Adapted from notes by
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ECE 5317-6351
Microwave Engineering

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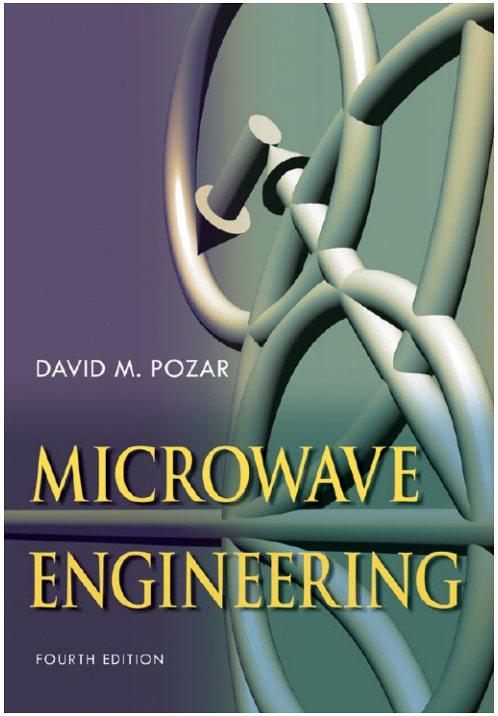
Dept. of ECE

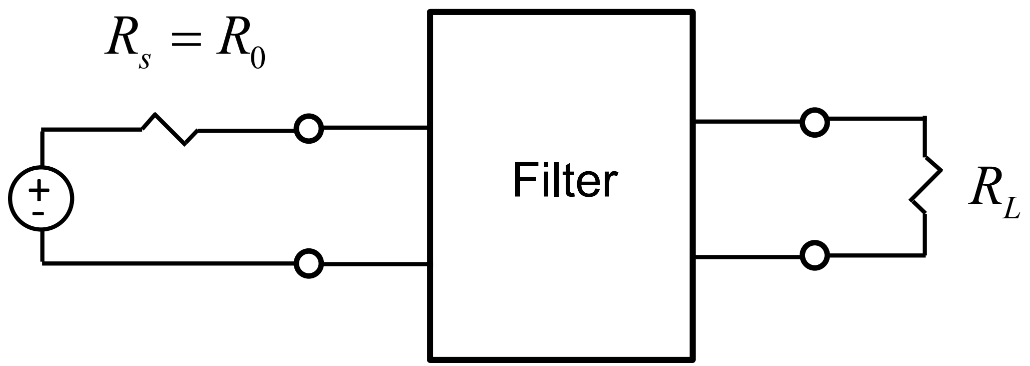


Notes 22

Filter Design Part 1: Insertion Loss Method

Filters

Consider the following circuit:

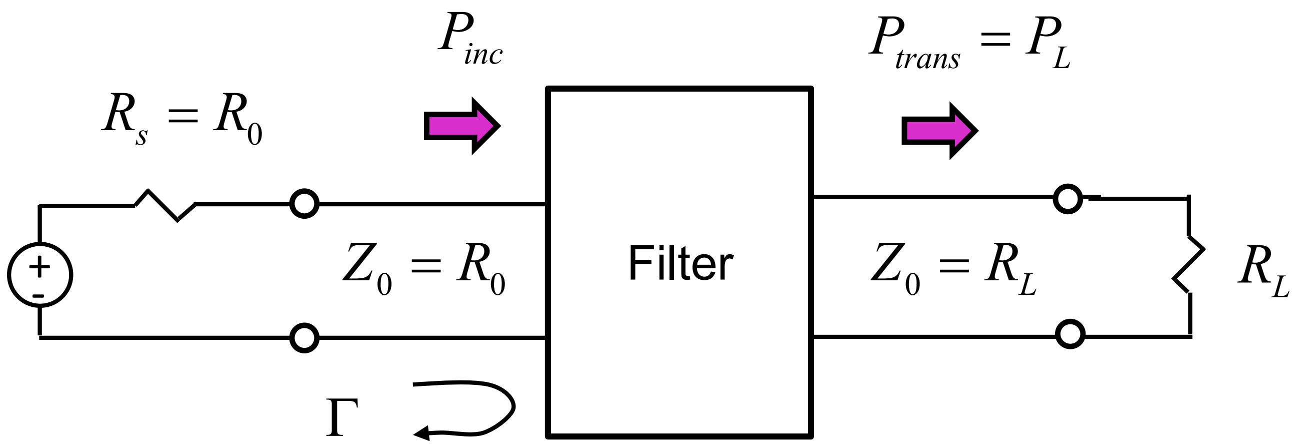


This is the basic filter circuit that we will study.

The filter is assumed to be lossless (consists of L and C elements).

Filters (cont.)

We can think of transmission lines connected at the input and output of the filter.



$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Power loss ratio:

$$P_{LR} \equiv \frac{P_{inc}}{P_L}$$

Lossless filter:

$$P_L = P_{in} = P_{inc}\left(1 - |\Gamma|^2\right)$$

$$P_{LR} = \frac{P_{inc}}{P_{inc}\left(1 - |\Gamma|^2\right)} = \frac{1}{\left(1 - |\Gamma|^2\right)}$$

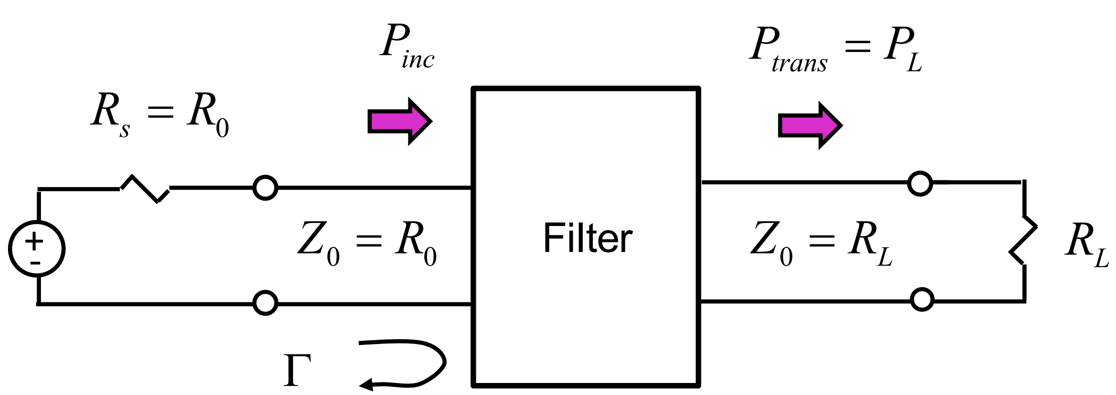
Filters (cont.)

Also

$$\left|S_{21}\right|^2 = \frac{P_2^-}{P_1^+} = \frac{P_{trans}}{P_{inc}} = \frac{P_L}{P_{inc}} = \frac{1}{P_{LR}} \quad \Rightarrow \quad P_{LR} = \frac{1}{\left|S_{21}\right|^2}$$

Insertion Loss (dB): $\text{IL} \equiv 10\log_{10} P_{LR} = -20\log_{10}\left|S_{21}\right|$

From conversation of energy: $\left|S_{21}\right|^2 = 1 - \left|S_{11}\right|^2$



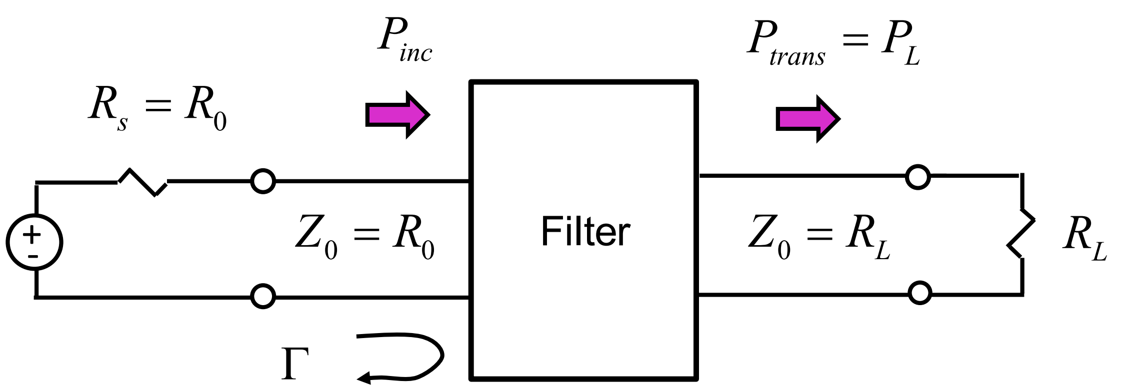
Filters (cont.)

The "insertion loss method" aims at designing a filter to achieve a particular type of response for the function $P_{LR}(\omega)$.

$$P_{LR} = \frac{P_{inc}}{P_L} = \frac{1}{\left(1-|\Gamma|^2\right)} = \frac{1}{|S_{21}|^2}$$

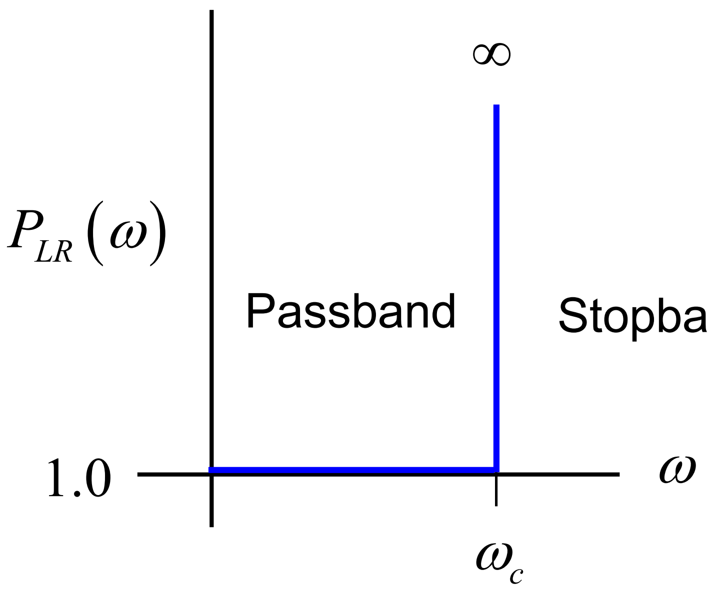
$$\text{IL} \equiv 10\log_{10} P_{LR} = -20\log_{10}|S_{21}|$$

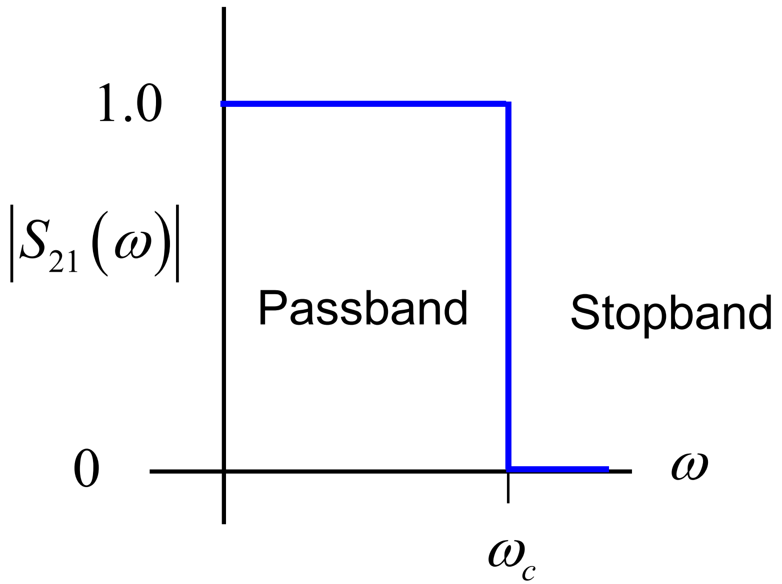
$$|S_{11}|^2 = 1-|S_{21}|^2$$



Low-Pass Filter

Ideal low-pass filter response





$$\text{Recall: } P_{LR} = \frac{1}{|S_{21}|^2}$$

Filters (cont.)

Types of filters:

- Low-pass
- High-pass
- Bandpass
- Bandstop

Approach:

Design the low-pass filter first. The other types of filters come from this by using a frequency transformation. (This is discussed in the next set of notes.)

Filters (cont.)

Common types of filters:

- Butterworth (binomial)*
- Chebyshev (type I)*
- Chebyshev (type II)
- Linear phase**
- Elliptic

*Discussed in this set of notes and the next set of notes.
** Discussed in the next set of notes.

Note on Reflection Coefficient

Property of the Power Loss Function

$$\Gamma(-\omega) = \Gamma^*(\omega)$$

(This property holds for the Fourier transform of any real-valued signal. A reflected time-domain signal must be real valued.)

$$\Rightarrow \quad \Gamma_R(-\omega) + j\Gamma_I(-\omega) = \Gamma_R(\omega) - j\Gamma_I(\omega)$$

$$\Rightarrow \quad \begin{cases} \Gamma_R(\omega) = \text{even function of } \omega \\ \Gamma_I(\omega) = \text{odd function of } \omega \end{cases}$$

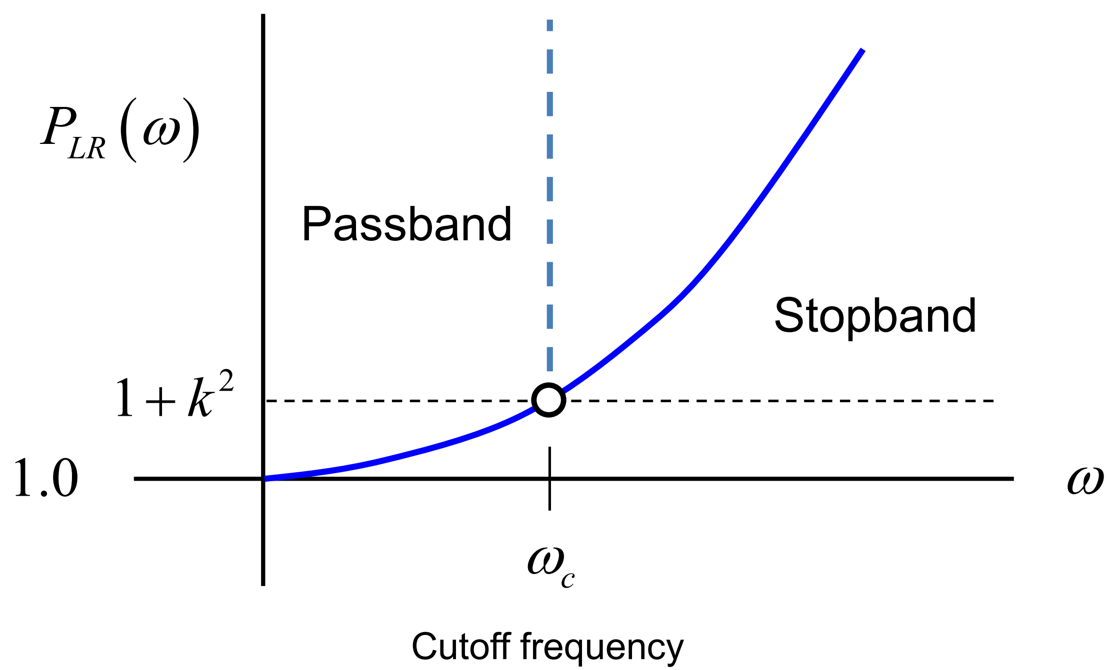
$$\Rightarrow \quad |\Gamma(\omega)|^2 = \text{even function of } \omega$$

$$\Rightarrow \quad P_{LR}(\omega) = \text{even function of } \omega$$

(When we design a filter, we have to keep this in mind!)

Low-Pass Filter

Low-pass filter response:

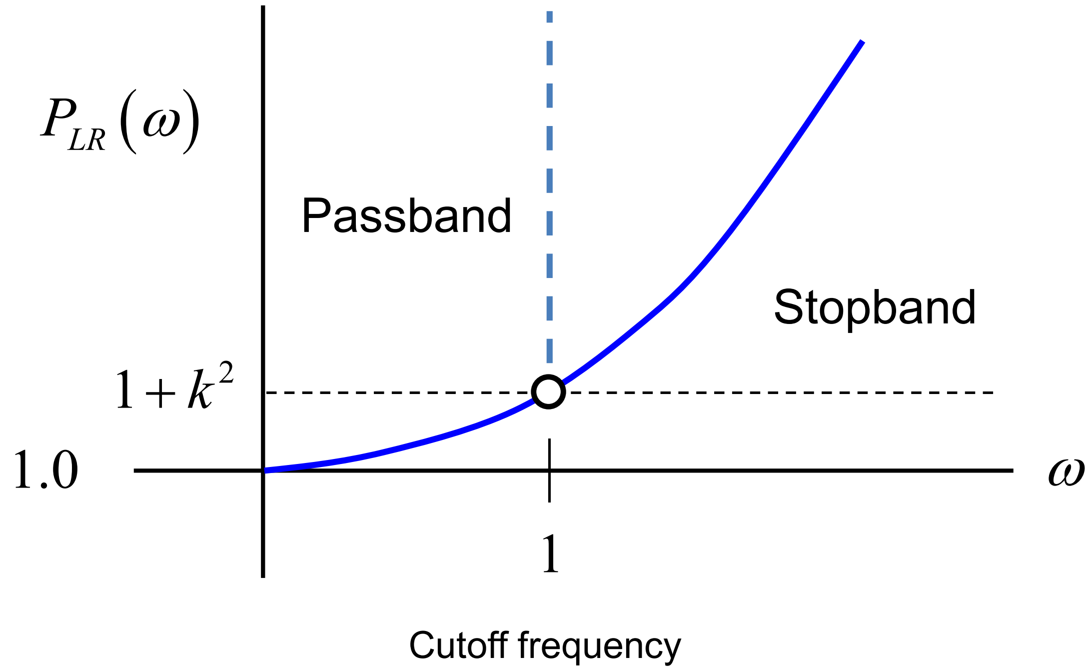


The constant k is somewhat arbitrary, and defines the cutoff frequency.

$$P_{LR}(\omega_c) = 1 + k^2$$

Low-Pass Filter

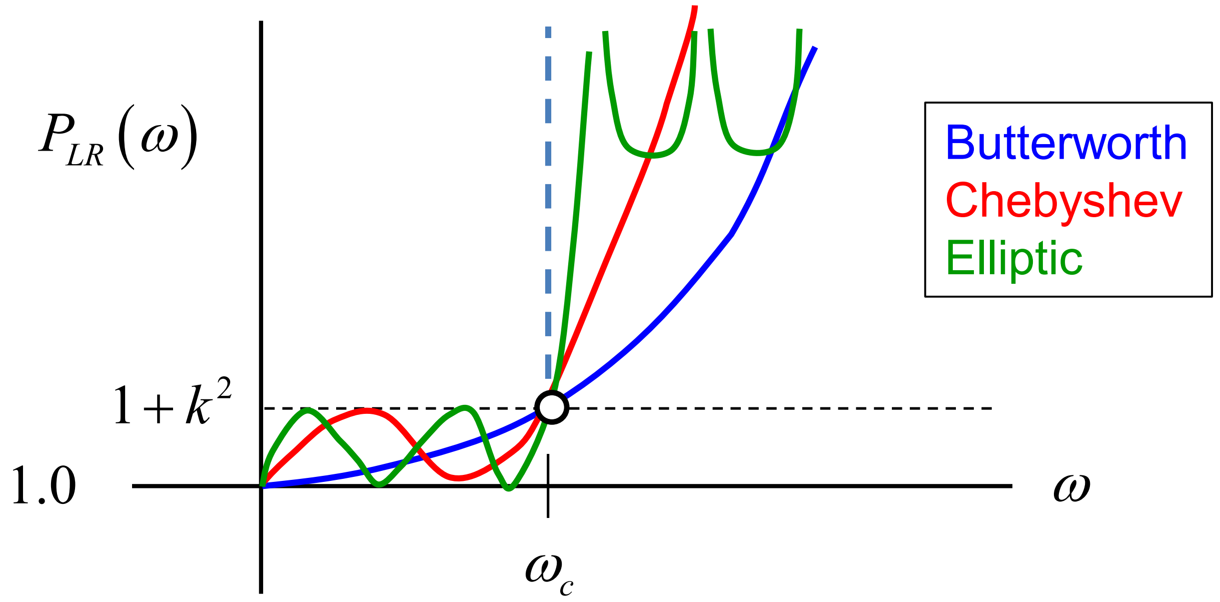
Normalized low-pass filter response:



We design the normalized low-pass filter first ($\omega_c = 1$), and then use scaling and frequency transformation to obtain the final filter (low-pass, high-pass, bandpass, bandstop).
(**This is done in the next set of notes.**)

Low-Pass Filter

Low-pass filter responses:



Note:
The Chebyshev response shown is called "type 1", with ripple in the passband. The "type 2" Chebyshev response has ripple in the stopband.

Low-Pass Filter

Comments:

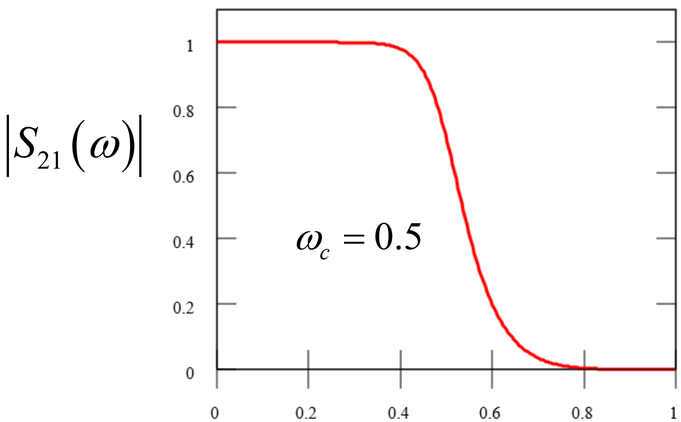
- The Butterworth design has the flattest response.
- The Chebyshev (type 1) design has a constant ripple in the passband.
- The Chebyshev (type 2) has a constant ripple in the stopband.
- The elliptic filter has ripple in both the passband and the stopband.

> The elliptic filter has the sharpest transition from the passband to the stopband, for given levels of ripple in the passband and stopband.

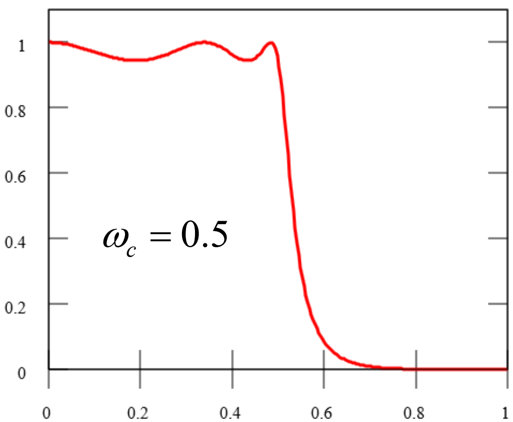
Low-Pass Filter

Comparison of Filter Responses

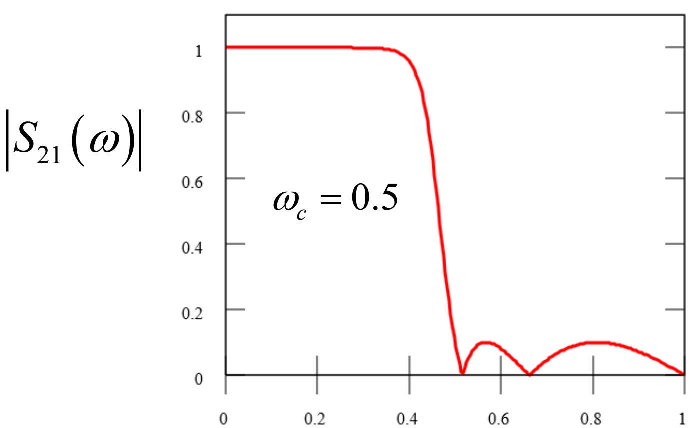
Butterworth



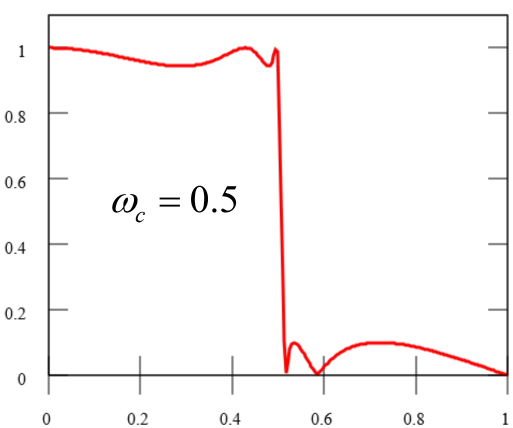
Chebyshev (type 1)



Chebyshev (type 2)



Elliptic



https://en.wikipedia.org/wiki/Electronic_filter

Butterworth Low-Pass Filter

Choose:

$$P_{LR}(\omega) = 1 + k^2 \left(\frac{\omega}{\omega_c} \right)^{2N}$$

N = order of the filter*

- The first N-1 derivatives of P_{LR} with respect to ω are zero.

At the band edge: $P_{LR}(\omega_c) = 1 + k^2$

Common choice: $k = 1$ $\left(P_{LR}(\omega_c) = 2, \ \mathrm{IL}(\omega_c) = 3\,\mathrm{dB} \right)$

* As seen later, this will also be the number of (L,C) elements in the filter.

Butterworth Low-Pass Filter (cont.)

High-frequency limit: $\omega / \omega_c \to \infty$

$$P_{LR}(\omega) \sim k^2 \left(\frac{\omega}{\omega_c} \right)^{2N}$$

Hence

$$\mathrm{IL} \equiv 10\log_{10} P_{LR} \sim 10\log_{10} \left(k^2 \left(\frac{\omega}{\omega_c} \right)^{2N} \right)$$

so

$$\mathrm{IL}(\omega) \sim 20\log_{10} k + 20N \log_{10} \left(\frac{\omega}{\omega_c} \right)$$

Conclusion: IL increases at $20N$ dB/decade in the stopband.

Chebyshev Low-Pass Filter

Choose:

$$P_{LR}(\omega) = 1 + k^2 T_N^2\left(\frac{\omega}{\omega_c}\right)$$

N = order of the filter*

- There is equal ripple in the passband.

At the band edge: $P_{LR}(\omega_c) = 1 + k^2$ Note : $T_n(1) = 1$

Ripple level: $1 + k^2 \quad \left(\text{dB} = 10\log_{10}\left(1+k^2\right)\right)$

$$T_n(x) \equiv \begin{cases} \cos\left(n\cos^{-1}x\right), & |x| \le 1 \\ \cosh\left(n\cosh^{-1}x\right), & |x| \ge 1 \end{cases}$$

$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$\vdots$$
$$T_n(x) = 2x\,T_{n\text{-}1}(x) - T_{n\text{-}2}(x)$$

* As seen later, this will also be the number of (L,C) elements in the filter.

Chebyshev Low-Pass Filter (cont.)

High-frequency limit: $\omega / \omega_c \to \infty$

$$P_{LR}(\omega) \sim \frac{k^2}{4} 2^{2N} \left(\frac{\omega}{\omega_c} \right)^{2N}$$

Note : $T_n(x) \sim 2^{n-1} x^n, \quad x \to \infty$

Hence

$$\mathrm{IL} \equiv 10 \log_{10} P_{LR} \sim 10 \log_{10} \left(\frac{k^2}{4} \right) + 20N \log_{10}(2) + 20N \log_{10} \left(\frac{\omega}{\omega_c} \right)$$

$$= 20 \log_{10}(k) - 10 \log_{10}(4) + 20N \log_{10}(2) + 20N \log_{10} \left(\frac{\omega}{\omega_c} \right)$$

so

$$\mathrm{IL} \sim 20 \log_{10}(k) + 20(N-1) \log_{10}(2) + 20N \log_{10} \left(\frac{\omega}{\omega_c} \right)$$

Chebyshev Low-Pass Filter (cont.)

Comparison:

Butterworth:

$$\mathrm{IL}(\omega) \sim 20\log_{10} k + 20N\log_{10}\left(\frac{\omega}{\omega_c}\right)$$

Chebyshev:

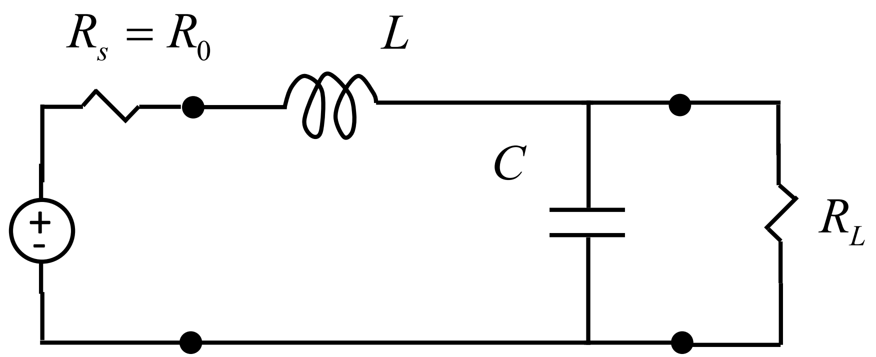
$$\mathrm{IL} \sim 20\log_{10}(k) + 20(N-1)\log_{10}(2) + 20N\log_{10}\left(\frac{\omega}{\omega_c}\right)$$

Conclusion:

In the stopband, the IL is larger than for the Chebyshev filter, but it increases at the same rate for both filters.

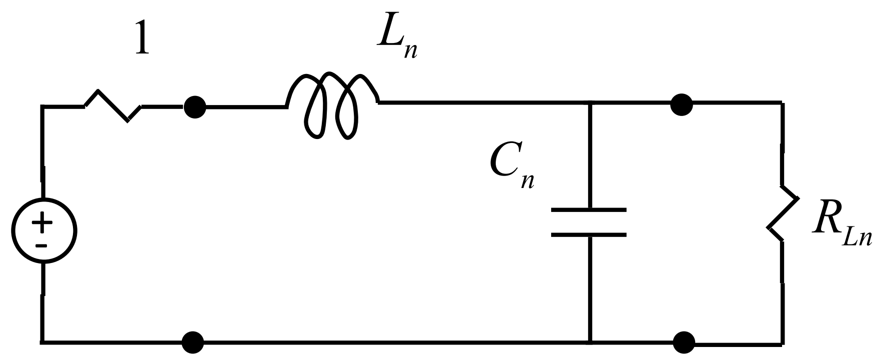
Two Element Low-Pass Filter

Original circuit:



Normalize impedances by R_0:

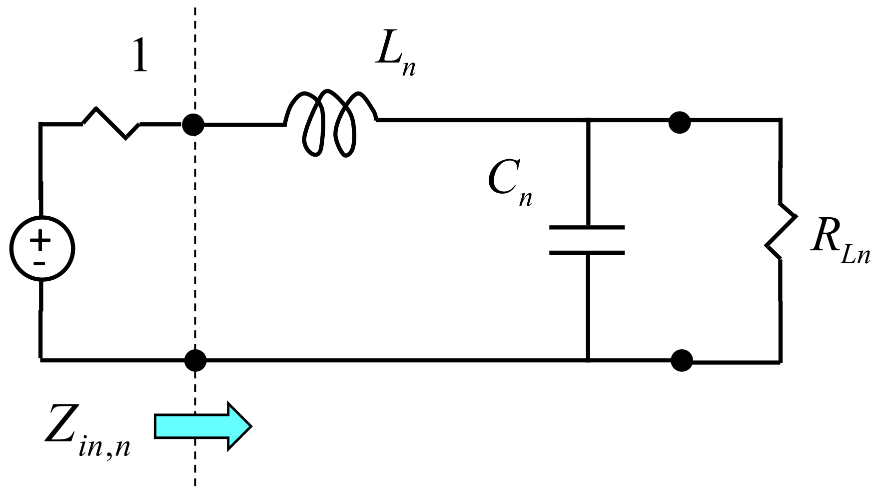
$$R_{Ln} = R_L / R_0$$
$$L_n = L / R_0$$
$$C_n = CR_0$$



This normalization does not change the reflection coefficient.

Two Element Low-Pass Filter (cont.)

Normalized circuit:



$$Z_{in,n} = j\omega L_n + \frac{1}{j\omega C_n + \frac{1}{R_{Ln}}}$$

$$= j\omega L_n + \frac{R_{Ln}}{1 + j\omega R_{Ln} C_n}$$

$$\Gamma_n = \frac{Z_{in,n} - 1}{Z_{in,n} + 1}$$

$$P_{LR} = \frac{1}{\left(1 - |\Gamma|^2\right)}$$

Two Element Low-Pass Filter (cont.)

After some algebra:

$$P_{LR} = \left(1 + \frac{(1 - R_{Ln})^2}{4R_{Ln}}\right) + \omega^2 \left(\left(\frac{1}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right)\right) + \omega^4 \left(\frac{1}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right)$$

or

$$P_{LR} = \left(1 + \frac{(1 - R_{Ln})^2}{4R_{Ln}}\right) + \left(\frac{\omega}{\omega_c}\right)^2 \left(\left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right)\right) + \left(\frac{\omega}{\omega_c}\right)^4 \left(\left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right)\right)$$

ω_c = specified cutoff frequency

Two Element Low-Pass Filter (cont.)

Hence we have:

$$P_{LR} = A_0 + A_2\left(\frac{\omega}{\omega_c}\right)^2 + A_4\left(\frac{\omega}{\omega_c}\right)^4$$

where

$$A_0 = \left(1 + \frac{(1-R_{Ln})^2}{4R_{Ln}}\right)$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right)$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right)$$

Two Element Low-Pass Butterworth

$$P_{LR} = A_0 + A_2\left(\frac{\omega}{\omega_c}\right)^2 + A_4\left(\frac{\omega}{\omega_c}\right)^4$$

Choose <u>Butterworth</u> response ($k = 1$):

$$P_{LR} = 1 + \left(\frac{\omega}{\omega_c}\right)^{2N} \quad \Rightarrow \quad N = 2$$

Hence

Note:
The order N of the filter is the same as the number of (L,C) elements.

$$A_0 = \left(1 + \frac{(1 - R_{Ln})^2}{4R_{Ln}}\right) = 1$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right) = 0$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right) = 1$$

Two Element Low-Pass Butterworth (cont.)

Three equations:

$$A_0 = \left(1 + \frac{(1 - R_{Ln})^2}{4R_{Ln}}\right) = 1$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right) = 0$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right) = 1$$

Solution:

A_0 equation:

$$R_{Ln} = 1$$

A_2 equation:

$$L_n = C_n$$

A_4 equation:

$$L_n = C_n = \frac{\sqrt{2}}{\omega_c}$$

Two Element Low-Pass Butterworth (cont.)

Unnormalizing:

$$R_{Ln} = R_L / R_0$$
$$L_n = L / R_0$$
$$C_n = C R_0$$

$\Rightarrow$

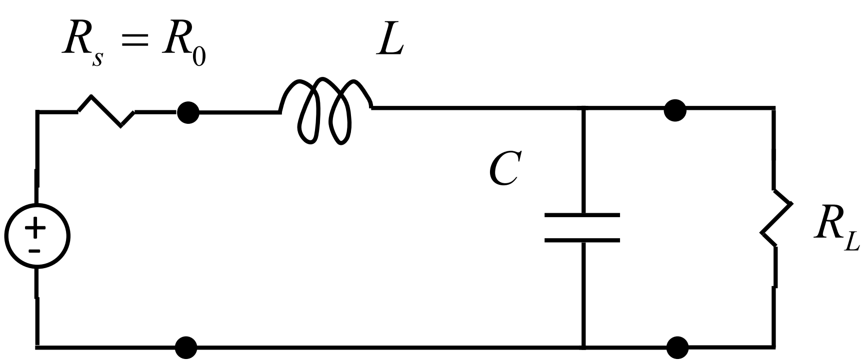
$$R_L = R_{Ln} R_0$$
$$L = L_n R_0$$
$$C = C_n / R_0$$

where

$$R_{Ln} = 1$$
$$L_n = C_n = \sqrt{2} / \omega_c$$

Hence:

$$R_L = R_0$$
$$L = R_0 \frac{\sqrt{2}}{\omega_c}$$
$$C = \frac{1}{R_0} \frac{\sqrt{2}}{\omega_c}$$

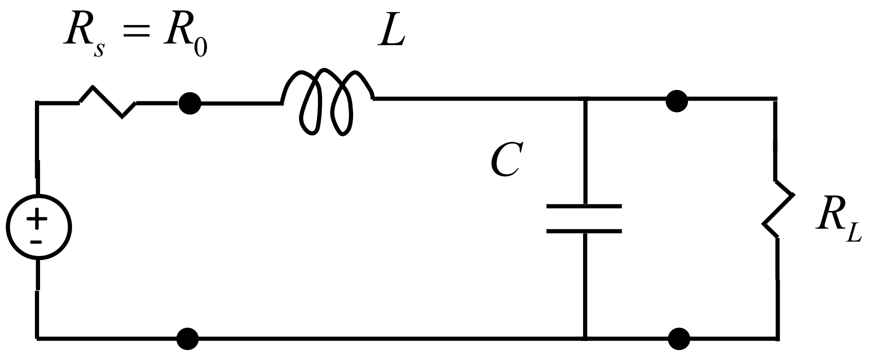


Two Element Low-Pass Butterworth (cont.)

Example

Design a low-pass Butterworth filter

$$N = 2$$
$$k = 1$$
$$f_c = 1\ \text{GHz}$$
$$R_0 = R_L = 50\,\Omega$$



Recall: $k = 1 \Rightarrow P_{LR}(\omega_c) = 2 \Rightarrow \text{IL}(\omega_c) = 3\,\text{dB}$

$$R_L = R_0$$
$$L = R_0 \frac{\sqrt{2}}{\omega_c}$$
$$C = \frac{1}{R_0}\frac{\sqrt{2}}{\omega_c}$$

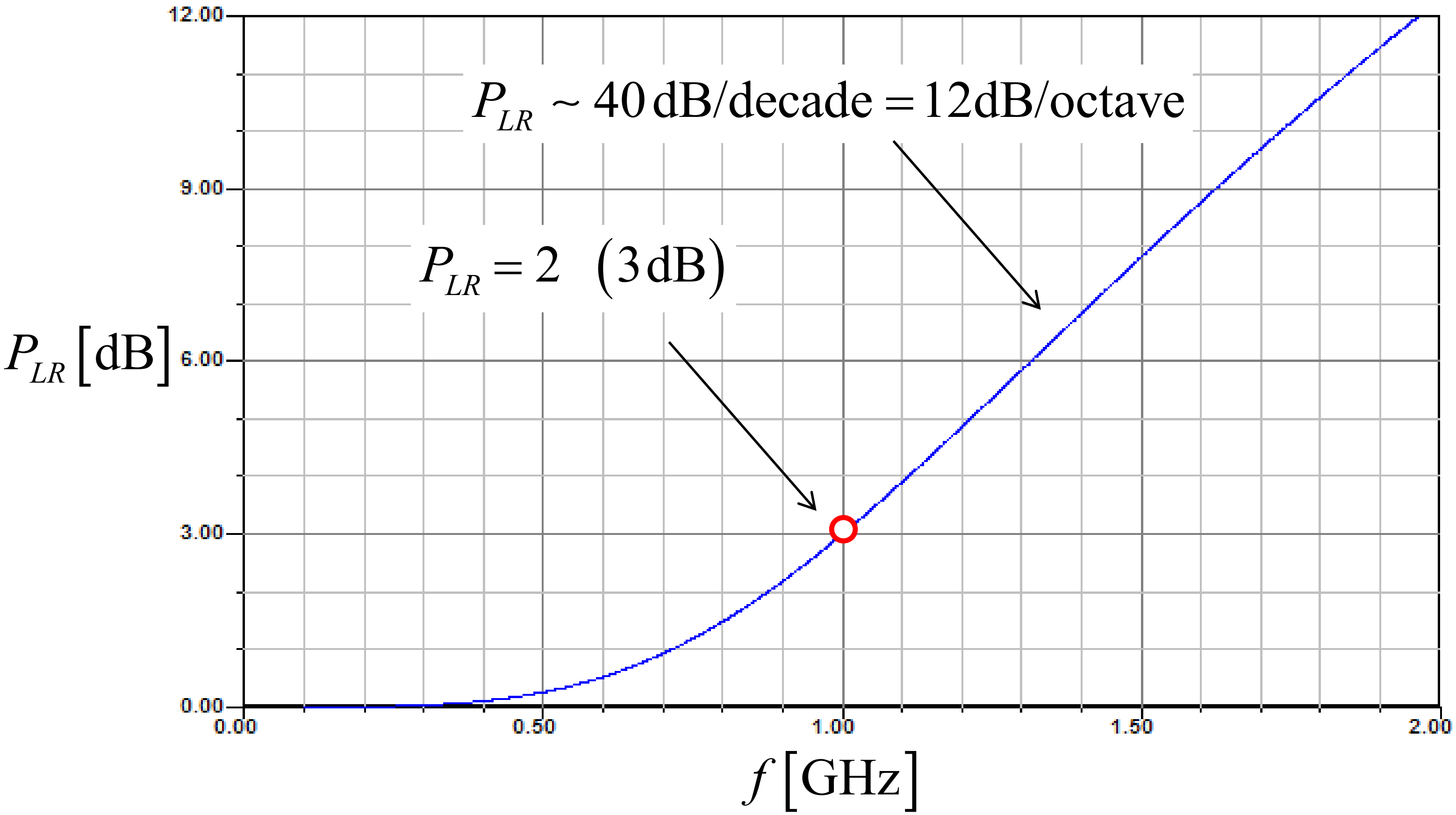
$\Rightarrow$

$$R_L = R_0 = 50\,[\Omega]$$
$$L = 50\frac{\sqrt{2}}{2\pi 10^9}\,[\text{H}]$$
$$C = \frac{1}{50}\frac{\sqrt{2}}{2\pi 10^9}\,[\text{F}]$$

$$R_L = R_0 = 50\,[\Omega]$$
$$L = 11.25\,[\text{nH}]$$
$$C = 4.50\,[\text{pF}]$$

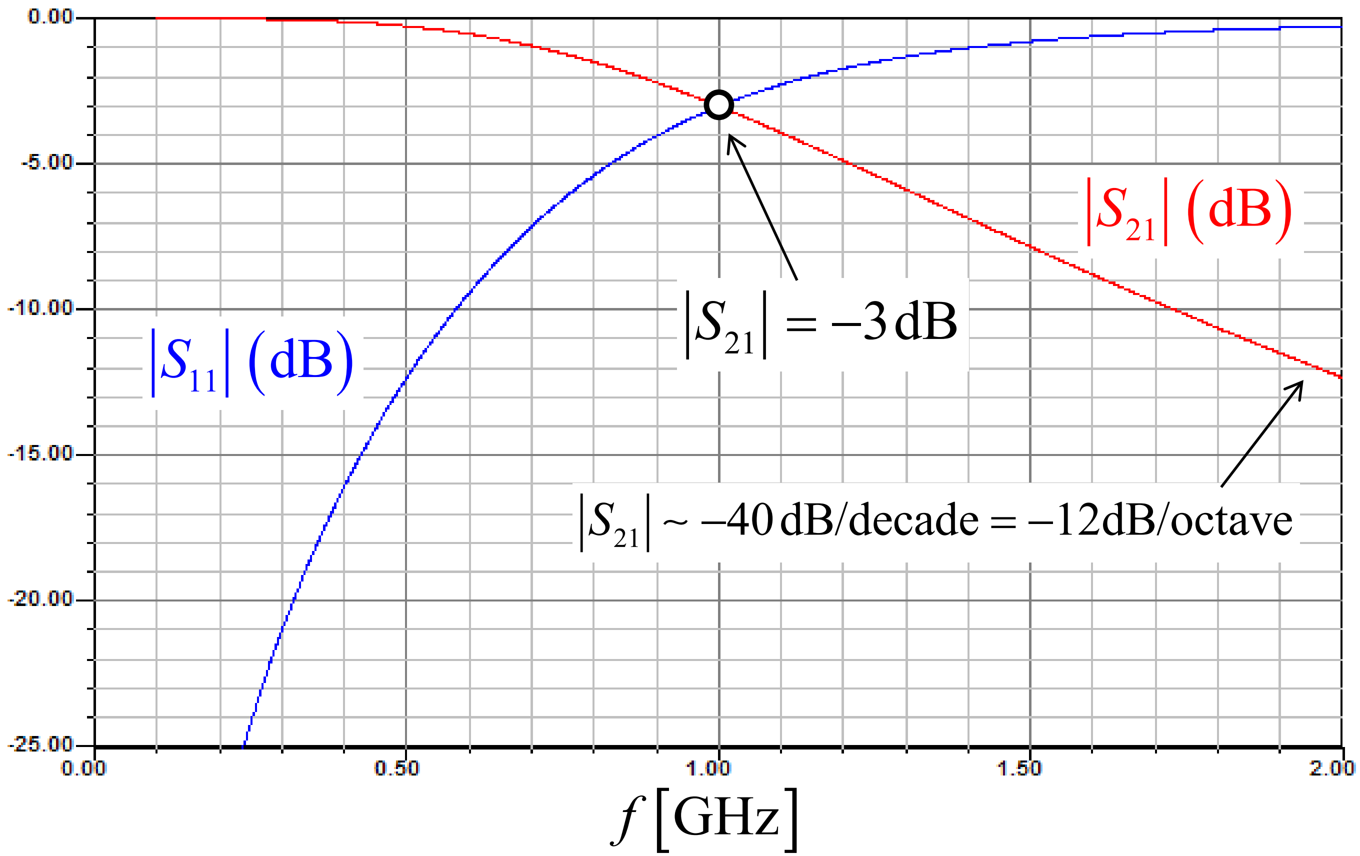
Two Element Low-Pass Butterworth (cont.)

Results from Ansys Designer



Two Element Low-Pass Butterworth (cont.)

Results from Ansys Designer



Recall: $\text{IL} = -S_{21}^{\text{dB}}, \quad |S_{11}|^2 = 1 - |S_{21}|^2$ (lossless)

Two Element Low-Pass Chebyshev

$$P_{LR} = A_0 + A_2\left(\frac{\omega}{\omega_c}\right)^2 + A_4\left(\frac{\omega}{\omega_c}\right)^4$$

Choose Chebyshev response (arbitrary k value):

$$P_{LR}(\omega) = 1 + k^2 T_N^2\left(\frac{\omega}{\omega_c}\right) \quad \Rightarrow \quad N = 2$$

so

$$P_{LR}(\omega) = 1 + k^2\left(4\left(\frac{\omega}{\omega_c}\right)^4 - 4\left(\frac{\omega}{\omega_c}\right)^2 + 1\right)$$

Note:

$$T_2(x) = 2x^2 - 1$$

$$T_2^2(x) = 4x^4 - 4x^2 + 1$$

Hence:

$$A_0 = 1 + k^2$$

$$A_2 = -4k^2$$

$$A_4 = 4k^2$$

Two Element Low-Pass Chebyshev (cont.)

Hence:

$$A_0 = \left(1 + \frac{\left(1 - R_{Ln}\right)^2}{4R_{Ln}}\right) = 1 + k^2$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right) = -4k^2$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right) = 4k^2$$

Solution (please see the Appendix):

$$R_{Ln} = 1 + 2k^2 \pm 2k\sqrt{1 + k^2}$$

$$L_n C_n = 4\left(\frac{k}{\omega_c^2}\right)\frac{1}{\sqrt{R_{Ln}}}$$

$$C_n^2 = \frac{1}{2R_{Ln}^2}\left(2pR_{Ln}^2 - 4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) \pm \sqrt{\left(4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) - 2pR_{Ln}^2\right)^2 - 4R_{Ln}^2 p^2}\right)$$

Appendix

Chebyshev solution:

$$A_0 = \left(1 + \frac{(1-R_{Ln})^2}{4R_{Ln}}\right) = 1 + k^2$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}}\right)\left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2\right) = -4k^2$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}}\right)\left(L_n^2 C_n^2 R_{Ln}^2\right) = 4k^2$$

A_0 equation:

$$1 + \frac{(1-R_{Ln})^2}{4R_{Ln}} = 1 + k^2$$

$$\Rightarrow \frac{(1-R_{Ln})^2}{4R_{Ln}} = k^2$$

$$\Rightarrow (1-R_{Ln})^2 = k^2 4R_{Ln}$$

$$1 - 2R_{Ln} + R_{Ln}^2 = k^2 4R_{Ln}$$

$$\Rightarrow R_{Ln}^2 - R_{Ln}\left(2 + 4k^2\right) + 1 = 0$$

$$\Rightarrow R_{Ln} = 1 + 2k^2 \pm 2k\sqrt{1+k^2}$$

Note :

$R_L \neq R_0$ for N = even.

(It turns out that $R_L = R_0$ for N = odd.)

Appendix (cont.)

$$A_0 = \left(1 + \frac{(1-R_{Ln})^2}{4R_{Ln}} \right) = 1 + k^2$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}} \right) \left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2 \right) = -4k^2$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}} \right) \left(L_n^2 C_n^2 R_{Ln}^2 \right) = 4k^2$$

A_4 equation:

$$L_n^2 C_n^2 = 16 \left(\frac{k^2}{\omega_c^4} \right) \frac{1}{R_{Ln}}$$

$$\Rightarrow \quad L_n C_n = 4 \left(\frac{k}{\omega_c^2} \right) \frac{1}{\sqrt{R_{Ln}}}$$

Appendix (cont.)

$$A_0 = \left(1 + \frac{(1 - R_{Ln})^2}{4R_{Ln}} \right) = 1 + k^2$$

$$A_2 = \left(\frac{\omega_c^2}{4R_{Ln}} \right) \left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2 \right) = -4k^2$$

$$A_4 = \left(\frac{\omega_c^4}{4R_{Ln}} \right) \left(L_n^2 C_n^2 R_{Ln}^2 \right) = 4k^2$$

A_2 equation:

$$\left(\frac{\omega_c^2}{4R_{Ln}} \right) \left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2 \right) = -4k^2$$

$$\Rightarrow \left(R_{Ln}^2 C_n^2 + L_n^2 - 2L_n C_n R_{Ln}^2 \right) = -4k^2 \left(\frac{4R_{Ln}}{\omega_c^2} \right)$$

$$\Rightarrow \left(R_{Ln}^2 C_n^2 + L_n^2 \right) = -4k^2 \left(\frac{4R_{Ln}}{\omega_c^2} \right) + 2\left(L_n C_n \right) R_{Ln}^2$$

(The RHS is known.)

Appendix (cont.)

A_2 equation (cont.):

$$\left(R_{Ln}^2 C_n^2 + L_n^2\right) = -4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) + 2pR_{Ln}^2 \qquad \left(p \equiv L_n C_n = 4\left(\frac{k}{\omega_c^2}\right)\frac{1}{\sqrt{R_{Ln}}}\right)$$

$$\Rightarrow \left(R_{Ln}^2 C_n^2 + \frac{p^2}{C_n^2}\right) = -4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) + 2pR_{Ln}^2$$

$$\Rightarrow \left(R_{Ln}^2 C_n^4 + p^2\right) = C_n^2\left(-4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) + 2pR_{Ln}^2\right)$$

$$\Rightarrow R_{Ln}^2 x^2 + x\left(4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) - 2pR_{Ln}^2\right) + p^2 = 0 \qquad \left(x \equiv C_n^2\right)$$

$$\Rightarrow x = \frac{1}{2R_{Ln}^2}\left(2pR_{Ln}^2 - 4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) \pm \sqrt{\left(4k^2\left(\frac{4R_{Ln}}{\omega_c^2}\right) - 2pR_{Ln}^2\right)^2 - 4R_{Ln}^2 p^2}\right)$$