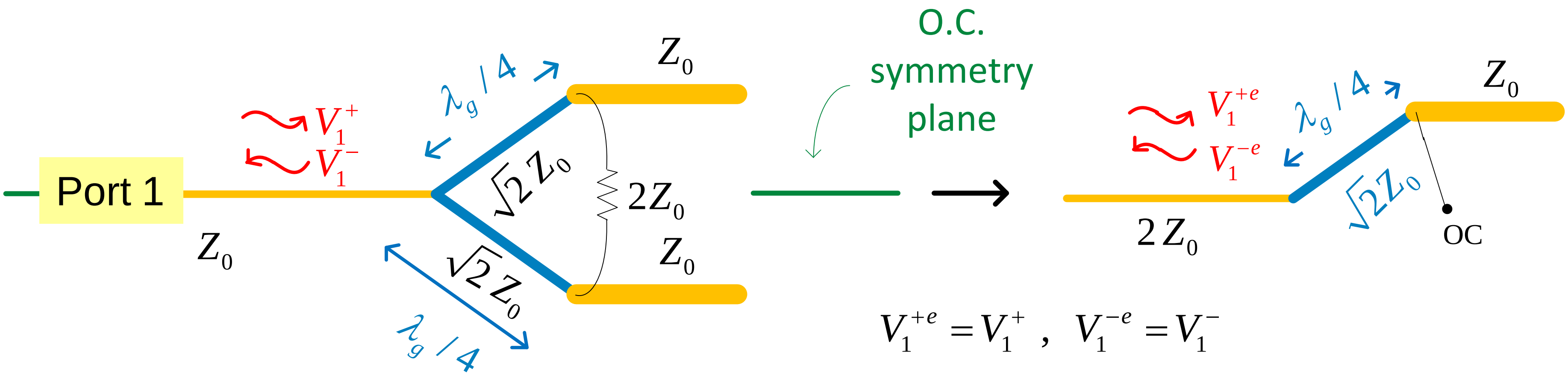


When port 1 is excited, the response, by symmetry, is even.
(Hence, the total voltages are the same as the even voltages.)

Appendix (cont.)

Port 1 Excitation
"even" problem
Analysis

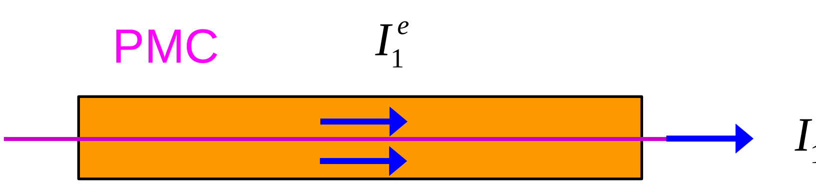
Top view



$$V_1^{+e} = V_1^+ \,,\quad V_1^{-e} = V_1^-$$

$$I_1^e = I_1 / 2$$

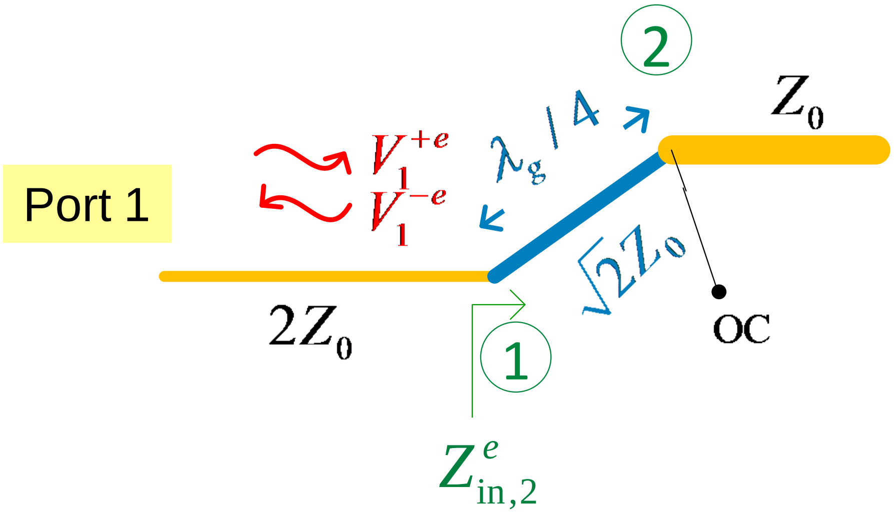
$$V_1^e = V_1$$



Z_0 microstrip line #1

Appendix (cont.)

Port 1 Excitation
"even" problem
Analysis (cont.)



$$Z_{\text{in}1}^e = \frac{\left(\sqrt{2}Z_0\right)^2}{Z_0} = 2Z_0$$

$$S_{11}^e = \frac{Z_{\text{in}1}^e - 2Z_0}{Z_{\text{in}1}^e + 2Z_0} = 0$$

Recall:

$$Z_{in} = \frac{Z_T^2}{Z_L}$$

(quarter-wave transformer)

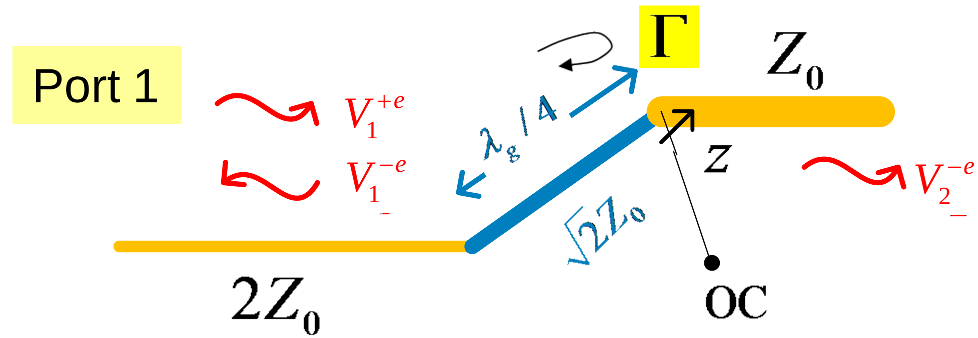
$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{a_2=a_3=0} = \left.\frac{V_1^{-e}}{V_1^{+e}}\right|_{a_2=0} = S_{11}^e = 0$$

Hence

$$\boxed{S_{11} = 0}$$

Appendix (cont.)

Port 1 Excitation
"even" problem
Analysis (cont.)



$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{a_2=a_3=0} = \left.\frac{V_2^{-e}}{V_1^{+e}}\right|_{a_2=a_3=0} = S_{21}^e$$

$$V_2^{-e} = V_2^e$$

$$V_1^e = V_1^{+e}\left(1 + \cancel{S_{11}^e}\right) = V_1^{+e}$$

Along $\lambda_g/4$ wave transformer:

$$V_T^e(z) = V_0^+ e^{-j\beta z}\left(1 + \Gamma e^{+j2\beta z}\right)$$

$z =$ distance from port 2

$$\Rightarrow S_{21}^e = \frac{V_2^{-e}}{V_1^{+e}} = \frac{V_2^e}{V_1^e} = -j\frac{(1+\Gamma)}{(1-\Gamma)} = -j\frac{2}{2\sqrt{2}}$$

$$\begin{cases} V_2^e = V_T^e(0) = V_0^+(1+\Gamma) \\ V_1^e = V_T^e(-\lambda_g/4) = V_0^+ j(1-\Gamma) \end{cases}$$

$$\Rightarrow \quad \boxed{S_{21} = \frac{-j}{\sqrt{2}} = S_{12}}$$

(symmetry of S matrix)

$$\Gamma = \frac{Z_0 - \sqrt{2}Z_0}{Z_0 + \sqrt{2}Z_0} = \frac{1-\sqrt{2}}{1+\sqrt{2}}$$

$$1+\Gamma = \frac{2}{1+\sqrt{2}} \qquad 1-\Gamma = \frac{2\sqrt{2}}{1+\sqrt{2}}$$

Appendix (cont.)

For the other two components:

By physical symmetry: $S_{31} = S_{21} = \dfrac{-j}{\sqrt{2}}$

By reciprocity (symmetry of S matrix): $S_{13} = S_{31} = \dfrac{-j}{\sqrt{2}}$

We then have the final S matrix:

$$[S] = \frac{-j}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$