Adapted from notes by
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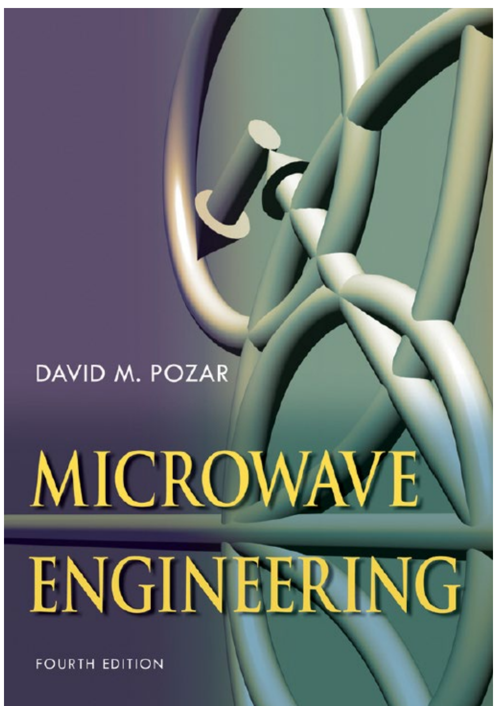
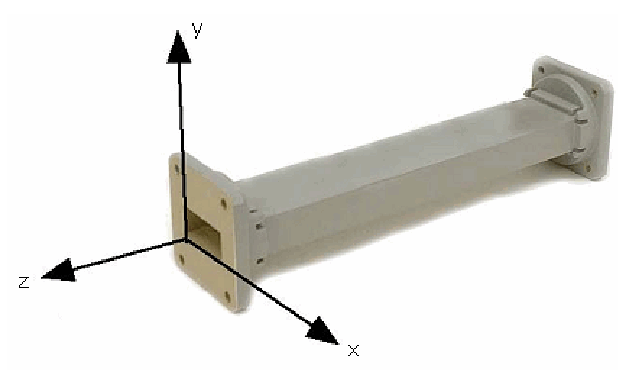


Notes 6

Waveguiding Structures
Part 1: General Theory



Waveguide Introduction

In general terms, a waveguiding system is a system that confines electromagnetic energy and channels it from one point to another.

(This is opposed to a wireless system that uses antennas.)

Examples

- Coax
- Twin lead (twisted pair)
- Printed circuit lines (e.g. microstrip)

Transmission Lines

- Parallel plate waveguide
- Rectangular waveguide
- Circular waveguide

Waveguides

- Optical fiber (dielectric waveguide)

Note: In microwave engineering, the term "waveguide" is often used to mean rectangular or circular waveguide (i.e., a hollow pipe of metal).

General Notation for Waveguiding Systems

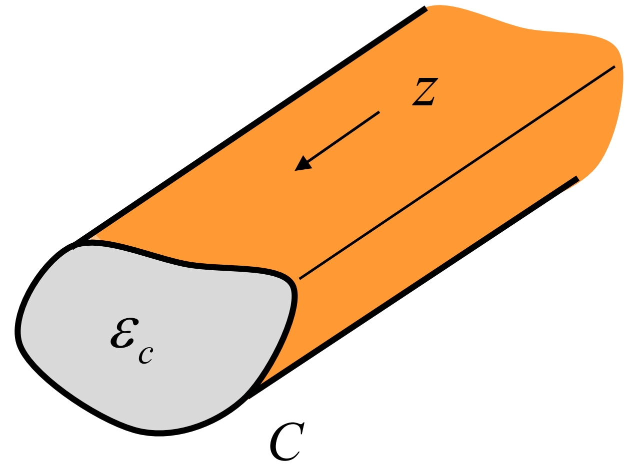
Assume $e^{j\omega t}$ time dependence and homogeneous source-free materials.

Assume wave propagation in the $\pm z$ direction:

$$e^{\mp \gamma z} = e^{\mp jk_z z}$$

$$\gamma = \alpha + j\beta, \quad k_z = \beta - j\alpha$$

$$\gamma = jk_z$$



Example of waveguiding system (a waveguide)

$$\underline{E}(x,y,z) = \left[\underline{e}_t(x,y) + \hat{z}\, e_z(x,y)\right] e^{\mp jk_z z}$$

Transverse (x,y) components

$$\underline{H}(x,y,z) = \left[\underline{h}_t(x,y) + \hat{z}\, h_z(x,y)\right] e^{\mp jk_z z}$$

$$\varepsilon_c \equiv \varepsilon - j\left(\frac{\sigma}{\omega}\right)$$

$$\varepsilon = \varepsilon' - j\varepsilon''$$

$$\varepsilon_c = \varepsilon_c' - j\varepsilon_c''$$

$$\tan\delta_d \equiv \frac{\varepsilon_c''}{\varepsilon_c'}$$

Note: Lower case letters denote 2-D fields (the z term is suppressed).

Helmholtz Equation

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \qquad \nabla \cdot \underline{E} = \frac{\rho_v}{\varepsilon}$$

$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E} + \underline{J} \qquad \nabla \cdot \underline{H} = 0$$

$$\Rightarrow \quad \begin{aligned} \nabla \times \nabla \times \underline{E} &= -j\omega\mu\left(\nabla \times \underline{H}\right) \\ &= -j\omega\mu\left(j\omega\varepsilon\underline{E} + \underline{J}\right) \\ &\;\; -j\omega\mu\left(j\omega\varepsilon\underline{E} + \sigma\underline{E}\right) \\ &= -j\omega\mu\left(j\omega\varepsilon_c\underline{E}\right) \\ &= k^2\underline{E} \end{aligned}$$

Recall: $\varepsilon_c \equiv \varepsilon - j\left(\dfrac{\sigma}{\omega}\right)$

where

$$k^2 \equiv \omega^2\mu\varepsilon_c \quad \text{(complex)}$$

Helmholtz Equation

$$\nabla \times \nabla \times \underline{E} = k^2 \underline{E}$$

Vector Laplacian definition: $\nabla^2 \underline{E} \equiv \nabla\left(\nabla \cdot \underline{E}\right) - \nabla \times \nabla \times \underline{E}$

where $\nabla^2 \underline{E} = \underline{\hat{x}}\left(\nabla^2 E_x\right) + \underline{\hat{y}}\left(\nabla^2 E_y\right) + \underline{\hat{z}}\left(\nabla^2 E_z\right)$

$$\Rightarrow \nabla\left(\nabla \cdot \underline{E}\right) - \nabla^2 \underline{E} = k^2 \underline{E}$$

$$\Rightarrow \nabla\left(\frac{\rho_v}{\varepsilon}\right) - \nabla^2 \underline{E} = k^2 \underline{E}$$

$$\Rightarrow \nabla^2 \underline{E} + k^2 \underline{E} = \nabla\left(\frac{\rho_v}{\varepsilon}\right)$$

Helmholtz Equation (cont.)

So far we have:

$$\nabla^2 \underline{E} + k^2 \underline{E} = \nabla \left(\frac{\rho_v}{\varepsilon} \right)$$

Next, we examine the term on the right-hand side.

Helmholtz Equation (cont.)

To do this, start with Ampere's law:

$$\nabla \times \underline{H} = j\omega\varepsilon_c \underline{E}$$

$$\Rightarrow \cancel{\nabla \cdot \left(\nabla \times \underline{H}\right)} = j\omega\varepsilon_c \left(\nabla \cdot \underline{E}\right)$$

$$\Rightarrow 0 = j\omega\varepsilon_c \left(\nabla \cdot \underline{E}\right)$$

$$\Rightarrow \nabla \cdot \underline{E} = 0$$

$$\Rightarrow \nabla \cdot \underline{D} = 0$$

$$\Rightarrow \rho_v = 0$$

In the time-harmonic (sinusoidal) steady state, there can never be any volume charge density inside of a linear, homogeneous, isotropic, source-free region that obeys Ohm's law.

Helmholtz Equation (cont.)

Hence, we have

$$\nabla^2 \underline{E} + k^2 \underline{E} = \underline{0}$$

Vector Helmholtz equation

Helmholtz Equation (cont.)

Similarly, for the magnetic field, we have

$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E} + \underline{J}$$

$$\Rightarrow \nabla \times \underline{H} = j\omega\varepsilon\underline{E} + \sigma\underline{E}$$

$$\Rightarrow \nabla \times \underline{H} = \left(\sigma + j\omega\varepsilon\right)\underline{E}$$

$$\Rightarrow \nabla \times \underline{H} = \left(j\omega\varepsilon_c\right)\underline{E}$$

Recall : $\varepsilon_c \equiv \varepsilon - j\left(\dfrac{\sigma}{\omega}\right)$

$$\Rightarrow \nabla \times \left(\nabla \times \underline{H}\right) = \left(j\omega\varepsilon_c\right)\nabla \times \underline{E}$$

$$\Rightarrow \nabla \times \left(\nabla \times \underline{H}\right) = \left(j\omega\varepsilon_c\right)\left(-j\omega\mu\underline{H}\right)$$

$$\Rightarrow \cancel{\nabla\left(\nabla \cdot \underline{H}\right)} - \nabla^2\underline{H} = \left(j\omega\varepsilon_c\right)\left(-j\omega\mu\underline{H}\right)$$

$$\Rightarrow -\nabla^2\underline{H} = k^2\underline{H}$$

Helmholtz Equation (cont.)

Hence, we have

$$\nabla^2 \underline{H} + k^2 \underline{H} = \underline{0}$$

Vector Helmholtz equation

Helmholtz Equation (cont.)

Summary

$$\nabla^2 \underline{E} + k^2 \underline{E} = \underline{0}$$

$$\nabla^2 \underline{H} + k^2 \underline{H} = \underline{0}$$

Vector Helmholtz equation

These equations are valid for a source-free, homogeneous, isotropic, linear material.

Helmholtz Equation (cont.)

From the property of the vector Laplacian, we have

$$\nabla^2 E_z + k^2 E_z = 0$$

$$\nabla^2 H_z + k^2 H_z = 0$$

Scalar Helmholtz equation

Recall: $$\nabla^2 \underline{E} = \underline{\hat{x}}\left(\nabla^2 E_x\right) + \underline{\hat{y}}\left(\nabla^2 E_y\right) + \underline{\hat{z}}\left(\nabla^2 E_z\right)$$

Field Representation

Assume a guided wave with a field variation $F(z)$ in the z direction of the form

$$F(z) = e^{\mp jk_z z}$$

(This is a property of any guided wave.)

Then all four of the transverse (x and y) field components can be expressed in terms of the two longitudinal ones:

$$(E_z, H_z)$$

Field Representation: Proof

Assume a source-free region with a variation $e^{\mp jk_z z}$

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \qquad \nabla \times \underline{H} = j\omega\varepsilon_c\underline{E}$$

Take (x,y,z) components

$$1)\ \frac{\partial E_z}{\partial y} \pm jk_z E_y = -j\omega\mu H_x \qquad 4)\ \frac{\partial H_z}{\partial y} \pm jk_z H_y = j\omega\varepsilon_c E_x$$

$$2)\ \mp jk_z E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \qquad 5)\ \mp jk_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_c E_y$$

$$3)\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad 6)\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\mu E_z$$

Field Representation: Proof (cont.)

Combining 1) and 5):

$$\Rightarrow \frac{\partial E_z}{\partial y} \pm jk_z \left(\frac{1}{j\omega\varepsilon_c} \right) \left(\mp jk_z H_x - \frac{\partial H_z}{\partial x} \right) = -j\omega\mu H_x$$

$$\Rightarrow \frac{\partial E_z}{\partial y} \mp \frac{k_z}{\omega\varepsilon_c} \frac{\partial H_z}{\partial x} = \left(-j\omega\mu - \frac{k_z^2}{j\omega\varepsilon_c} \right) H_x$$

$$\Rightarrow j\omega\varepsilon_c \frac{\partial E_z}{\partial y} \mp jk_z \frac{\partial H_z}{\partial x} = \underbrace{\left(k^2 - k_z^2\right)}_{k_c^2} H_x$$

$$\Rightarrow H_x = \frac{1}{k_c^2} \left(j\omega\varepsilon_c \frac{\partial E_z}{\partial y} \mp jk_z \frac{\partial H_z}{\partial x} \right)$$

$$k_c \equiv \sqrt{k^2 - k_z^2}$$

Cutoff wavenumber
(real number, as discussed later)

A similar derivation holds for the other three transverse field components.

Field Representation (cont.)

Summary of Results

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial y} \mp k_z \frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial x} \pm k_z \frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\pm k_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\mp k_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right)$$

These equations give the transverse field components in terms of the longitudinal components, E_z and H_z.

$$k^2 = \omega^2 \mu \varepsilon_c$$

$$k_c = \sqrt{k^2 - k_z^2}$$

Field Representation (cont.)

Therefore, we only need to solve the Helmholtz equations for the longitudinal field components (E_z and H_z).

$$\nabla^2 E_z + k^2 E_z = 0$$

$$\nabla^2 H_z + k^2 H_z = 0$$

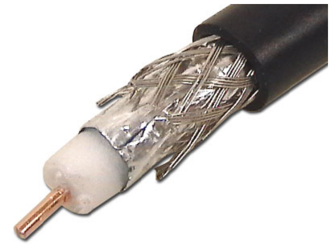
$$\Big\Downarrow \quad \text{From table}$$

$$E_x, E_y, H_x, H_y$$

Types of Waveguiding Systems

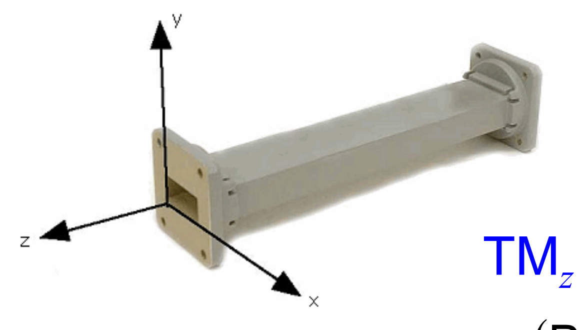
Types of guided waves:

- TEM_z: $E_z = 0, \quad H_z = 0$
- TM_z: $E_z \neq 0, \quad H_z = 0$
- TE_z: $E_z = 0, \quad H_z \neq 0$
- Hybrid: $E_z \neq 0, \quad H_z \neq 0$



TEM_z

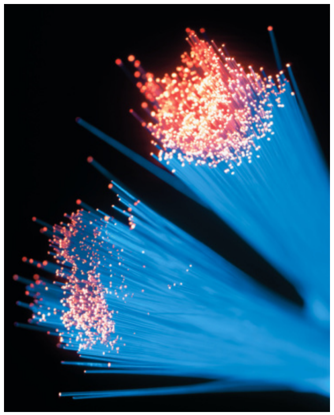
$(R = 0)$



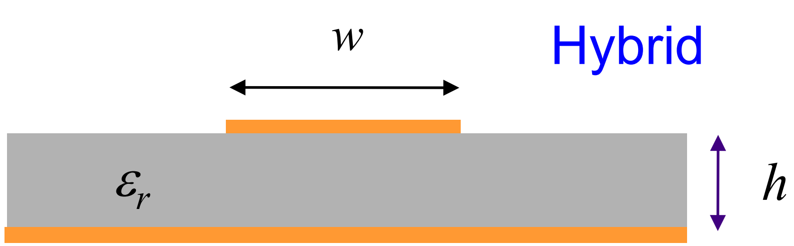
TM_z , TE_z

(PEC)

Hybrid



Hybrid

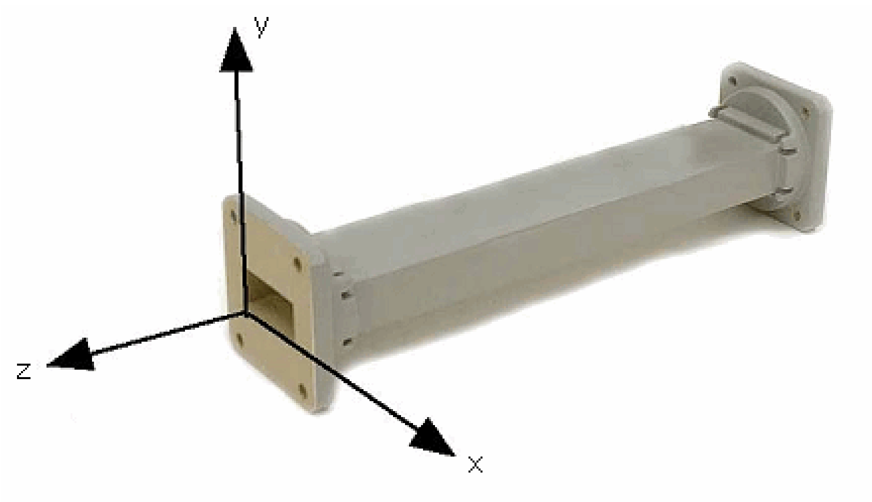


Microstrip

Waveguides

❖ We assume that the boundary is PEC.

❖ We assume that the inside is filled with a homogenous isotropic linear material (could be air)



An example of a waveguide (rectangular waveguide)

Two types of modes: TE_z , TM_z

Transverse Electric (TE_z) Waves

$\Rightarrow \; E_z = 0$ The electric field is "transverse" (perpendicular) to z.

In general, $E_x, E_y, H_x, H_y, H_z \neq 0$

To find the TE_z field solutions (away from any sources), solve

$$(\nabla^2 + k^2) H_z = 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) H_z = 0$$

Transverse Electric (TE_z) Waves (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)H_z=0$$

Recall that the field solutions we seek are assumed to vary as $F(z)=e^{\mp jk_z z}$

$$\Rightarrow H_z(x,y,z)=h_z(x,y)\,e^{\mp jk_z z}$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\underbrace{-k_z^2+k^2}_{k_c^2}\right)h_z(x,y)=0 \qquad k_c^2\equiv k^2-k_z^2$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)h_z(x,y)=0$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2\,h_z(x,y)$$

Transverse Electric (TE_z) Waves (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2\,h_z(x,y)$$

$\Downarrow$ Change notation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y)=\lambda\,\psi(x,y)$$

(2D Eigenvalue problem)

$\psi = h_z(x,y) =$ eigenfunction

$\lambda = -k_c^2 =$ eigenvalue

For this type of eigenvalue problem, the eigenvalue is always <u>real</u>.

(A proof of this may be found in the ECE 6340 notes.)

We need to solve the eigenvalue problem subject to the appropriate boundary conditions.

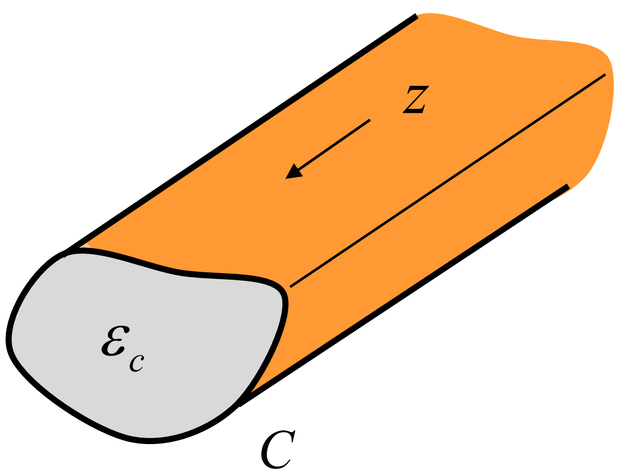
Transverse Electric (TE_z) Waves (cont.)

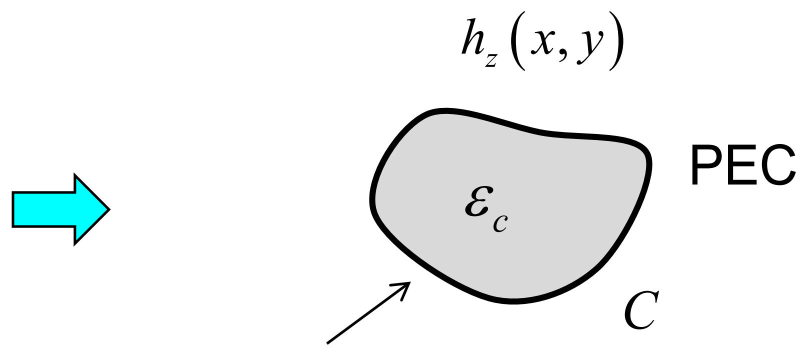
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y) = -k_c^2\,h_z(x,y)$$

Neumann boundary condition
(see below)

A solution to the eigenvalue problem can always be found for a PEC boundary (proof omitted).

Hence, TE_z modes exist inside of a waveguide (conducting pipe).





$$\frac{\partial h_z}{\partial n} = 0$$

Neumann boundary condition

Transverse Electric (TE_z) Waves (cont.)

Once the solution for H_z is obtained, we use

$$H_x = \mp \frac{jk_z}{k_c^2}\frac{\partial H_z}{\partial x} \qquad E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$H_y = \mp \frac{jk_z}{k_c^2}\frac{\partial H_z}{\partial y} \qquad E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

For a wave propagating in the positive z direction (top sign):

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{k_z}$$

For a wave propagating in the negative z direction (bottom sign):

$$-\frac{E_x}{H_y} = \frac{E_y}{H_x} = \frac{\omega\mu}{k_z}$$

TE wave impedance

$$\boxed{Z_{TE} \equiv \frac{\omega\mu}{k_z}}$$

Transverse Electric (TE_z) Waves (cont.)

For a wave propagating in the positive z direction, we also have:

$$\underline{e}_t(x,y) = \underline{\hat{x}}\, e_x(x,y) + \underline{\hat{y}}\, e_y(x,y)$$

$$\underline{\hat{z}} \times \underline{e}_t = \hat{y} e_x - \hat{x} e_y$$

$$\Rightarrow \underline{\hat{z}} \times \underline{e}_t = Z_{TE}\left(\hat{y} h_y + \hat{x} h_x\right)$$

$$= Z_{TE}\, \underline{h}_t$$

$$\Rightarrow \underline{h}_t = \frac{1}{Z_{TE}}(\underline{\hat{z}} \times \underline{e}_t)$$

Recall:

$$e_x = Z_{TE} h_y$$

$$e_y = -Z_{TE} h_x$$

Similarly, for a wave propagating in the negative z direction,

$$\underline{h}_t = \frac{1}{Z_{TE}}(-\underline{\hat{z}} \times \underline{e}_t)$$

Transverse Electric (TE_z) Waves (cont.)

Summarizing both cases, we have

$$\underline{h}_t(x,y) = \pm \frac{1}{Z_{TE}} \left(\underline{\hat{z}} \times \underline{e}_t(x,y) \right)$$

+ sign: wave propagating in the + z direction
- sign: wave propagating in the - z direction

Transverse Magnetic (TM_z) Waves

$\Rightarrow \; H_z = 0$ The magnetic field is "transverse" (perpendicular) to z.

In general, E_x, E_y, E_z ,H_x, $H_y \neq 0$

To find the TE_z field solutions (away from any sources), solve

$$(\nabla^2 + k^2) E_z = 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) E_z = 0$$

Transverse Magnetic (TM_z) Waves (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_z=0$$

Recall that the field solutions we seek are assumed to vary as $F(z)=e^{\mp jk_z z}$

$$\Rightarrow E_z(x,y,z)=e_z(x,y)\,e^{\mp jk_z z}$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\underbrace{-k_z^2+k^2}_{k_c^2}\right)e_z(x,y)=0 \qquad k_c^2\equiv k^2-k_z^2$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)e_z(x,y)=0$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_z(x,y)=-k_c^2e_z(x,y)$$

Transverse Electric (TE_z) Waves (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_z\left(x,y\right)=-k_c^2\,e_z\left(x,y\right)$$

(Eigenvalue problem)

$e_z\left(x,y\right)=$ eigenfunction

$-k_c^2 =$ eigenvalue

We need to solve the eigenvalue problem subject to the appropriate boundary conditions.

For this type of eigenvalue problem, the eigenvalue is always <u>real</u>.

(A proof of this may be found in the ECE 6340 notes.)

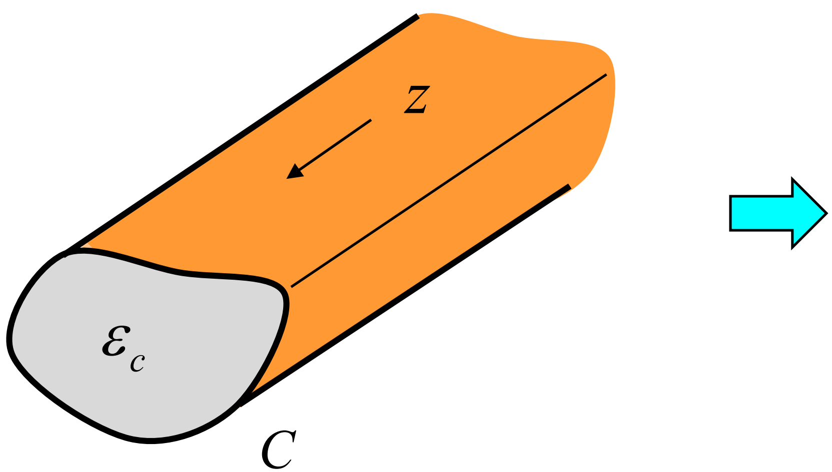
Transverse Magnetic (TM_z) Waves (cont.)

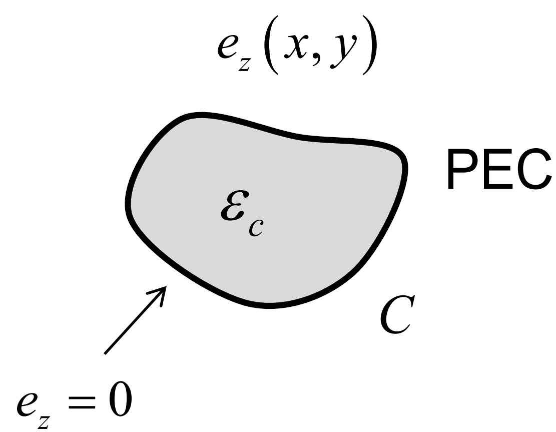
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_z(x,y)=-k_c^2\,e_z(x,y)$$

Dirichlet boundary condition
(see below)

A solution to the eigenvalue problem can always be found for a PEC boundary (proof omitted).

Hence, TM_z modes exist inside of a waveguide (conducting pipe).





Dirichlet boundary condition

Transverse Magnetic (TM_z) Waves (cont.)

Once the solution for E_z is obtained, we use

$$H_x = \frac{j\omega\varepsilon_c}{k_c^2}\frac{\partial E_z}{\partial y} \qquad E_x = \mp\frac{jk_z}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$H_y = -\frac{j\omega\varepsilon_c}{k_c^2}\frac{\partial E_z}{\partial x} \qquad E_y = \mp\frac{jk_z}{k_c^2}\frac{\partial E_z}{\partial y}$$

For a wave propagating in the positive z direction (top sign):

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{k_z}{\omega\varepsilon_c}$$

For a wave propagating in the negative z direction (bottom sign):

$$-\frac{E_x}{H_y} = \frac{E_y}{H_x} = \frac{k_z}{\omega\varepsilon_c}$$

TM wave impedance

$$Z_{TM} \equiv \frac{k_z}{\omega\varepsilon_c}$$

Transverse Magnetic (TM_z) Waves (cont.)

For a wave propagating in the positive z direction, we also have:

$$\underline{e}_t(x,y) = \underline{\hat{x}}e_x(x,y) + \underline{\hat{y}}e_y(x,y)$$

$$\underline{\hat{z}} \times \underline{e}_t = \hat{y}e_x - \hat{x}e_y$$

$$\Rightarrow \underline{\hat{z}} \times \underline{e}_t = Z_{TM}\left(\hat{y}h_y + \hat{x}h_x\right)$$

$$= Z_{TM}\,\underline{h}_t$$

$$\Rightarrow \underline{h}_t = \frac{1}{Z_{TM}}(\underline{\hat{z}} \times \underline{e}_t)$$

Recall:

$$e_x = Z_{TM}h_y$$

$$e_y = -Z_{TM}h_x$$

Similarly, for a wave propagating in the negative z direction,

$$\underline{h}_t = \frac{1}{Z_{TM}}(-\underline{\hat{z}} \times \underline{e}_t)$$

Transverse Magnetic (TM_z) Waves (cont.)

Summarizing both cases, we have

$$\underline{h}_t(x,y) = \pm \frac{1}{Z_{TM}}\left(\underline{\hat{z}} \times \underline{e}_t(x,y)\right)$$

+ sign: wave propagating in the + z direction
- sign: wave propagating in the - z direction

Transverse ElectroMagnetic (TEM) Waves

$$\Rightarrow \; E_z = 0,\; H_z = 0$$

In general, $E_x, E_y, H_x, H_y \neq 0$

From the previous table for the transverse field components, all of them are equal to zero if E_z and H_z are both zero.

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial y} \mp k_z \frac{\partial H_z}{\partial x} \right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial x} \pm k_z \frac{\partial H_z}{\partial y} \right)$$

$$E_x = \frac{-j}{k_c^2}\left(\pm k_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y} \right)$$

$$E_y = \frac{j}{k_c^2}\left(\mp k_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x} \right)$$

<u>Unless</u> $\boxed{k_c^2 = 0}$

⇨ For TEM waves $k_c^2 \equiv k^2 - k_z^2 = 0$

Hence, we have

$$\boxed{k_z = k = \omega\sqrt{\mu\varepsilon_c}}$$

Transverse ElectroMagnetic (TEM) Waves (cont.)

$$\underline{H}(x,y,z) = \underline{h}(x,y)\,e^{-jk_z z} = \underline{h}_t(x,y)\,e^{-jkz}$$

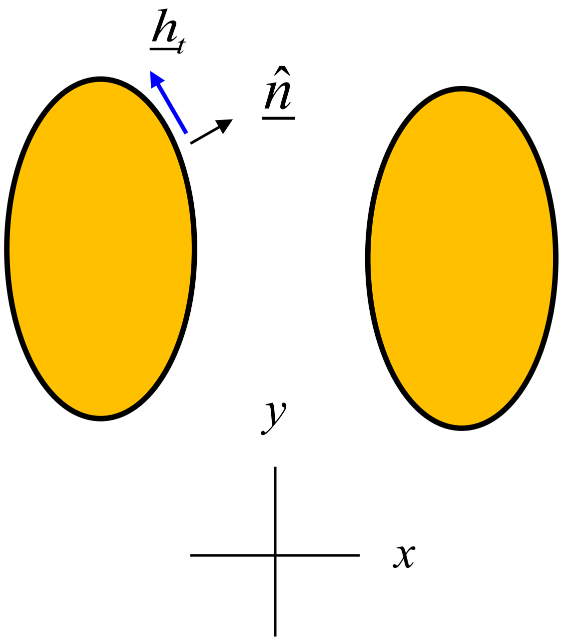
From EM boundary conditions, we have:

$$\underline{J}_s = \underline{\hat{n}} \times \underline{H}$$

so

$$\underline{J}_s = \left(\underline{\hat{n}} \times \underline{h}_t(x,y)\right)e^{-jkz}$$

$$\Rightarrow \quad \boxed{\underline{J}_s = \underline{\hat{z}}\, J_{sz}}$$



The current flows purely in the z direction.

Transverse ElectroMagnetic (TEM) Waves (cont.)

In a linear, isotropic, homogeneous source-free region,

$$\nabla \cdot \underline{E} = 0$$

In rectangular coordinates, we have

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \cancel{\frac{\partial E_z}{\partial z}} = 0$$

Notation:

$$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

$$\nabla_t \cdot \underline{E} = 0$$

$$\Rightarrow \nabla_t \cdot \left(\underline{e}_t(x,y) e^{\mp j k_z z} \right) = 0$$

$$\Rightarrow e^{\mp j k_z z} \nabla_t \cdot \left(\underline{e}_t(x,y) \right) - \cancel{\underline{e}_t(x,y) \cdot \nabla \left(e^{\mp j k_z z} \right)} = 0$$

$$\Rightarrow \nabla_t \cdot \left(\underline{e}_t(x,y) \right) = 0$$

Hence, we have

$$\boxed{\nabla_t \cdot \left(\underline{e}_t(x,y) \right) = 0}$$

Transverse ElectroMagnetic (TEM) Waves (cont.)

Also, for the TEM$_z$ mode, we have from Faraday's law (taking the z component):

$$\underline{\hat{z}} \cdot \left(\nabla \times \underline{E} \right) = \underline{\hat{z}} \cdot \left(-j\omega\mu \underline{H} \right) = -j\omega\mu \cancel{H_z} = 0$$

Using the formula for the z component of the curl of $\underline{E}$, we have:

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

Hence

$$\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y} = 0$$

Note:

$$\nabla_t = \underline{\hat{x}} \frac{\partial}{\partial x} + \underline{\hat{y}} \frac{\partial}{\partial y}$$

$$\underline{e}_t(x, y) = \underline{\hat{x}} e_x(x, y) + \underline{\hat{y}} e_y(x, y)$$

$$\Downarrow$$

$$\nabla_t \times \underline{e}_t(x, y) = \underline{\hat{z}} \left(\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y} \right)$$

Hence, we have:

$$\boxed{\nabla_t \times \left(\underline{e}_t(x, y) \right) = \underline{0}}$$

Transverse ElectroMagnetic (TEM) Waves (cont.)

$$\nabla_t \times \left(\underline{e}_t \left(x, y \right) \right) = \underline{0}$$

$$\Downarrow$$

$$\boxed{\underline{e}_t \left(x, y \right) = -\nabla_t \Phi \left(x, y \right)}$$

$$\Downarrow$$

$$\nabla_t \cdot \left(\underline{e}_t \left(x, y \right) \right) = 0 \quad \Rightarrow \quad \nabla_t \cdot \left(-\nabla_t \Phi \left(x, y \right) \right) = 0$$

Hence

$$\boxed{\nabla_t^2 \Phi \left(x, y \right) = 0}$$

Transverse ElectroMagnetic (TEM) Waves (cont.)

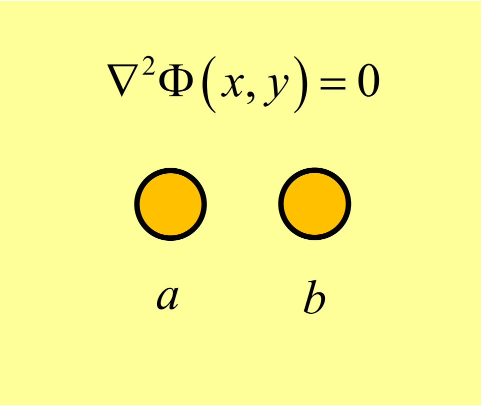
Since the potential function that describes the electric field in the cross-sectional plane is two dimensional, we can drop the "t" subscript if we wish:

$$\nabla^2 \Phi(x, y) = 0$$

Boundary Conditions:

$$\Phi(x, y) = \Phi_a \quad \text{conductor "}a\text{"}$$

$$\Phi(x, y) = \Phi_b \quad \text{conductor "}b\text{"}$$



PEC conductors

This is enough to make the potential function <u>unique</u>.

Hence, the potential function is the same for DC as it is for a high-frequency microwave signal.

> The field of a TEM mode <u>does not change shape with frequency</u>: it has the same shape as a DC field.

Transverse ElectroMagnetic (TEM) Waves (cont.)

Notes:

- A TEM_z mode has an electric field that has exactly the same shape as a static (DC) field. (A similar proof holds for the magnetic field.)
- This implies that the C and L for the TEM_z mode on a transmission line are independent of frequency.
- This also implies that the voltage drop between the two conductors of a transmission line carrying a TEM_z mode is path independent.
- A TEM_z mode requires two or more conductors: a static electric field cannot exist inside of a waveguide (hollow metal pipe) due to the Faraday cage effect.

Transverse ElectroMagnetic (TEM) Waves (cont.)

For a TEM mode, both wave impedances are the same:

Recall: $k_z = k$

$$Z_{TE} = \frac{\omega\mu}{k_z} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon_c}} = \eta$$

$$Z_{TM} = \frac{k_z}{\omega\varepsilon_c} = \frac{k}{\omega\varepsilon_c} = \frac{\omega\sqrt{\mu\varepsilon_c}}{\omega\varepsilon_c} = \sqrt{\frac{\mu}{\varepsilon_c}} = \eta$$

Note: η is complex for lossy media.

TEM Solution Process

A) Solve Laplace's equation subject to appropriate B.C.s.:

$$\nabla^2 \Phi(x, y) = 0$$

B) Find the transverse electric field: $\underline{e}_t(x, y) = -\nabla \Phi(x, y)$

C) Find the total electric field: $\underline{E}(x, y, z) = \underline{e}_t(x, y)\, e^{\mp j k_z z}, \;\; k_z = k$

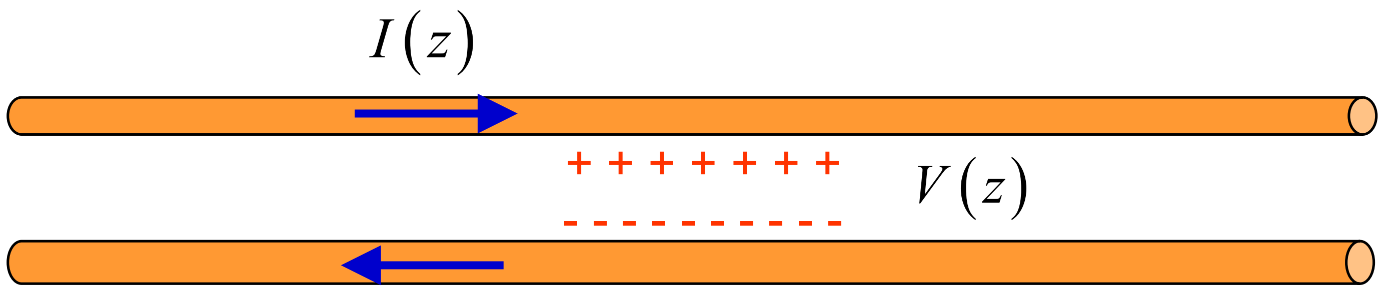
D) Find the magnetic field: $\underline{H} = \dfrac{1}{\eta}\left(\pm \hat{z} \times \underline{E}\right); \quad \pm z$ propagating

Note: The only frequency dependence is in the wavenumber $k_z = k$.

Note on Lossy Transmission Line

- A TEM mode can have an arbitrary amount of dielectric loss.
- A TEM mode cannot have conductor loss.

Inside conductors: $\underline{J} = \sigma \underline{E}$



$$\boxed{J_z \Rightarrow E_z}$$

If there is conductor loss, there must be an E_z field.

In practice, a small conductor loss will not change the shape of the fields too much, and the mode is approximately TEM.

Note on Lossy Transmission Line (cont.)

If there is only dielectric loss (an exact TEM mode):

$$\gamma = \sqrt{\left(\cancel{R} + j\omega L\right)\left(G + j\omega C\right)}$$

$$\gamma = jk_z = jk \quad \left(k_z = k\right)$$

$$\gamma = \sqrt{\left(j\omega L\right)\left(G + j\omega C\right)}$$

$$\begin{aligned}\gamma &= jk \\ &= j\omega\sqrt{\mu\varepsilon_c} \\ &= j\omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r\left(1 - j\tan\delta_d\right)}\end{aligned}$$

$$\left(j\omega L\right)\left(G + j\omega C\right) = \left(-\omega^2\right)\mu_0\mu_r\varepsilon_0\varepsilon_r\left(1 - j\tan\delta_d\right)$$

Equate real and imaginary parts

$$LC = \mu_0\mu_r\varepsilon_0\varepsilon_r$$

$$\left(\omega L\right)\left(G\right) = \omega^2\mu_0\mu_r\varepsilon_0\varepsilon_r\left(\tan\delta_d\right)$$

Note: ε_r denotes ε_r' (real)

Note on Lossy Transmission Line (cont.)

Equations for a TEM mode:

$$LC = \mu_0 \mu_r \varepsilon_0 \varepsilon_r$$

$$(\omega L)(G) = \omega^2 \mu_0 \mu_r \varepsilon_0 \varepsilon_r (\tan \delta_d)$$

$$\Rightarrow \quad \frac{G}{\omega C} = \frac{\omega^2 \mu_0 \mu_r \varepsilon_0 \varepsilon_r (\tan \delta_d)}{\omega^2 LC} = \frac{\mu_0 \mu_r \varepsilon_0 \varepsilon_r (\tan \delta_d)}{LC} = \frac{\mu_0 \mu_r \varepsilon_0 \varepsilon_r (\tan \delta_d)}{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}$$

From these two equations we have:

$$LC = \mu_0 \mu_r \varepsilon_0 \varepsilon_r$$

$$\frac{G}{\omega C} = \tan \delta_d$$

Note:
These formulas were assumed previously in Notes 3.

Note on Lossy Transmission Line (cont.)

This general formula accounts for <u>both dielectric and conductor loss</u>:

$$\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)}$$

Conductor loss

Dielectric loss

When R is present the mode is not <u>exactly</u> TEM, but we usually ignore this.

Summary

TEM Mode:

$$E_z = H_z = 0$$

$$k_z = k = k_0\sqrt{\mu_r \varepsilon_{rc}} = k_0\sqrt{\mu_r \varepsilon_r \left(1 - j\tan\delta_d\right)}$$

$$\underline{h}_t\left(x,y\right) = \pm\frac{1}{\eta}\left(\underline{\hat{z}} \times \underline{e}_t\left(x,y\right)\right)$$

$$\varepsilon_{rc} = \varepsilon_r\left(1 - j\tan\delta_d\right)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_{rc}}}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7303\ \Omega$$

Summary (cont.)

Transmission line mode (approximate TEM mode):

- The mode requires <u>two</u> conductors.
- The mode is purely TEM only when $R = 0$.

$$\gamma = jk_z = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\left.\begin{aligned} LC &= \mu_0 \mu_r \varepsilon_0 \varepsilon_r \\ \frac{G}{\omega C} &= \tan \delta_d \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} L &= Z_0^{lossless} \sqrt{\mu_0 \varepsilon_0} \sqrt{\mu_r \varepsilon_r} \\ C &= \sqrt{\mu_0 \varepsilon_0} \sqrt{\mu_r \varepsilon_r} \,/\, Z_0^{lossless} \\ G &= (\omega C) \tan \delta_d \\ R &= R \end{aligned}$$

$$\underline{J}_s = \underline{\hat{z}}\, J_{sz}$$

$$Z_0^{lossless} \equiv \sqrt{\frac{L}{C}}$$

Summary (cont.)

TE_z Mode:

❖ The mode can exist inside of a <u>single</u> pipe (waveguide).

$$E_z = 0,\ H_z \neq 0$$

$$k_z = \sqrt{k^2 - k_c^2} \qquad k = k_0\sqrt{\mu_r \varepsilon_{rc}} = k_0\sqrt{\mu_r \varepsilon_r\left(1 - j\tan\delta_d\right)}$$

$k_c =$ real number (depends on geometry and mode number)

$$\underline{h}_t\left(x,y\right) = \pm\frac{1}{Z_{TE}}\left(\underline{\hat{z}}\times\underline{e}_t\left(x,y\right)\right) \qquad Z_{TE} = \frac{\omega\mu}{k_z}$$

$$\nabla_t^2 h_z\left(x,y\right) = -k_c^2\, h_z\left(x,y\right)$$

Summary (cont.)

TM$_z$ Mode:

❖ The mode can exist inside a <u>single</u> pipe (waveguide).

$$H_z = 0,\ E_z \neq 0$$

$$k_z = \sqrt{k^2 - k_c^2} \qquad k = k_0\sqrt{\mu_r \varepsilon_{rc}} = k_0\sqrt{\mu_r \varepsilon_r \left(1 - j\tan\delta_d\right)}$$

$k_c =$ real number (depends on geometry and mode number)

$$\underline{h}_t(x,y) = \pm\frac{1}{Z_{TM}}\left(\underline{\hat{z}} \times \underline{e}_t(x,y)\right) \qquad Z_{TM} = \frac{k_z}{\omega\varepsilon}$$

$$\nabla_t^2 e_z(x,y) = -k_c^2\, e_z(x,y)$$