Adapted from notes by
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Microwave Engineering

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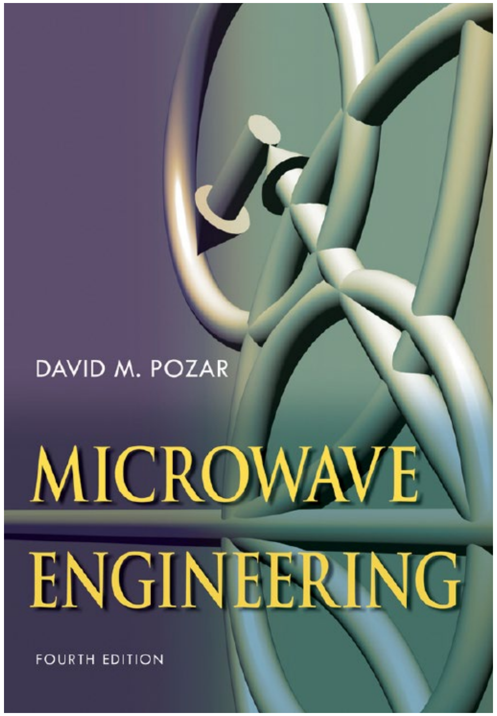
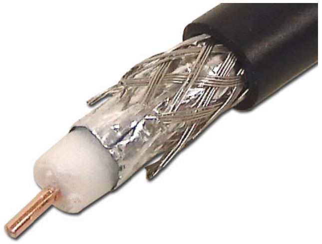
Dept. of ECE



Notes 3

Transmission Lines
Part 2: TL Formulas

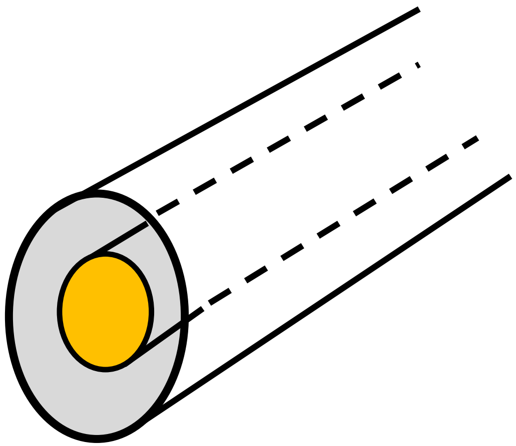


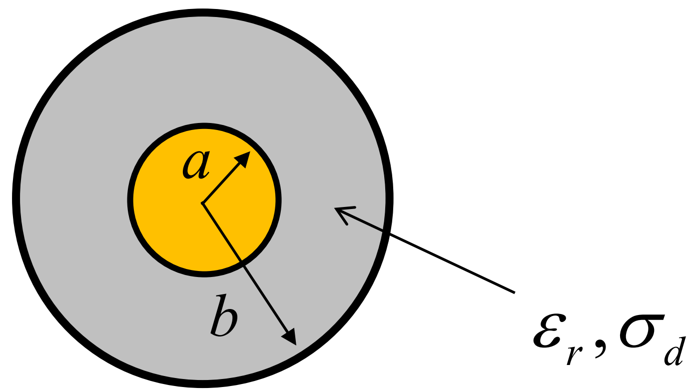
Overview

- In this set of notes we develop some general formulas that hold for any transmission line.
- We first examine the coaxial cable as an example.

Coaxial Cable

Here we present a "case study" of one particular transmission line, the coaxial cable.



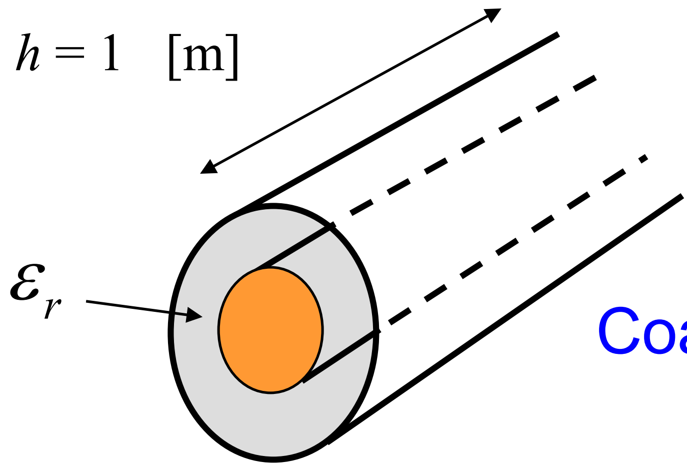


Find C, L, G, R

For a TEM_z mode, the shape of the fields is <u>independent of frequency</u>, and hence we can perform the calculation of C and L using electrostatics and magnetostatics.

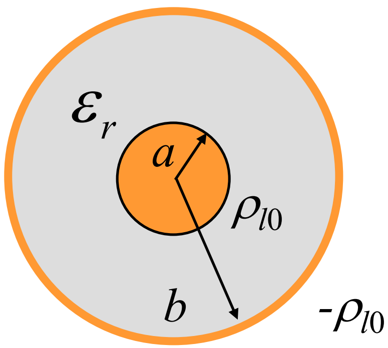
We will assume no variation in the z direction, and take a length of one meter in the z direction in order to calculate the per-unit-length parameters.

Coaxial Cable (cont.)



Find C (capacitance / length)

Coaxial cable



From Gauss's law:

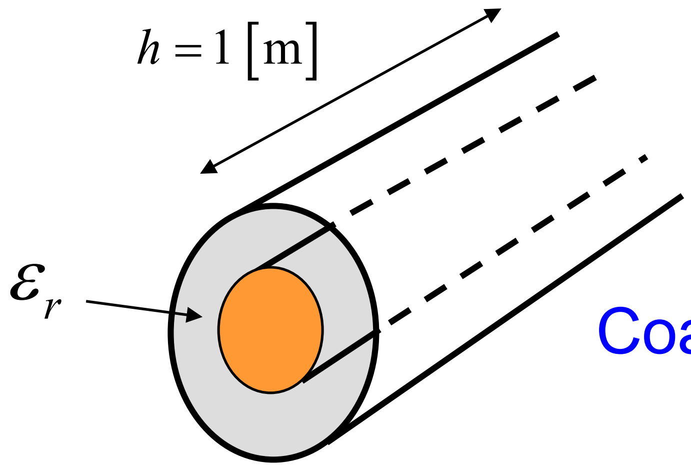
$$\underline{E} = \underline{\hat{\rho}}\left(\frac{\rho_{\ell 0}}{2\pi\varepsilon\rho}\right) = \underline{\hat{\rho}}\left(\frac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r\rho}\right)$$

$$V = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E}\cdot\underline{dr}$$

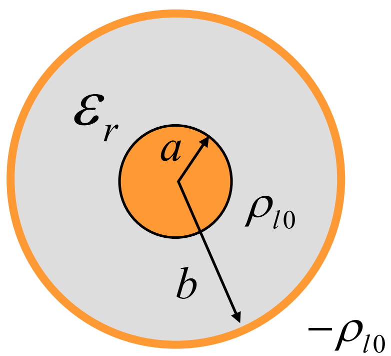
$$= \int_a^b E_\rho \, d\rho = \frac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r}\ln\left(\frac{b}{a}\right)$$

$\rho_{\ell 0}$ = line charge density (C/m) on the inner conductor

Coaxial Cable (cont.)



Coaxial cable



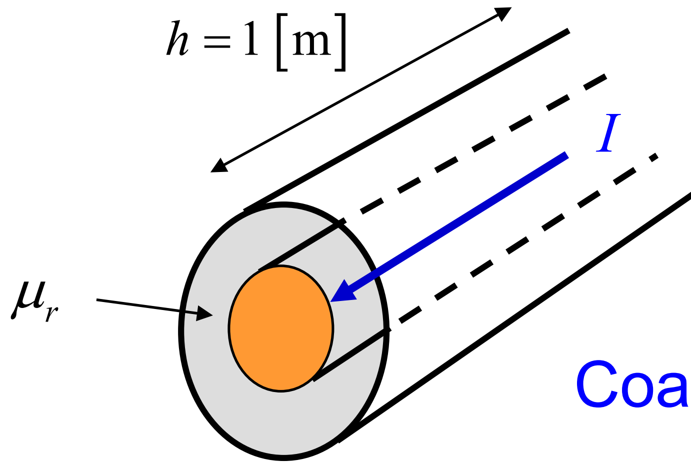
Hence

$$C = \frac{Q}{V} = \frac{\rho_{\ell 0}(1)}{\left(\dfrac{\rho_{\ell 0}}{2\pi\varepsilon_0\varepsilon_r}\right)\ln\left(\dfrac{b}{a}\right)}$$

We then have:

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\dfrac{b}{a}\right)} \quad [\mathrm{F/m}]$$

Coaxial Cable (cont.)



Coaxial cable

Find L (inductance / length)

From Ampere's law:

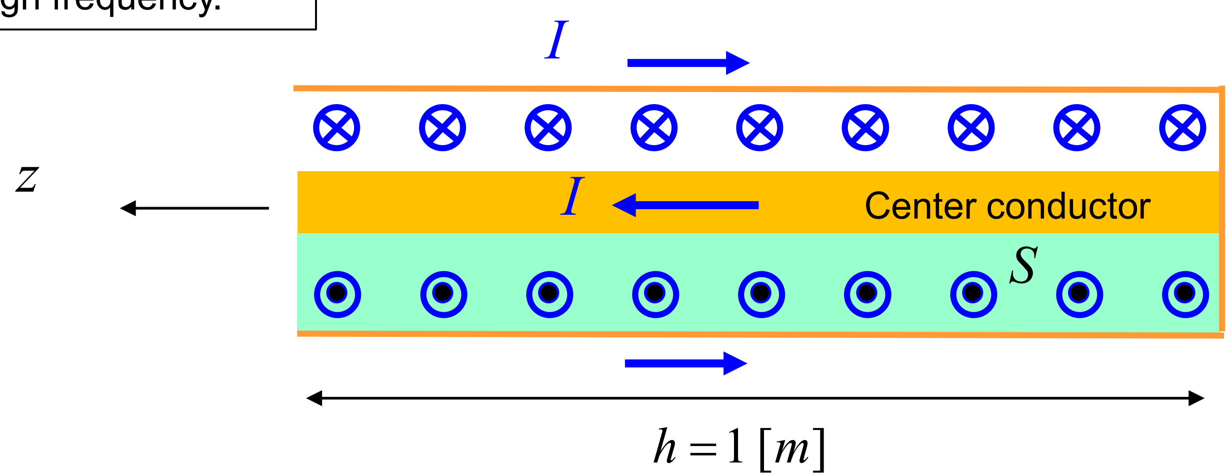
$$\underline{H} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right)$$

$$\underline{B} = \underline{\hat{\phi}}\left(\frac{I}{2\pi\rho}\right)\mu_0\mu_r$$

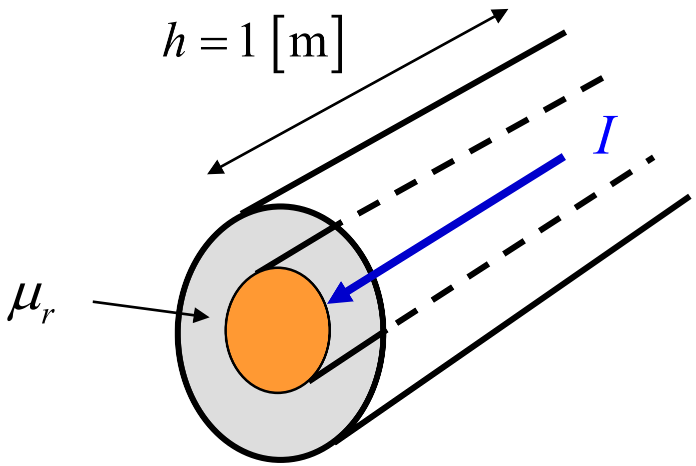
Note:
We ignore "internal inductance" here, and only look at the magnetic field *between* the two conductors (accurate for high frequency.

Magnetic flux:

$$\psi = (1)\int_a^b B_\phi \, d\rho$$



Coaxial Cable (cont.)



$$\psi = (1)\,\mu_0\mu_r \int_a^b H_\phi \, d\rho$$
$$= \mu_0\mu_r \int_a^b \frac{I}{2\pi\rho} \, d\rho$$
$$= \mu_0\mu_r \frac{I}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\psi}{I} = \mu_0\mu_r \frac{1}{2\pi} \ln\left(\frac{b}{a}\right)$$

Hence $$L = \frac{\mu_0\mu_r}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\mathrm{H/m}]$$

Coaxial Cable (cont.)

Observations:

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\dfrac{b}{a}\right)} \quad [\mathrm{F/m}]$$

(independent of frequency)

$$L = \frac{\mu_0\mu_r}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\mathrm{H/m}]$$

(independent of frequency)

$$LC = \mu\varepsilon = \mu_0\varepsilon_0\left(\mu_r\varepsilon_r\right)$$

This result actually holds for <u>any</u> transmission line that is homogenously filled* (proof omitted).

*This result assumes that the permittivity is real. To be more general, for a lossy line, we replace the permittivity with the real part of the permittivity in this result.

Coaxial Cable (cont.)

For a lossless (or low loss) cable: $$Z_0 = \sqrt{\frac{L}{C}}$$

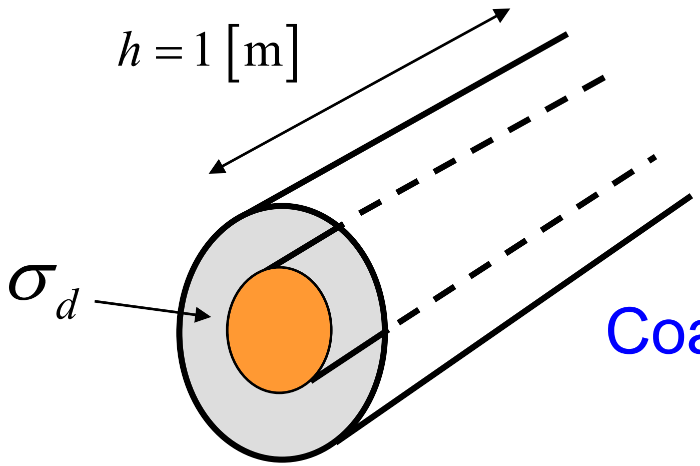
$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\text{F/m}]$$

$$L = \frac{\mu_0\mu_r}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\text{H/m}]$$

$$Z_0 = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}\frac{1}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\Omega]$$

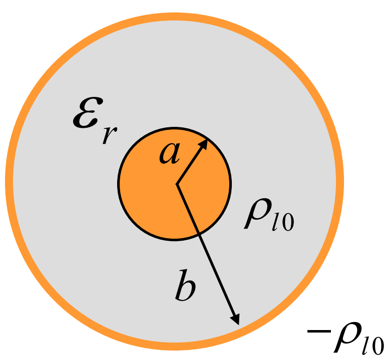
$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7303\ [\Omega]$$

Coaxial Cable (cont.)



Coaxial cable

Find G (conductance / length)



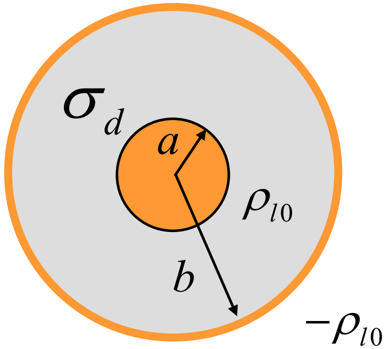
From Gauss's law:

$$\underline{E} = \hat{\underline{\rho}} \left(\frac{\rho_{\ell 0}}{2\pi \varepsilon \rho} \right) = \hat{\underline{\rho}} \left(\frac{\rho_{\ell 0}}{2\pi \varepsilon_0 \varepsilon_r \rho} \right)$$

$$V = V_{AB} = \int_{\underline{A}}^{\underline{B}} \underline{E} \cdot \underline{dr}$$

$$= \int_a^b E_\rho \, d\rho = \frac{\rho_{\ell 0}}{2\pi \varepsilon_0 \varepsilon_r} \ln\left(\frac{b}{a} \right)$$

Coaxial Cable (cont.)



$$\underline{J} = \sigma_d \underline{E}$$

$$I_{leak} = J_\rho \big|_{\rho=a} \left[(1)\, 2\pi a\right]$$
$$= 2\pi a \sigma_d \, E_\rho \big|_{\rho=a}$$
$$= 2\pi a \sigma_d \left(\frac{\rho_{\ell 0}}{2\pi \varepsilon_0 \varepsilon_r a} \right)$$

We then have $G = \dfrac{I_{leak}}{V}$

$$G = \frac{2\pi a \sigma_d \left(\dfrac{\rho_{\ell 0}}{2\pi \varepsilon_0 \varepsilon_r a} \right)}{\dfrac{\rho_{\ell 0}}{2\pi \varepsilon_0 \varepsilon_r} \ln\left(\dfrac{b}{a} \right)}$$

or

$$G = \frac{2\pi \sigma_d}{\ln\left(\dfrac{b}{a} \right)} \quad [\text{S/m}]$$

Coaxial Cable (cont.)

Observation:

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{F/m}] \qquad \varepsilon = \varepsilon_0 \varepsilon_r$$

$$G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{S/m}]$$

$$G = C\left(\frac{\sigma_d}{\varepsilon}\right)$$

This result actually holds for <u>any</u> transmission line that is homogenously filled* (proof omitted).

*This result assumes that the G term arises only from conductivity, and not polarization loss.

Coaxial Cable (cont.)

As just derived,

$$G = C\left(\frac{\sigma_d}{\varepsilon}\right)$$

Hence:

$$\frac{G}{\omega C} = \left(\frac{\sigma_d}{\omega\varepsilon}\right) = \tan\delta_d$$

This is the loss tangent that would arise from conductivity.

or

$$\frac{G}{\omega C} = \tan\delta_d$$

This result actually holds for any transmission line that is homogenously filled* (proof omitted).

*This result is very general, and allows the G term to come from either conductivity or polarization loss.

Accounting for Dielectric Loss

Complex Permittivity

The permittivity becomes complex when there is polarization (molecular friction) loss.

$$\varepsilon = \left(\varepsilon' - j\varepsilon''\right)$$

↑

Loss term due to polarization (molecular friction)

Example: Distilled water heats up in a microwave oven, even though there is essentially no conductivity!

Accounting for Dielectric Loss (cont.)

Effective Complex Permittivity

The <u>effective</u> permittivity accounts for conductive loss.

$$\varepsilon_c \equiv \varepsilon - j\left(\frac{\sigma}{\omega}\right)$$

← Effective permittivity that accounts for conductivity

Hence

$$\varepsilon_c = \left(\varepsilon' - j\varepsilon''\right) - j\left(\frac{\sigma}{\omega}\right)$$

$$= \varepsilon_c{}' - j\varepsilon_c{}''$$

We then have

$$\varepsilon_c{}' = \varepsilon'$$

$$\varepsilon_c{}'' = \varepsilon'' + \left(\frac{\sigma}{\omega}\right)$$

Accounting for Dielectric Loss (cont.)

Most general expression for loss tangent:

$$\varepsilon_c = \varepsilon_c{}' - j\varepsilon_c{}''$$

$$\varepsilon_c{}' = \varepsilon'$$

$$\varepsilon_c{}'' = \varepsilon'' + \left(\frac{\sigma}{\omega}\right)$$

$$\tan\delta \equiv \frac{\varepsilon_c{}''}{\varepsilon_c{}'} = \frac{\varepsilon'' + \left(\frac{\sigma}{\omega}\right)}{\varepsilon'}$$

Loss due to molecular friction (ε'')

Loss due to conductivity (σ/ω)

The loss tangent accounts for <u>both</u> molecular friction and conductivity.

Accounting for Dielectric Loss (cont.)

For most practical insulators (e.g., Teflon), we have

$$\sigma \approx 0$$

$$\Rightarrow \quad \tan\delta = \frac{\varepsilon_c''}{\varepsilon_c'} \approx \frac{\varepsilon''}{\varepsilon'}$$

Note: The loss tangent is usually (approximately) constant for practical insulating materials, over a wide range of frequencies.

Typical microwave insulating material (e.g., Teflon): $\tan\delta = 0.001$.

Accounting for Dielectric Loss (cont.)

Important point about notation:

In most books, ε_r' is denoted as simply ε_r

(We will adopt this convention also.)

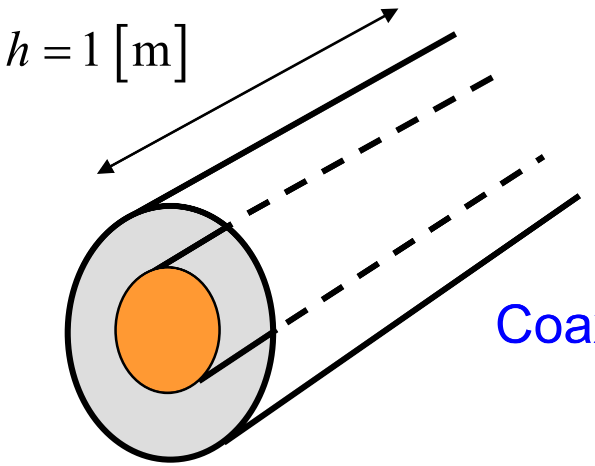
In this case we then write:

$$\varepsilon_{rc} = \varepsilon_r \left(1 - j \tan\delta\right)$$

Means ε_r' (real part of ε_{rc}')

The effective complex relative permittivity

Coaxial Cable (cont.)



Find R (resistance / length)

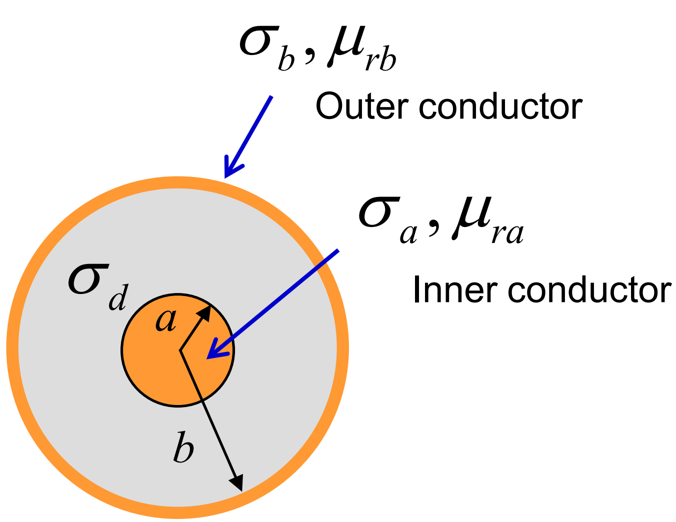
$$R = R_a + R_b$$

$$R_a = R_{sa}\left(\frac{1}{2\pi a}\right)$$

$$R_b = R_{sb}\left(\frac{1}{2\pi b}\right)$$

R_s = surface resistance of metal

(This is discussed later.)



$$R_{sa} = \frac{1}{\sigma_a \delta_a} \qquad R_{sb} = \frac{1}{\sigma_b \delta_b}$$

$$\delta_a = \sqrt{\frac{2}{\omega \mu_0 \mu_{ra} \sigma_a}} \qquad \delta_b = \sqrt{\frac{2}{\omega \mu_0 \mu_{rb} \sigma_b}}$$

δ = skin depth of metal

General Formulas for (L,G,C)

The three per-unit-length parameters (L, G, C) can be found from

$$Z_0^{lossless}\,,\;\; \varepsilon_r\,,\; \tan\delta_d$$

These values are usually known from the manufacturer.

$$L = Z_0^{lossless}\sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r}$$

$$C = \sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r}\;/\;Z_0^{lossless}$$

$$G = \left(\omega C\right)\tan\delta_d$$

These formulas hold for <u>any</u> homogeneously-filled transmission line.

$Z_0^{lossless} \equiv$ characteristic impedance of **lossless** line $= \sqrt{\dfrac{L}{C}}$

General Formulas for (L,G,C) (cont.)

The derivation of the previous results follows from:

$$LC = \mu_0 \mu_r \varepsilon_0 \varepsilon_r$$

$$\frac{L}{C} = \left(Z_0^{lossless} \right)^2$$

Multiply and divide

Here ε_r denotes ε_r'.

General Formulas for (L,G,C) (cont.)

Example:

A transmission line has the following properties:

$$\varepsilon_r = 2.1 \left(\text{Teflon}\right)$$

$$Z_0^{lossless} = 50 \ \Omega$$

$$\tan\delta = 0.001$$

$$f = 10 \ [\text{GHz}]$$ (frequency is only needed for G)

Results:

$$L = 2.4169 \times 10^{-7} \ [\text{H/m}]$$

$$C = 9.6677 \times 10^{-11} \ [\text{F/m}]$$

$$G = 6.0744 \times 10^{-3} \ [\text{S/m}]$$

Note:
We cannot determine R without knowing the type of transmission line and the dimensions (and the conductivity of the metal).

Wavenumber Formulas

General case (R,L,G,C):

$$k_z = \beta - j\alpha = -j\sqrt{(R + j\omega L)(G + j\omega C)}$$

Lossless case (L,C):

$$k_z = \beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon} = k = k_0\sqrt{\mu_r\varepsilon_r}$$

Dielectric loss only (L,C,G):

$$k_z = k = \omega\sqrt{\mu\varepsilon_c} = k' - jk'' = k_0\sqrt{\mu_r\varepsilon_{rc}}$$ (please see next slide)

Wavenumber (cont.)

Dielectric loss only (L,C,G):

$$
\begin{aligned}
k_z &= -j\sqrt{(j\omega L)(G+j\omega C)} \\
&= -j\sqrt{(j\omega L)(\omega C\tan\delta_d + j\omega C)} \\
&= -j\omega\sqrt{(jL)(C\tan\delta_d + jC)} \\
&= -j\omega\sqrt{LC}\sqrt{(j)(\tan\delta_d + j)} \\
&= -j\omega\sqrt{LC}\sqrt{-(1-j\tan\delta_d)} \\
&= \omega\sqrt{LC}\sqrt{(1-j\tan\delta_d)} \\
&= \omega\sqrt{\mu\varepsilon'}\sqrt{(1-j\tan\delta_d)} \\
&= \omega\sqrt{\mu\varepsilon_c} \\
&= k
\end{aligned}
$$

$$R = 0$$

Note:
The mode stays a perfect TEM_z mode if $R = 0$.
$k_z = k$ for <u>any</u> TEM_z mode.

Notes :

$$\varepsilon' = \varepsilon_c'$$

$$\tan\delta = \frac{\varepsilon_c''}{\varepsilon_c'}$$

Common Transmission Lines

Coax

$$Z_0^{lossless} = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} \frac{1}{2\pi} \ln\left(\frac{b}{a}\right) \quad [\Omega]$$

$$R = R_{sa}\left(\frac{1}{2\pi a}\right) + R_{sb}\left(\frac{1}{2\pi b}\right)$$

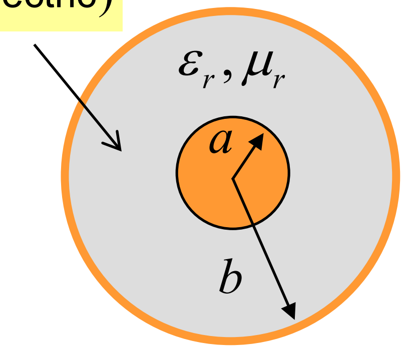
$$R_{sa} = \frac{1}{\sigma_a \delta_a} \qquad R_{sb} = \frac{1}{\sigma_b \delta_b}$$

(surface resistance of metal for inner or outer conductors)

$$\delta_a = \sqrt{\frac{2}{\omega \mu_0 \mu_{ra} \sigma_a}} \qquad \delta_b = \sqrt{\frac{2}{\omega \mu_0 \mu_{rb} \sigma_b}}$$

(skin depth of metal for inner or outer conductors)

$\tan\delta_d$ (dielectric)



σ_a = conductivity of inner conductor metal

σ_b = conductivity of outer conductor metal

$$L = Z_0^{lossless} \sqrt{\mu_0 \varepsilon_0} \sqrt{\mu_r \varepsilon_r}$$

$$C = \sqrt{\mu_0 \varepsilon_0} \sqrt{\mu_r \varepsilon_r} \, / \, Z_0^{lossless}$$

$$G = (\omega C) \tan\delta_d$$

$$R = R$$

Common Transmission Lines

Twin-lead

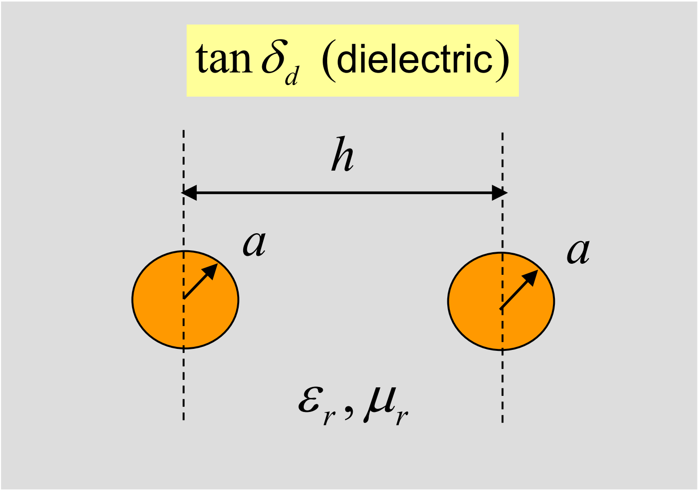
$$Z_0^{lossless} = \frac{\eta_0}{\pi}\sqrt{\frac{\mu_r}{\varepsilon_r}}\cosh^{-1}\left(\frac{h}{2a}\right) \quad [\Omega]$$

$$R = R_s\left[\frac{1}{\pi a}\frac{\left(\frac{h}{2a}\right)}{\sqrt{\left(\frac{h}{2a}\right)^2-1}}\right]$$

$$R_s = \frac{1}{\sigma_m \delta}$$ (surface resistance of metal)

$$\delta = \sqrt{\frac{2}{\omega\mu_0\mu_{rm}\sigma_m}}$$ (skin depth of metal)

Two identical conductors



σ_m = conductivity of metal

$$L = Z_0^{lossless}\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}$$

$$C = \sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}\,/\,Z_0^{lossless}$$

$$G = (\omega C)\tan\delta_d$$

$$R = R$$

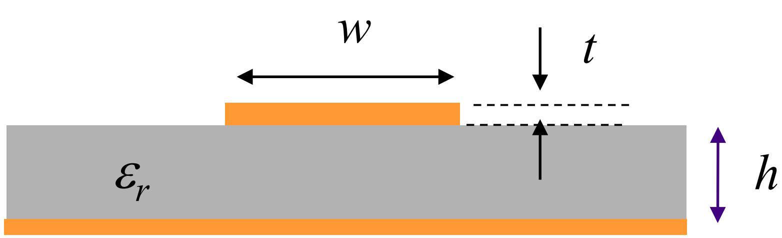
Common Transmission Lines (cont.)

Microstrip $(w/h \le 1)$

Approximate CAD formula

$$Z_0 = \frac{60}{\sqrt{\varepsilon_r^{eff}}} \ln\left(\frac{8h}{w} + \frac{w}{4h}\right)$$

$$\varepsilon_r^{eff} = \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\left(\frac{1}{\sqrt{1 + 12\dfrac{h}{w}}}\right)$$



Common Transmission Lines (cont.)

Microstrip $(w/h \geq 1)$

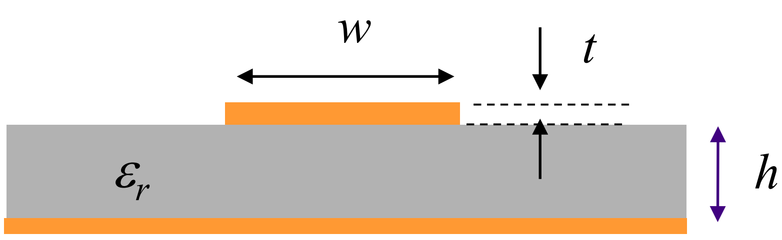
Approximate CAD formula

$$Z_0(f) = Z_0(0)\left(\frac{\varepsilon_r^{eff}(f)-1}{\varepsilon_r^{eff}(0)-1}\right)\sqrt{\frac{\varepsilon_r^{eff}(0)}{\varepsilon_r^{eff}(f)}}$$

$$Z_0(0) = \frac{120\pi}{\sqrt{\varepsilon_r^{eff}(0)}\left[(w'/h)+1.393+0.667\ln\left((w'/h)+1.444\right)\right]}$$

$$w' = w + \frac{t}{\pi}\left(1+\ln\left(\frac{2h}{t}\right)\right)$$

Note:
Usually $w/h > 1$ for a 50 Ω line.



Common Transmission Lines (cont.)

Microstrip $(w/h \geq 1)$

Approximate CAD formula

$$\varepsilon_r^{eff}(f) = \left(\sqrt{\varepsilon_r^{eff}(0)} + \frac{\sqrt{\varepsilon_r} - \sqrt{\varepsilon_r^{eff}(0)}}{1+4F^{-1.5}} \right)^2$$

$$\varepsilon_r^{eff}(0) = \frac{\varepsilon_r+1}{2} + \left(\frac{\varepsilon_r-1}{2}\right)\left(\frac{1}{\sqrt{1+12(h/w)}}\right) - \left(\frac{\varepsilon_r-1}{4.6}\right)\left(\frac{t/h}{\sqrt{w/h}}\right)$$

$$F = 4\left(\frac{h}{\lambda_0}\right)\sqrt{\varepsilon_r-1}\left(0.5+\left(1+0.868\ln\left(1+\frac{w}{h}\right)\right)^2\right)$$