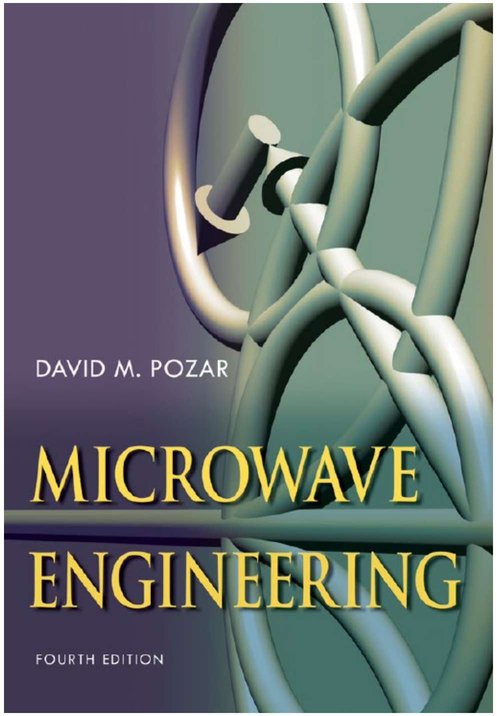
Adapted from notes by
Prof. Jeffery T. Williams

ECE 5317-6351
Microwave Engineering

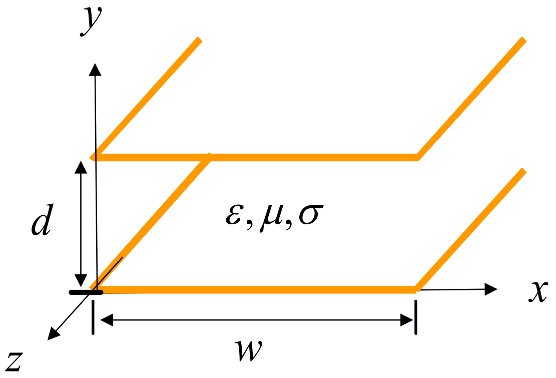
Fall 2019

Prof. David R. Jackson
Dept. of ECE



Notes 8

Waveguiding Structures
Part 3: Parallel Plates



Field Equations (from Notes 6)

Summary

$$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial y} \mp k_z \frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial x} \pm k_z \frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\pm k_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\mp k_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right)$$

This table of fields will be useful to us in the present discussion.

Assumption:

$$e^{\mp j k_z z}$$

$$k^2 = \omega^2 \mu \varepsilon_c$$

(k can be complex)

$$k_c = \sqrt{k^2 - k_z^2}$$

(k_c is always real)

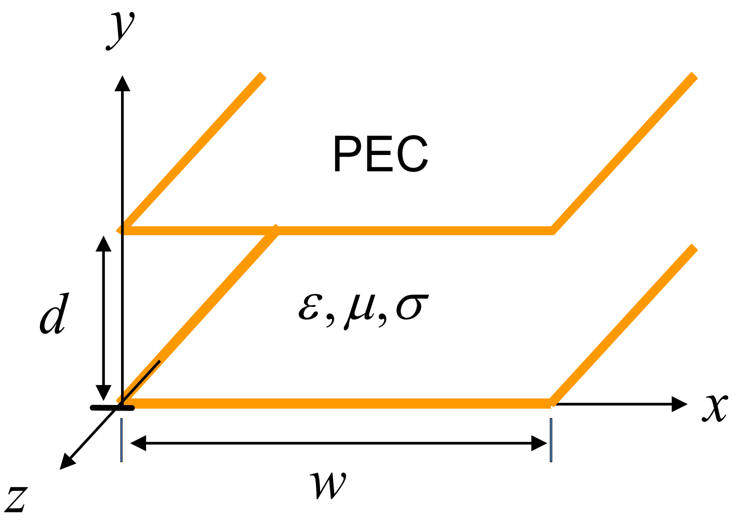
$$\begin{aligned}
\varepsilon_c &= \varepsilon - j\frac{\sigma}{\omega} \\
&= \varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega} \\
&= \varepsilon_c' - j\varepsilon_c'' \\
&= \varepsilon_c'\left(1 - j\frac{\varepsilon_c''}{\varepsilon_c'}\right) \\
&= \varepsilon_c'(1 - j\tan\delta_d) \\
&= \varepsilon_0\varepsilon_r(1 - j\tan\delta\ )
\end{aligned}$$

Parallel-Plate Waveguiding Structure

- Both plates assumed PEC
- $w >> d$

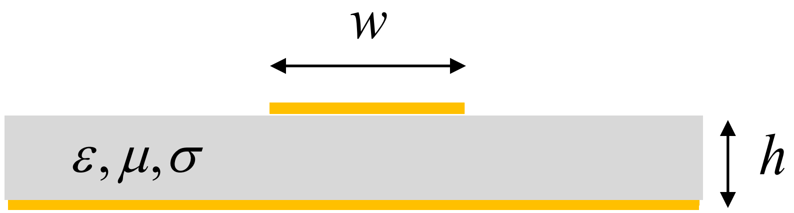
$$\Rightarrow \frac{\partial}{\partial x} \approx 0$$

(neglect edge effects)



The parallel-plate structure is a good approximate model for a wide microstrip line.

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$



TEM Solution Process

A) Solve Laplace's equation subject to appropriate B.C.s.:

$$\nabla^2\Phi(x,y)=0$$

B) Find the transverse electric field: $\underline{e}_t(x,y)=-\nabla\Phi(x,y)$

C) Find the total electric field: $\underline{E}(x,y,z)=\underline{e}_t(x,y)e^{\mp jk_z z},\ k_z=k$

D) Find the magnetic field: $\underline{H}=\dfrac{1}{\eta}(\pm\hat{z}\times\underline{E});\quad \pm z$ propagation

Note: The only frequency dependence is in the wavenumber $k_z=k$.

TEM Mode

2 conductors $\Rightarrow$ 1 TEM mode

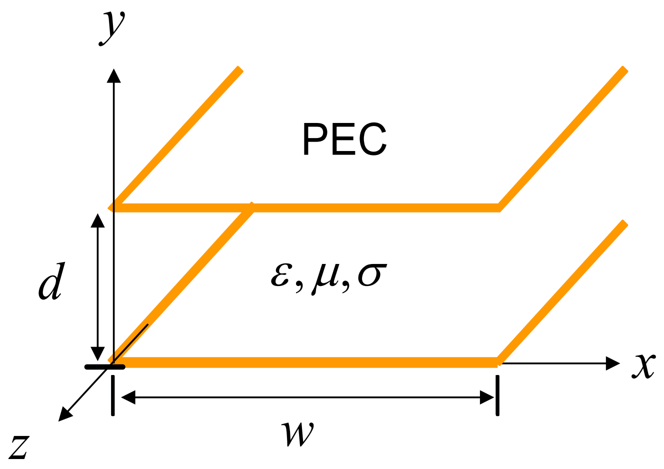
To solve for TEM mode:

$$\nabla^2 \Phi(x, y) = 0$$

$$\nabla^2 \Phi = \left(\cancel{\frac{\partial^2}{\partial x^2}} + \frac{\partial^2}{\partial y^2} \right) \Phi = 0 \qquad \begin{array}{l} 0 \le x \le w \\ 0 \le y \le d \end{array}$$

Boundary conditions:

$$\Phi(x,0) = 0 \;\; ; \;\; \Phi(x,d) = V_0$$



$$k_z = \beta - j\alpha = k = \omega\sqrt{\mu\varepsilon_c} = k' - jk''$$

$$\beta = k'$$
$$\alpha = k''$$

TEM Mode (cont.)

$$\frac{\partial^2 \Phi}{\partial y^2} = 0$$ where $$\Phi(x,0) = 0 \quad \& \quad \Phi(x,d) = V_0$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\Phi(x,y) = A + By \qquad A = 0 \qquad B = \frac{V_0}{d}$$

Hence $$\boxed{\Phi(x,y) = \frac{V_0}{d} y}$$

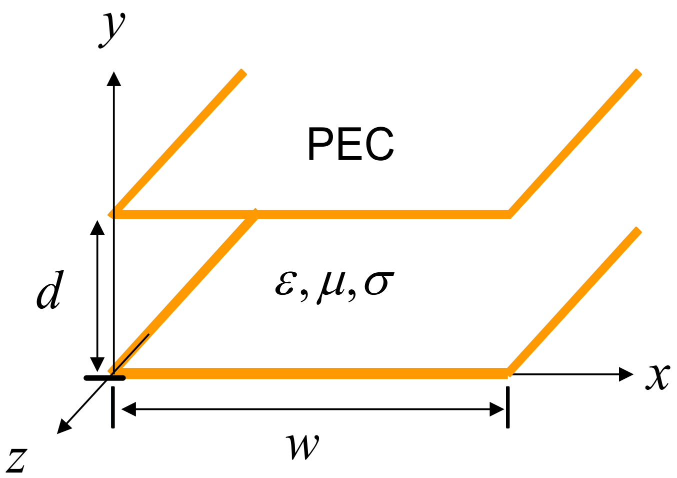
We then have

$$\underline{e}_t(x,y) = -\nabla\Phi = -\hat{\underline{y}} \frac{\partial}{\partial y} \Phi = -\hat{\underline{y}} \frac{V_0}{d}$$

$$\Rightarrow \quad \underline{E}(x,y,z) = \underline{e}_t(x,y) e^{\mp jkz} = -\hat{y} \frac{V_0}{d} e^{\mp jkz}$$

TEM Mode (cont.)

$$\underline{E}(x,y,z) = -\hat{y}\frac{V_0}{d}e^{\mp jkz}$$



Recall

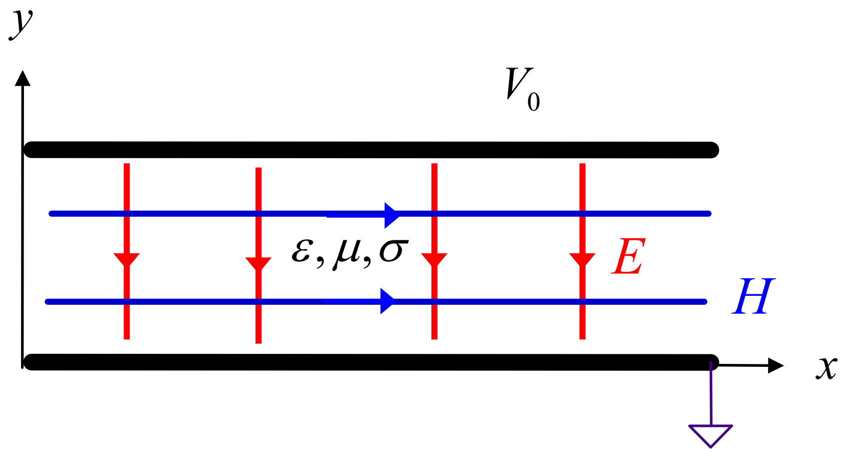
$$\underline{H} = \frac{1}{\eta}(\pm\hat{z} \times \underline{E})$$

$$\Rightarrow \underline{H}(x,y,z) = \pm\hat{x}\frac{V_0}{\eta d}e^{\mp jkz}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

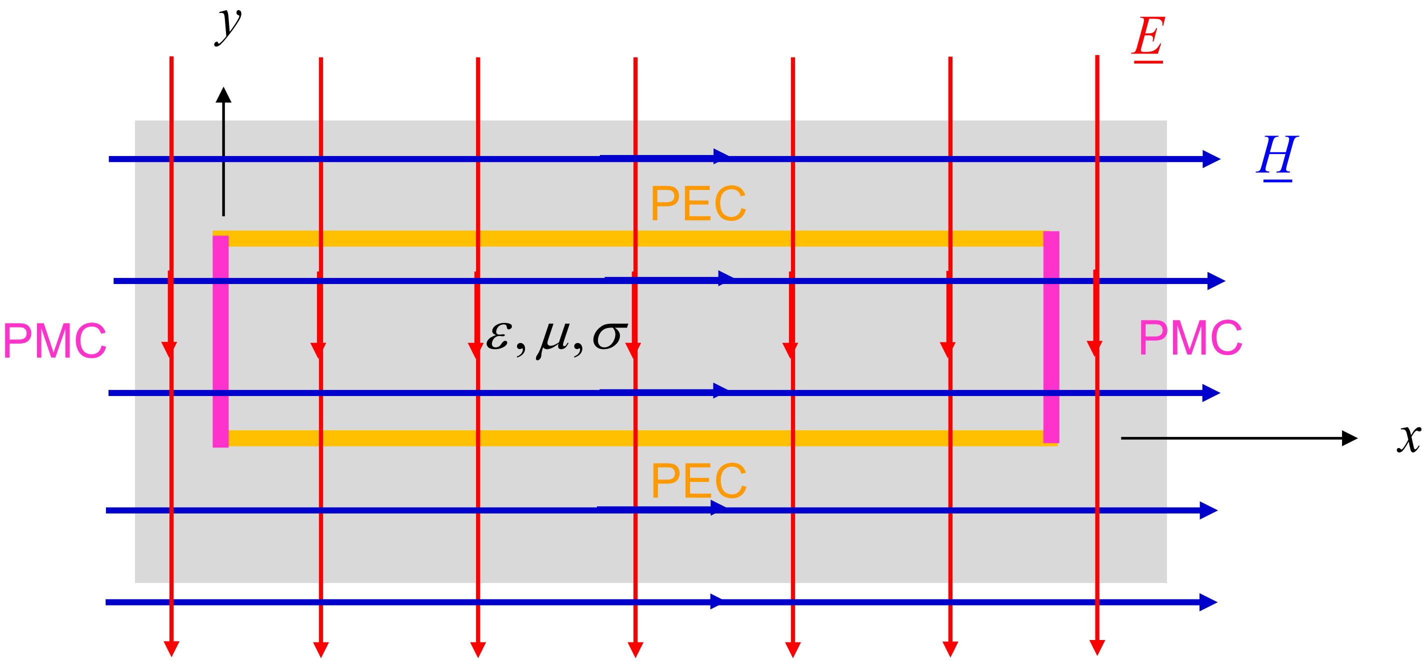
$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

Fields for $+z$ mode:



TEM Mode (cont.)

We can view the TEM mode in a parallel-plate waveguiding structure as a rectangular “slice” of a plane wave.



The PEC and PMC walls do not disturb the fields of the plane wave.

PEC : $\underline{\hat{n}} \times \underline{E} = \underline{0}$

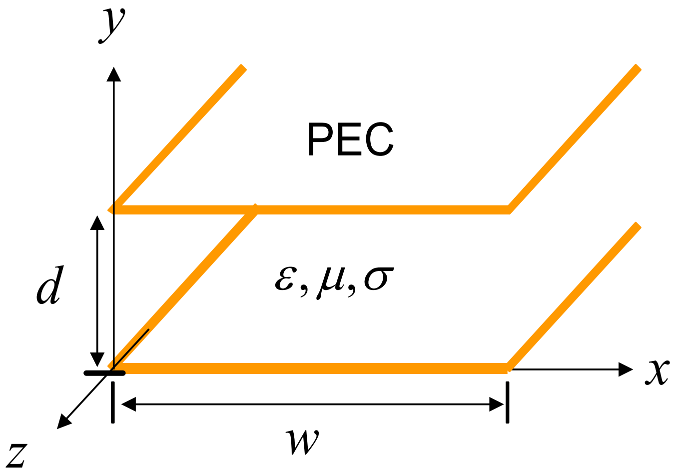
PMC : $\underline{\hat{n}} \times \underline{H} = \underline{0}$

PEC: Perfect Electric Conductor

PMC: Perfect Magnetic Conductor

TEM Mode (cont.)

Assume a wave propagating the in + z direction henceforth.



$$\underline{E}(x, y, z) = -\hat{y}\frac{V_0}{d}e^{-jkz}$$

$$\underline{H}\left(x, y, z\right) = \hat{x}\frac{V_0}{\eta d}e^{-jkz}$$

Time-ave power flow in $+z$ direction:

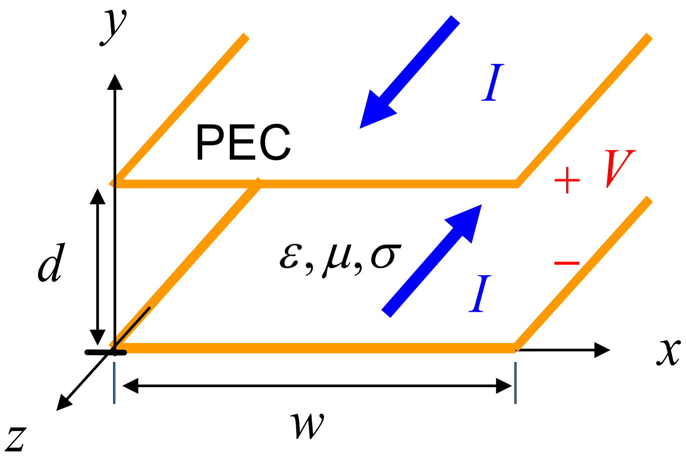
$$P^{+} = \frac{1}{2}\mathrm{Re}\left\{\iint_{s}(\underline{E}\times\underline{H}^{*})\cdot\underline{\hat{z}}\,dS\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\int_0^w\int_0^d\left(\frac{|V_0|^2}{\eta^* d^2}\,\underline{\hat{z}}\right)\cdot\underline{\hat{z}}\;e^{-2k''z}dydx\right\}$$

$$= \frac{1}{2}|V_0|^2\left(\frac{1}{d^2}\right)(wd)\mathrm{Re}\left(\frac{1}{\eta^*}\right)e^{-2k''z}$$

$$P^{+} = \frac{1}{2}|V_0|^2\left(\frac{w}{d}\right)\mathrm{Re}\left(\frac{1}{\eta^*}\right)e^{-2k''z}$$

TEM Mode (cont.)



Transmission line voltage

$$V(z) = \int_{d}^{0} \underline{E} \cdot \left(\underline{\hat{y}}\, dy \right)$$

$$\Rightarrow \quad V(z) = V_0\, e^{-jkz}$$

Transmission line current

$$I(z) = \int_{0}^{w} J_{sz}^{\text{top}}\, dx = \int_{0}^{w} H_x(x, d, z)\, dx$$

$$\Rightarrow I(z) = \overbrace{\frac{V_0}{\eta d} w}^{I_0} e^{-jkz}$$

Characteristic Impedance

$$Z_0 = \frac{V_0}{I_0}\frac{e^{-jkz}}{e^{-jkz}}$$

$$\boxed{Z_0 = \eta \frac{d}{w}} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon_c}}$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

Note : On PEC $\underline{J}_s = \underline{\hat{n}} \times \underline{H} \quad \Rightarrow \quad J_{sz}^{\text{top}} = H_x$ (on top plate)

TEM Mode (cont.)

Time-ave power flow in $+z$ direction:

(calculated using the voltage and current)

$$P^{+} = \frac{1}{2}\mathrm{Re}\left\{VI^{*}\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{V_0\left(\frac{V_0}{\eta d}w\right)^{*} e^{-2k''z}\right\}$$

$$P^{+} = \frac{1}{2}\left|V_0\right|^{2}\left(\frac{w}{d}\right)\mathrm{Re}\left(\frac{1}{\eta^{*}}\right)e^{-2k''z}$$

Recall that we found from the fields that:

Same

$$P^{+} = \frac{1}{2}\left|V_0\right|^{2}\left(\frac{w}{d}\right)\mathrm{Re}\left(\frac{1}{\eta^{*}}\right)e^{-2k''z}$$

This is expected, since a TEM mode is a transmission-line type of mode, which is described by voltage and current.

TM$_z$ Modes ($H_z = 0$)

Recall:

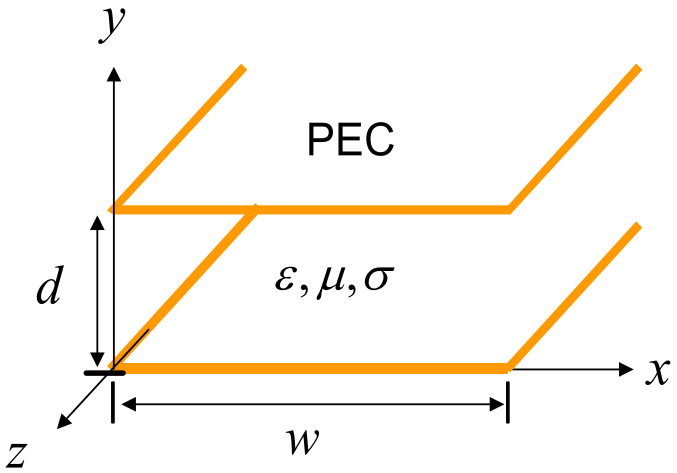
$$E_z(x,y,z) = e_z(x,y)\,e^{-jk_z z}$$

where

$$\nabla_t^2 e_z(x,y) = -k_c^2 e_z(x,y)$$

eigenvalue problem

$$k_c \equiv \sqrt{k^2 - k_z^2}$$



Note:
Solving the eigenvalue problem
(using appropriate boundary conditions)
will tell us what the eigenvalue k_c is.

$$\left(\cancel{\frac{\partial^2}{\partial x^2}} + \frac{\partial^2}{\partial y^2}\right) e_z(x,y) = -k_c^2 e_z(x,y)$$

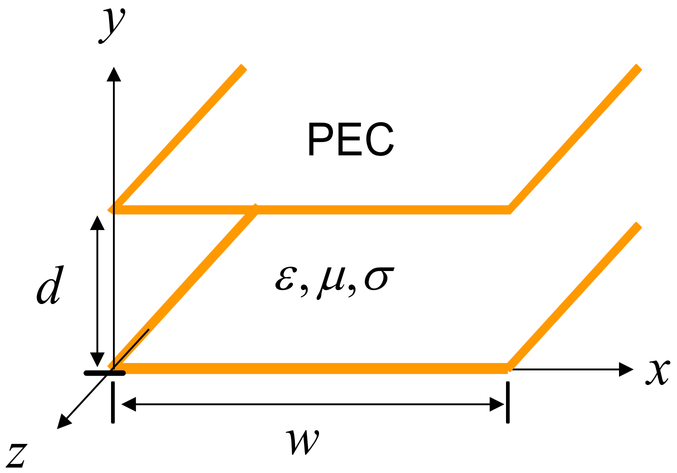
(Assume no x variation)

so $$\frac{d^2}{dy^2} e_z(y) = -k_c^2 e_z(y)$$

TM$_z$ Modes (cont.)

$$\frac{d^2}{dy^2} e_z(y) = -k_c^2 e_z(y)$$

subject to B.C.'s $E_z = 0 \quad @\, y = 0, d$



Solving the above equation:

$$e_z(y) = A\sin(k_c y) + B\cos(k_c y)$$

Apply B.C.'s :

$$@\, y = 0 \quad \Rightarrow B = 0$$

$$@\, y = d \quad \Rightarrow k_c d = n\pi \qquad n = 1, 2, \ldots \quad \Rightarrow k_c = \frac{n\pi}{d}$$

TM$_z$ Modes (cont.)

$$e_z(y) = A\sin\left(\frac{n\pi}{d}y\right) \qquad n = 1, 2, \ldots$$

$$\Rightarrow E_z(y,z) = A_n \sin\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$k_z = \left(k^2 - k_c^2\right)^{1/2} = \left(k^2 - \left(\frac{n\pi}{d}\right)^2\right)^{1/2}$$

$k^2 = \omega^2 \mu \varepsilon_c$

For a wave propagating in the $+z$ direction:

$$H_x = \frac{j\omega\varepsilon_c}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\varepsilon_c}{k_c^2}A_n\left(\frac{n\pi}{d}\right)\cos\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$E_y = -\frac{jk_z}{k_c^2}\frac{\partial E_z}{\partial y} = -\frac{jk_z}{k_c^2}A_n\left(\frac{n\pi}{d}\right)\cos\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$E_x = 0 \qquad\qquad H_y = 0 \qquad\qquad H_z = 0$$

No x variation

TM$_z$ mode

TM$_z$ Modes (cont.)

Summary

$$E_z = A_n \sin\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$E_y = -\frac{jk_z}{k_c} A_n \cos\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

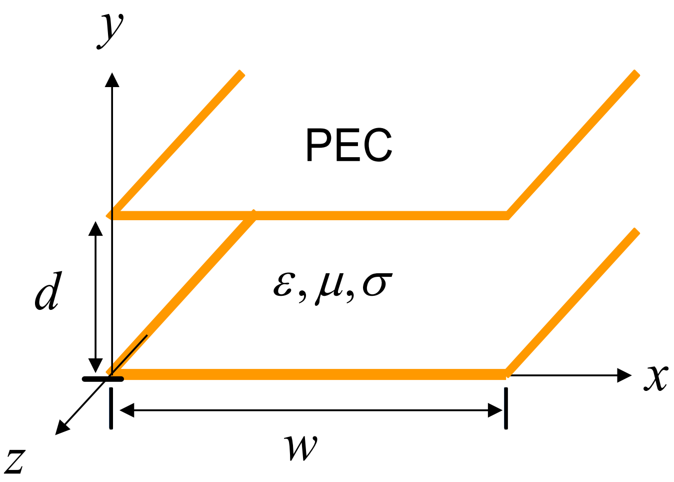
$$H_x = \frac{j\omega\varepsilon_c}{k_c} A_n \cos\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$E_x = H_y = H_z = 0$$

$$k_c = \frac{n\pi}{d} \; ; \quad n = 1, 2, \ldots$$

$$k_z = \left(k^2 - \left(\frac{n\pi}{d}\right)^2\right)^{1/2}$$

$$k^2 = \omega^2 \mu \varepsilon_c$$



Each value of n corresponds to a unique TM$_z$ field solution or "mode" in the waveguide.

$\Rightarrow$ TM$_n$ mode

Note:

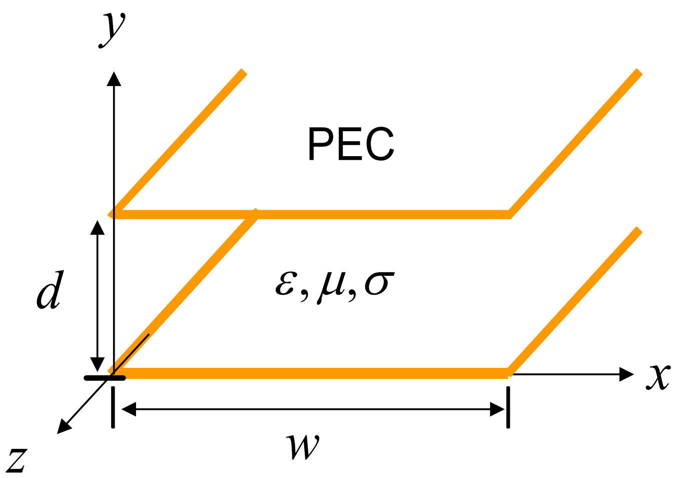
$$n = 0 \Rightarrow k_z = k$$
$$\Rightarrow \text{TM}_0 = \text{TEM}$$

The TEM mode can be thought of as a TM$_0$ mode.

TM$_z$ Modes (cont.)

Lossless Case

$$\varepsilon_c = \varepsilon = \varepsilon'$$



$$k_z = \left(k^2 - \overbrace{\left(\frac{n\pi}{d}\right)^2}^{k_c^2} \right)^{1/2} \qquad n = 0, 1, 2, \ldots$$

$$= \left(k^2 - k_c^2\right)^{1/2} \qquad k^2 = \omega^2 \mu \varepsilon \;\; (\text{real})$$

For $k^2 > k_c^2$

$$\Rightarrow k_z = \beta = \sqrt{k^2 - k_c^2}$$

$\Rightarrow$ propagating mode

For $k^2 < k_c^2$

$$\Rightarrow k_z = -j\sqrt{k_c^2 - k^2} = -j\alpha$$

$$\Rightarrow e^{-jk_z z} = e^{-\alpha z}$$

Fields decay exponentially

$\Rightarrow$ "evanescent" mode

TM$_z$ Modes (cont.)

Cutoff frequency (for lossless case)

This is the frequency that defines the border between evanescence and propagation.

$f_c \equiv$ cutoff frequency

$$\varepsilon_c = \varepsilon = \varepsilon'$$

$@\, f = f_c \qquad k = k_c \;\Rightarrow\; \omega_c \sqrt{\mu\varepsilon} = \dfrac{n\pi}{d}$

$$f_c = f_{cn} = \frac{n}{2d} \frac{1}{\sqrt{\mu\varepsilon}}$$

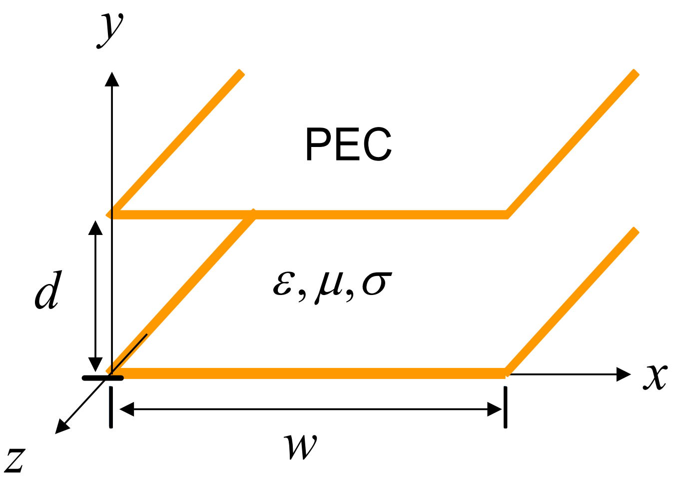
← cutoff frequency for TM$_n$ mode

Note: For a lossy waveguide, there is no sharp definition of cutoff frequency.

TM$_z$ Modes (cont.)

Time average power flow in z direction (lossless case):

$$\varepsilon_c = \varepsilon = \varepsilon'$$



$$P_{TMn} = \frac{1}{2}\mathrm{Re}\left[\int_0^w\int_0^d \left(\underline{E}\times\underline{H}^*\right)\cdot\underline{\hat{z}}\,dydx\right]$$

$$= -\frac{1}{2}\mathrm{Re}\left[\int_0^w\int_0^d E_y H_x^* dydx\right]$$

$$= \frac{\omega\varepsilon}{2k_c^2}\mathrm{Re}\{k_z\}\left|A_n\right|^2 w\int_0^d \cos^2\left(\frac{n\pi}{d}y\right)dy\; e^{-2\alpha z}$$

$$E_y = -\frac{jk_z}{k_c}A_n\cos\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$H_x = \frac{j\omega\varepsilon_c}{k_c}A_n\cos\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$P_{TMn} = \frac{\omega\varepsilon}{2k_c^2}\mathrm{Re}\{k_z\}\left|A_n\right|^2 w\left(\frac{d}{2}\right)e^{-2\alpha z}$$

$$n = 1, 2, \ldots$$

k_z is real for $f > f_c$

k_z is imaginary for $f < f_c$

TE$_z$ Modes ($E_z = 0$)

Recall:

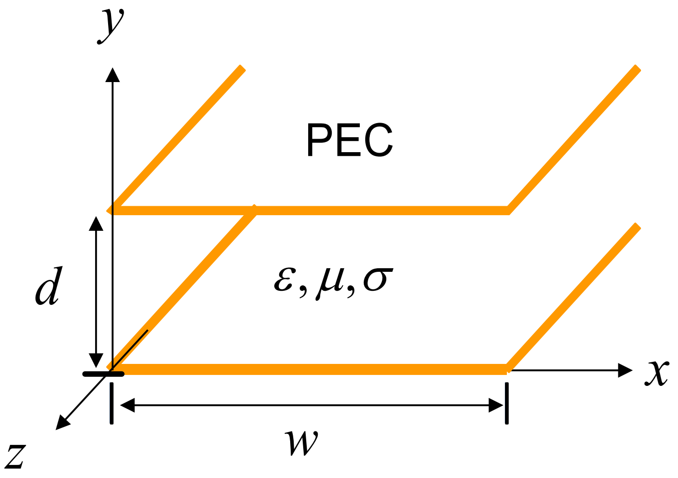
$$H_z(x,y,z) = h_z(x,y)\,e^{-jk_z z}$$

where

$$\nabla_t^2 h_z(x,y) = -k_c^2 h_z(x,y)$$

eigenvalue problem

$$k_c \equiv \sqrt{k^2 - k_z^2}$$



Note:
Solving the eigenvalue problem
(using appropriate boundary conditions)
will tell us what the eigenvalue k_c is.

$$\left(\cancel{\frac{\partial^2}{\partial x^2}} + \frac{\partial^2}{\partial y^2}\right) h_z(x,y) = -k_c^2 h_z(x,y)$$

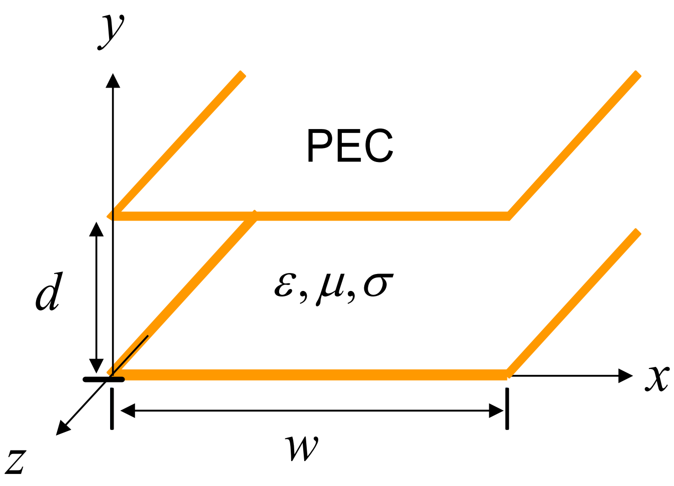
(Assume no x variation)

so $$\frac{d^2}{dy^2} h_z(y) = -k_c^2 h_z(y)$$

TE$_z$ Modes (cont.)

$$\frac{d^2}{dy^2}h_z\left(y\right)=-k_c^2h_z\left(y\right)$$

subject to B.C.'s $E_x = 0$ @ $y=0,\ d$



Solving the above equation:

$$h_z = A\sin(k_c y)+B\cos(k_c y)$$

$$h_z' = k_c A\cos(k_c y)-k_c B\sin(k_c y)$$

$$E_x=\frac{1}{j\omega\varepsilon_c}\left(\frac{\partial H_z}{\partial y}-\frac{\partial \cancel{H_y}}{\partial z}\right)$$

Apply B.C.'s:

$$@\, y=0 \;\Rightarrow\; A=0$$

$$@\, y=d \;\Rightarrow\; k_c d = n\pi,\;\; n=1,2,3,\ldots \quad \Rightarrow k_c=\frac{n\pi}{d}$$

PEC: $\underline{H}\cdot\underline{\hat{n}}=0,\quad \underline{E}_t=\underline{0}$

$$\Downarrow$$

$$\frac{\partial h_z}{\partial y}=0$$

TE$_z$ Modes (cont.)

$$h_z\left(y\right)=B_n\cos\left(\frac{n\pi}{d}y\right)\qquad n=1,2,3,\ldots$$

$$H_z\left(y,z\right)=B_n\cos\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$k_z=\left(k^2-k_c^2\right)^{1/2}$$

$$=\left(k^2-\left(\frac{n\pi}{d}\right)^2\right)^{1/2}$$

$$k^2=\omega^2\mu\varepsilon_c$$

For a wave propagating in the $+z$ direction:

$$E_x=\frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}=\frac{j\omega\mu}{k_c^2}B_n\left(\frac{n\pi}{d}\right)\sin\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$H_y=-\frac{jk_z}{k_c^2}\frac{\partial H_z}{\partial y}=\frac{jk_z}{k_c^2}B_n\left(\frac{n\pi}{d}\right)\sin\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$H_x=0\qquad\qquad E_y=0\qquad\qquad E_z=0$$

No x variation

TE$_z$ mode

TE$_z$ Modes (cont.)

Summary

$$H_z = B_n \cos\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$E_x = \frac{j\omega\mu}{k_c} B_n \sin\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

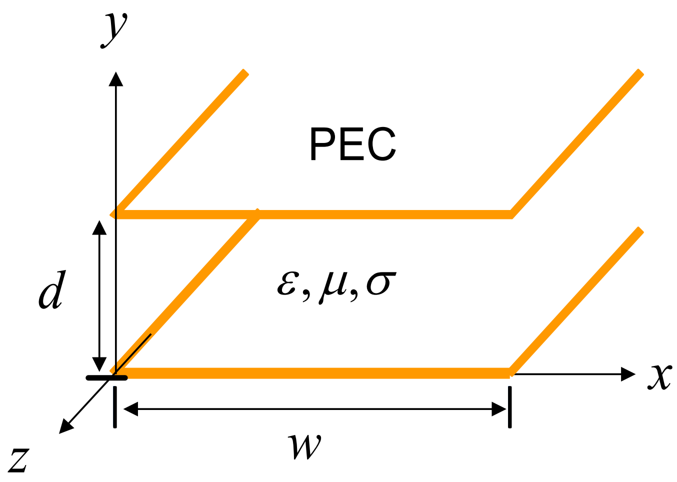
$$H_y = \frac{jk_z}{k_c} B_n \sin\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$H_x = E_y = E_z = 0$$

$$k_c = \frac{n\pi}{d} \; ; \quad n = 1, 2, \ldots$$

$$k_z = \left(k^2 - \left(\frac{n\pi}{d}\right)^2\right)^{1/2}$$

$$k^2 = \omega^2 \mu \varepsilon_c$$



Each value of n corresponds to a unique TE$_z$ field solution or "mode."

$\Rightarrow$ TE$_n$ mode

Cutoff frequency

$$f_c = f_{cn} = \frac{n}{2d}\left(\frac{1}{\sqrt{\mu\varepsilon}}\right)$$

Note: There is no TE$_0$ mode

(This mode would be a plane wave having E_x and H_y, but would not be supported by this system. This mode would require PMC on top and bottom, and PEC on the sides.)

Power in TE_z Mode

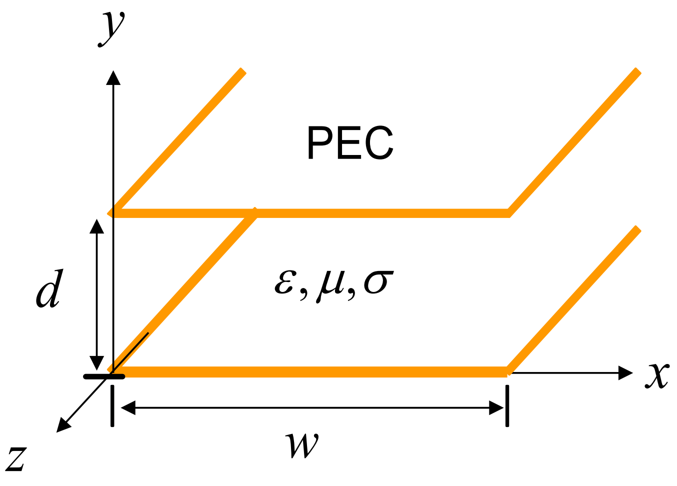
Time average power flow in z direction (lossless case):

$$\varepsilon_c = \varepsilon = \varepsilon'$$

$$P_{TEn}^{+} = \frac{1}{2}\operatorname{Re}\left[\int_0^w\int_0^d \left(\underline{E}\times\underline{H}^*\right)\cdot\underline{\hat{z}}\,dydx\right]$$

$$= \frac{1}{2}\operatorname{Re}\left[\int_0^w\int_0^d E_x H_y^*\,dydx\right]$$

$$= \frac{\omega\mu}{2k_c^2}\operatorname{Re}\{k_z\}\left|B_n\right|^2 w\int_0^d \sin^2\left(\frac{n\pi}{d}y\right)dy\; e^{-2\alpha z}$$



$$P_{TEn}^{+} = \frac{\omega\mu}{4k_c^2}\operatorname{Re}\left\{k_z\right\}\left|B_n\right|^2\left(wd\right)e^{-2\alpha z}$$

$$E_x = \frac{j\omega\mu}{k_c}B_n \sin\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$H_y = \frac{jk_z}{k_c}B_n \sin\left(\frac{n\pi}{d}y\right)e^{-jk_z z}$$

$$\uparrow$$

k_z is real for $f > f_c$

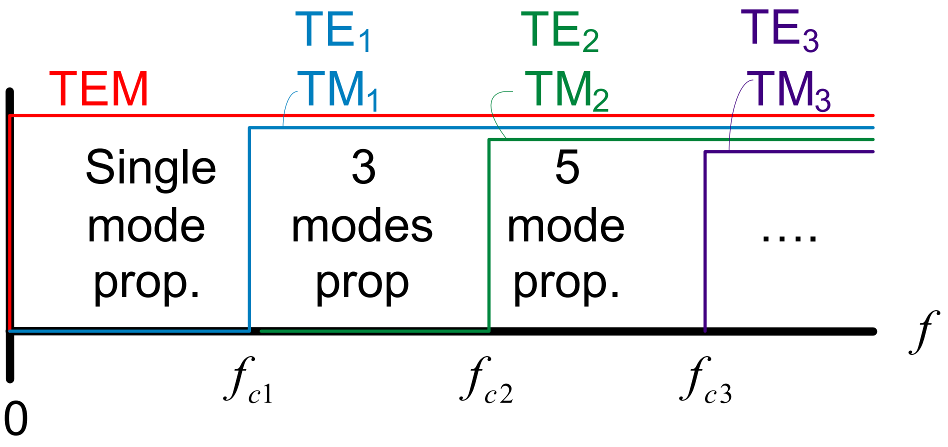
k_z is imaginary for $f < f_c$

Mode Chart

For all the modes of a parallel-plate waveguiding structure, we have

$$f_{cn} = \frac{n}{2d}\left(\frac{1}{\sqrt{\mu\varepsilon}}\right)$$

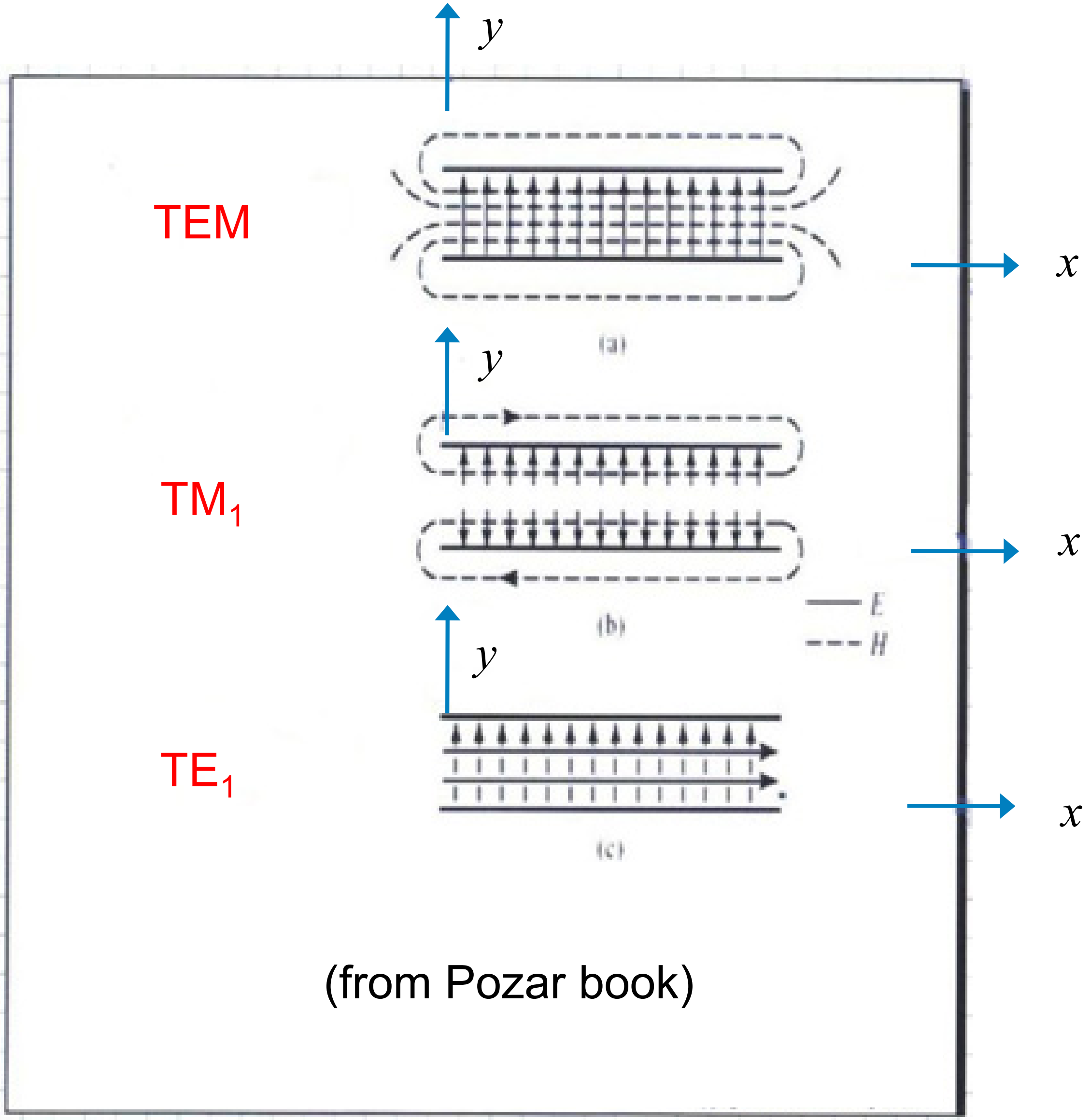
$$\varepsilon_c = \varepsilon = \varepsilon'$$



The mode with lowest cutoff frequency is called the "dominant" mode of the waveguide.

Important conclusion: If we want to use the structure as a transmission line, we need to operate in the region $f < f_{c1}$.

Field Plots



(from Pozar book)

Plane Wave Interpretation

TM$_z$ waveguide mode propagating in +z direction:

$$E_z = A_n \sin\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$E_y = -\frac{jk_z}{k_c} A_n \cos\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$H_x = \frac{j\omega\varepsilon_c}{k_c} A_n \cos\left(\frac{n\pi}{d} y\right) e^{-jk_z z}$$

$$k_c = \frac{n\pi}{d}$$

Re-label this as k_y

$$E_z = A_n \sin\left(k_y y\right) e^{-jk_z z}$$

$$E_y = -\frac{jk_z}{k_c} A_n \cos\left(k_y y\right) e^{-jk_z z}$$

$$H_x = \frac{j\omega\varepsilon_c}{k_c} A_n \cos\left(k_y y\right) e^{-jk_z z}$$

⇨

$$E_z = -A_n \left(\frac{1}{2j}\right)\left(e^{-jk_y y} e^{-jk_z z} - e^{+jk_y y} e^{-jk_z z}\right)$$

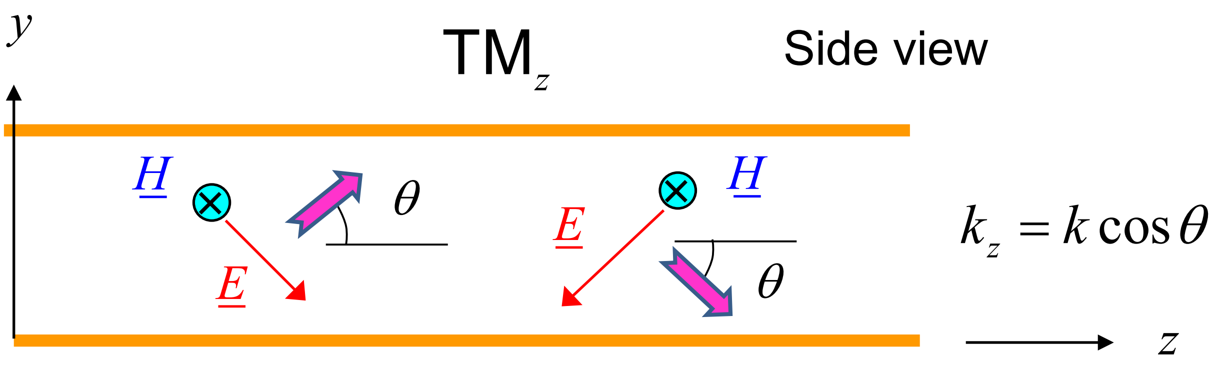
$$E_y = -\frac{jk_z}{2k_c} A_n \left(e^{-jk_y y} e^{-jk_z z} + e^{+jk_y y} e^{-jk_z z}\right)$$

$$H_x = \frac{j\omega\varepsilon_c}{2k_c} A_n \left(e^{-jk_y y} e^{-jk_z z} + e^{+jk_y y} e^{-jk_z z}\right)$$

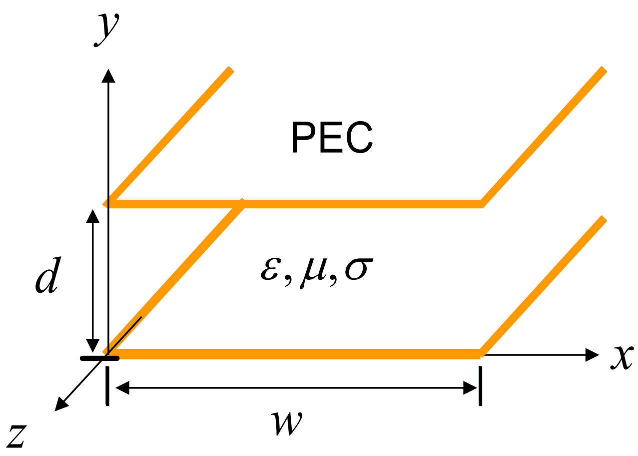
$$\cos\left(x\right) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \qquad \sin\left(x\right) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

Plane Wave Interpretation (cont.)

The TM$_z$ waveguide mode is a sum of two plane waves*:



$$k_z = k\cos\theta$$



$$E_z = -A_n\left(\frac{1}{2j}\right)\left(e^{-jk_y y}e^{-jk_z z} - e^{+jk_y y}e^{-jk_z z}\right)$$

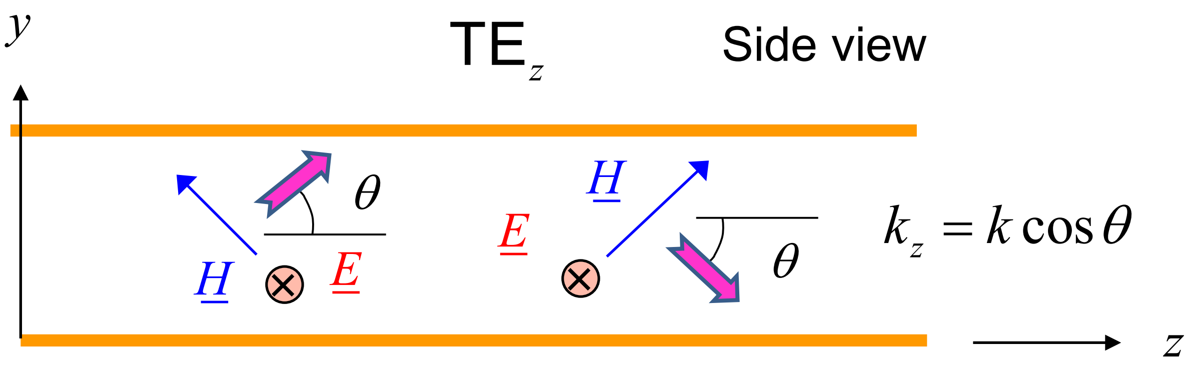
$$E_y = -\frac{jk_z}{2k_c}A_n\left(e^{-jk_y y}e^{-jk_z z} + e^{+jk_y y}e^{-jk_z z}\right)$$

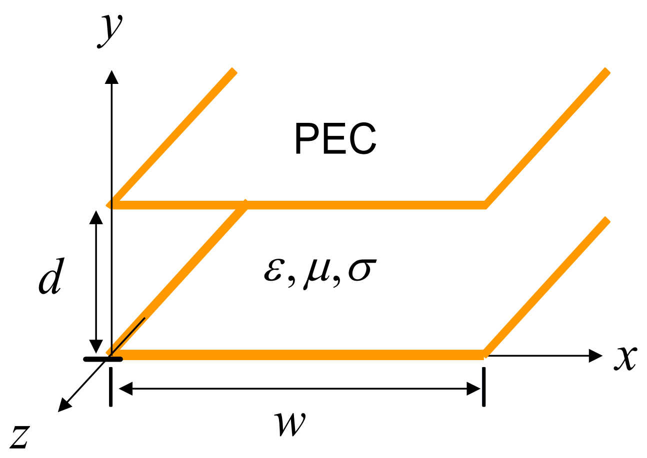
$$H_x = \frac{j\omega\varepsilon_c}{2k_c}A_n\left(e^{-jk_y y}e^{-jk_z z} + e^{+jk_y y}e^{-jk_z z}\right)$$

*We can also think of one a single plane wave bouncing up and down.

Plane Wave Interpretation (cont.)

The TE$_z$ waveguide mode is a sum of two plane waves*:





$$H_z = B_n\left(\frac{1}{2}\right)\left(e^{-jk_y y}e^{-jk_z z} + e^{+jk_y y}e^{-jk_z z}\right)$$

$$E_x = -\frac{j\omega\mu}{(2j)k_c}B_n\left(e^{-jk_y y}e^{-jk_z z} - e^{+jk_y y}e^{-jk_z z}\right)$$

$$H_y = -\frac{jk_z}{(2j)k_c}B_n\left(e^{-jk_y y}e^{-jk_z z} - e^{+jk_y y}e^{-jk_z z}\right)$$

*We can also think of one a single plane wave bouncing up and down.

Conductor Attenuation on Parallel Plates

TEM Mode

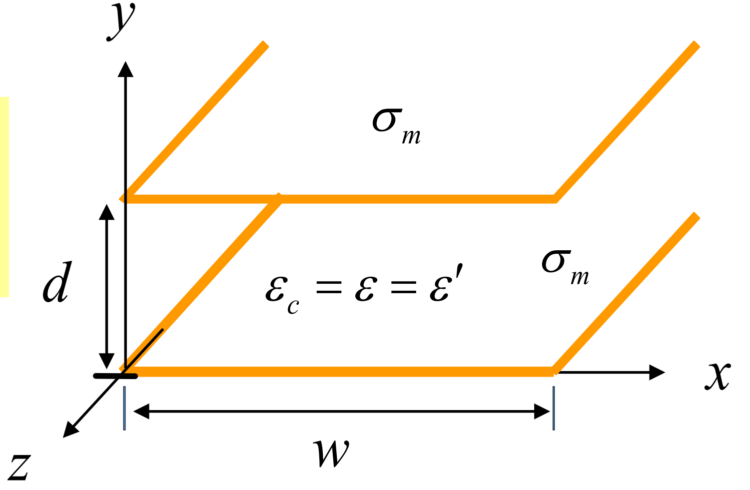
$$P_0 = \mathrm{Re}\left\{\int_0^w\int_0^d \frac{1}{2}\left(\underline{E}\times\underline{H}^*\right)\cdot\hat{z}\,dydx\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\int_0^w\int_0^d \left(\frac{|V_0|^2}{\eta d^2}\hat{z}\right)\cdot\hat{z}\,dydx\right\}$$

$$= \frac{1}{2}|V_0|^2\left(\frac{w}{d}\right)\left(\frac{1}{\eta}\right)$$

Assume no dielectric loss for the calculation of conductor attenuation.

$$k,\eta \in \mathrm{Real}$$



$$\underline{E} = -\hat{y}\frac{V_0}{d}e^{-jkz}$$

$$\underline{H} = +\hat{x}\frac{V_0}{\eta d}e^{-jkz}$$

On the top plate:

$$\underline{J}_s^{\mathrm{top}} = -\underline{\hat{y}}\times\underline{H}$$

$$= \hat{z}\frac{V_0}{\eta d}e^{-jkz}$$

On the bottom plate:

$$\underline{J}_s^{\mathrm{bot}} = \underline{\hat{y}}\times\underline{H} = -\hat{z}\frac{V_0}{\eta d}e^{-jkz}$$

$$\alpha_c = \frac{P_l(0)}{2P_0}$$

Conductor Attenuation on Parallel Plates (cont.)

$$P_l(0) = \frac{R_s}{2}\left\{ \int_0^w \left| \underline{J}_s^{\text{top}} \right|^2 \Big|_{z=0} dx \quad + \quad \int_0^w \left| \underline{J}_s^{\text{bot}} \right|^2 \Big|_{z=0} dx \right\}$$

$$= R_s \int_0^w \left| \frac{V_0}{\eta d} \right|^2 dx$$

(equal contributions from both plates)

$$= R_s \frac{\left|V_0\right|^2}{(\eta d)^2} w$$

$$P_l(0) = \frac{1}{2}\int_{C_1} R_{s1} \left|\underline{J}_s\right|^2 \Big|_{z=0} d\ell \quad + \quad \frac{1}{2}\int_{C_2} R_{s1} \left|\underline{J}_s\right|^2 \Big|_{z=0} d\ell$$

We then have:

$$\alpha_c = \frac{P_l(0)}{2P_0} = \frac{R_s \left|V_0\right|^2 \left(\dfrac{w}{(\eta d)^2} \right)}{2\left(\dfrac{1}{2}\right)\left|V_0\right|^2 \left(\dfrac{w}{\eta d}\right)}$$

The final result is then

$$\alpha_c = \frac{R_s}{\eta d}$$

Conductor Attenuation on Parallel Plates (cont.)

Let’s try the same calculation using the Wheeler incremental inductance rule.

$$\alpha_c^{\mathrm{cond}} = \left(\frac{R_s}{2Z_0\eta}\right)\frac{\partial Z_0}{\partial \ell}$$

We apply the formula for each conductor and then add the results: $\alpha_c = \alpha_c^{top} + \alpha_c^{bot}$

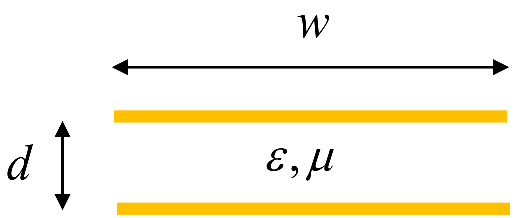
From previous calculations:

$$Z_0 = \eta\left(\frac{d}{w}\right)$$

In this formula, ℓ (for a given conductor) is the distance by which the conducting boundary is receded away from the field region.

$$\frac{\partial Z_0}{\partial \ell} = \frac{dZ_0}{\partial d}$$

(since $\partial\ell = \partial d$ for either plate)



$k, \eta \in \mathrm{Real}$

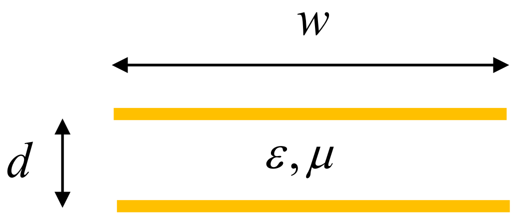
Conductor Attenuation on Parallel Plates (cont.)

$$\alpha_c^{\text{cond}} = \left(\frac{R_s}{2Z_0\eta}\right)\frac{\partial Z_0}{\partial \ell} \qquad \frac{\partial Z_0}{\partial \ell} = \frac{dZ_0}{\partial d} \qquad Z_0 = \eta\left(\frac{d}{w}\right)$$

Hence, we have:

$$\alpha_c^{\text{top}} = \left(\frac{R_s}{2Z_0\eta}\right)\frac{\partial}{\partial d}\left(\eta\frac{d}{w}\right) = \frac{R_s}{2wZ_0} = \frac{R_s}{2w\left(\eta\dfrac{d}{w}\right)} = \frac{R_s}{2\eta d}$$

$$\alpha_c^{\text{bot}} = \left(\frac{R_s}{2Z_0\eta}\right)\frac{\partial}{\partial d}\left(\eta\frac{d}{w}\right) = \frac{R_s}{2wZ_0} = \frac{R_s}{2w\left(\eta\dfrac{d}{w}\right)} = \frac{R_s}{2\eta d}$$



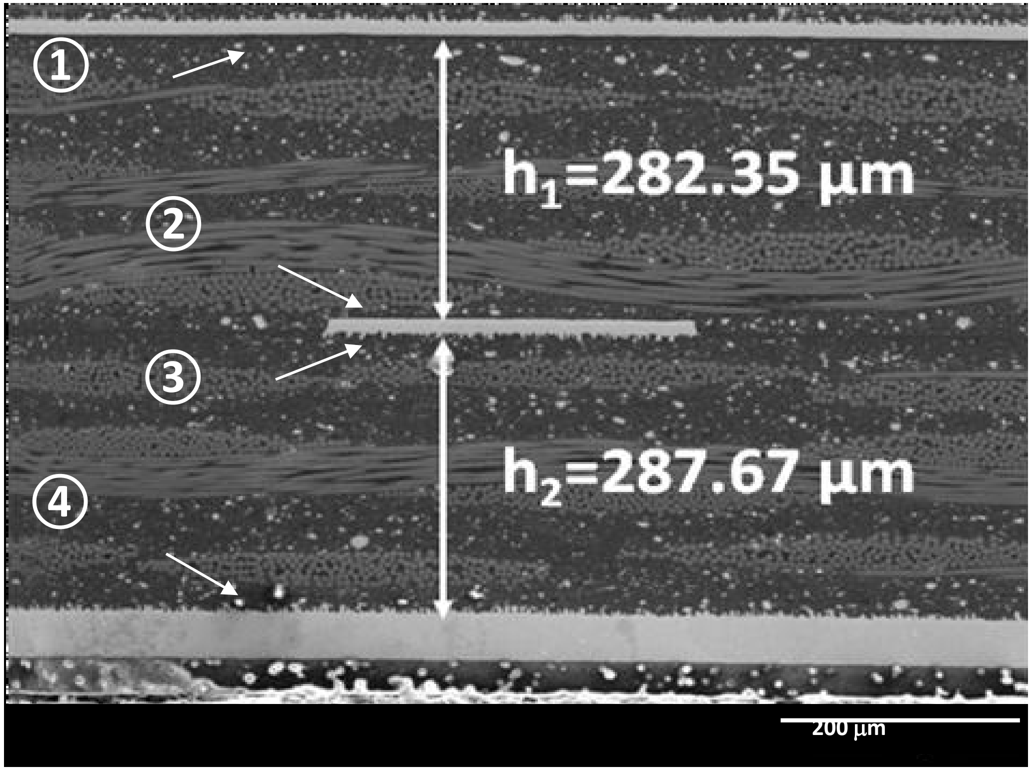
$k, \eta \in \text{Real}$

$$\Rightarrow \quad \boxed{\alpha_c = \frac{R_s}{\eta d}}$$

Surface Roughness

Conductor attenuation will increase due to surface roughness effects.

Stripline



Surfaces 3 and 4 are rough.

Surface Roughness (cont.)

We can use an effective conductivity to account for surface roughness.

Example:

Pure copper

$$\sigma = 5.8 \times 10^7 \ [\mathrm{S/m}]$$

Practical copper

$$\sigma = 3.0 \times 10^7 \ [\mathrm{S/m}]$$

Surface Roughness (cont.)

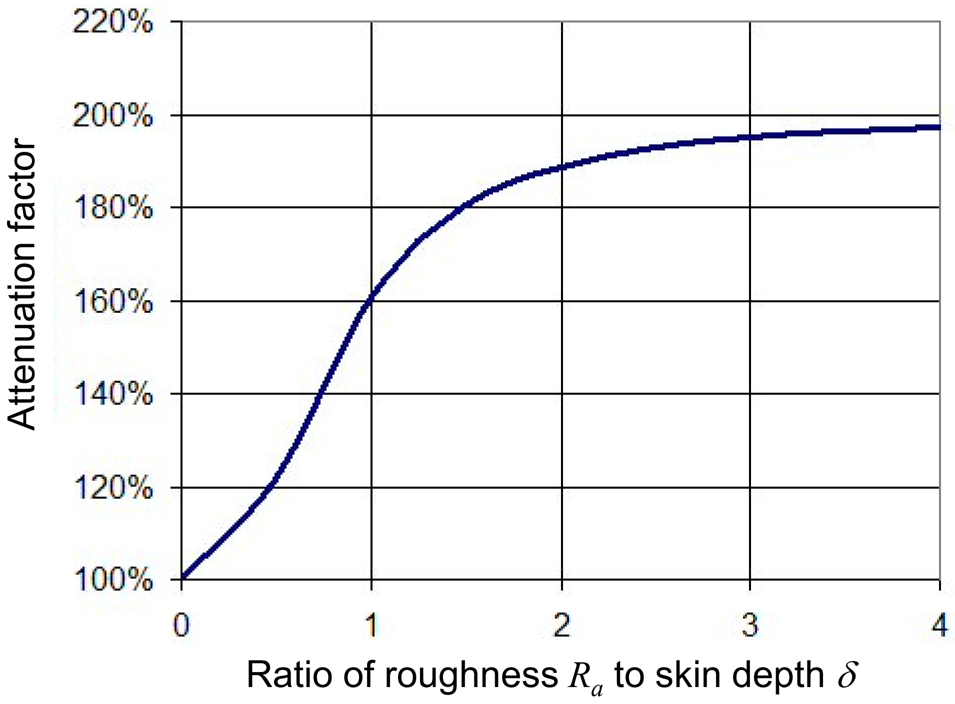
Hammerstad and Jensen formula

$$K_{\text{rough}} = 1 + \frac{2}{\pi}\tan^{-1}\left(1.4\left(\frac{R_a}{\delta}\right)^2\right)$$

R_a = height of surface roughness

This is a factor that gives the increase in the attenuation constant α.

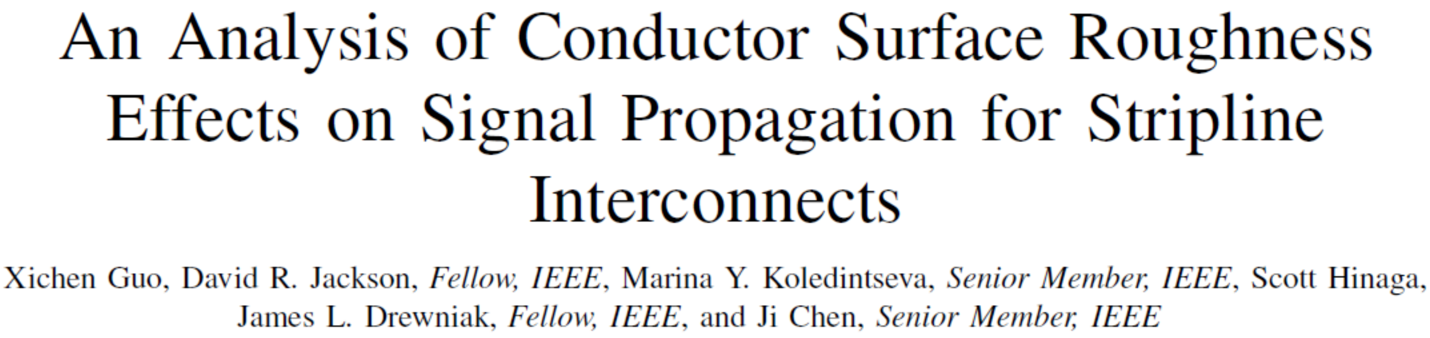
Attenuation factor K_{rough} vs. surface roughness



E. Hammerstad and O. Jensen, "Accurate models for microstrip computer-aided design," in Microwave Symp. Digest, IEEE MTT-S International, 1980, vol. 1, no. 12, pp. 407–409.

https://www.microwaves101.com/encyclopedias/surface-roughness

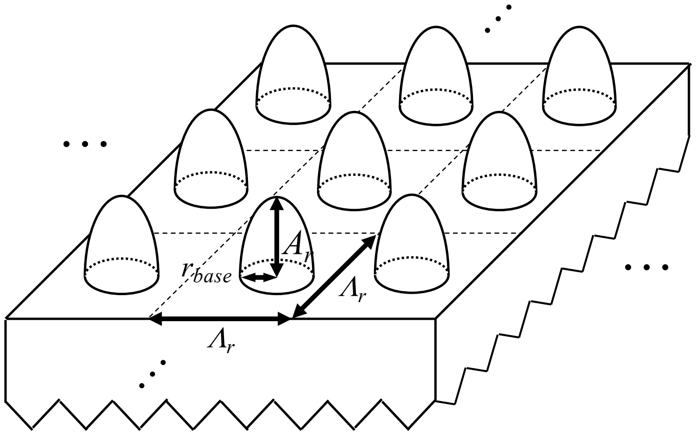
Surface Roughness (cont.)

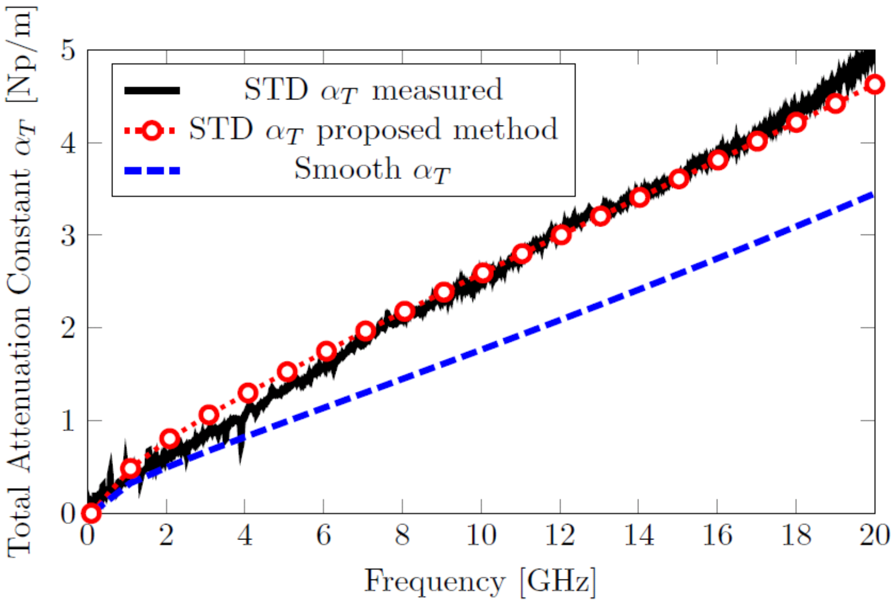


An Analysis of Conductor Surface Roughness Effects on Signal Propagation for Stripline Interconnects

Xichen Guo, David R. Jackson, *Fellow, IEEE*, Marina Y. Koledintseva, *Senior Member, IEEE*, Scott Hinaga, James L. Drewniak, *Fellow, IEEE*, and Ji Chen, *Senior Member, IEEE*

$r_{\text{base}} / \Lambda_r = 1/3$



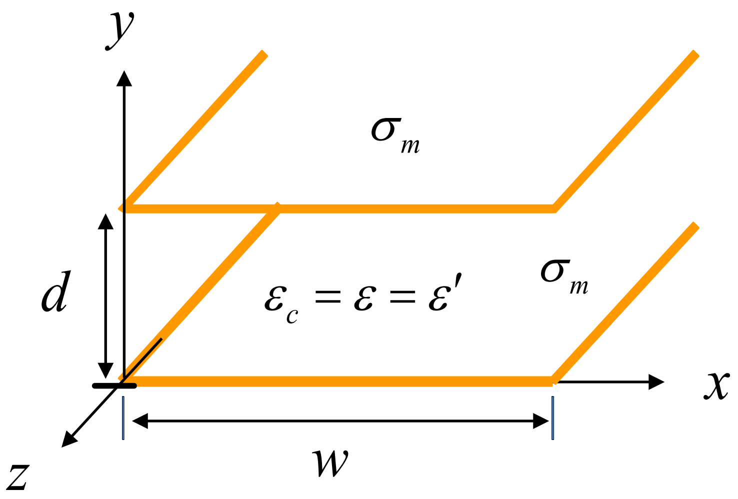


A_r: Hemispheroid height
r_{base}: Hemispheroid radius
Λ_r: Period

X. Guo, D. R. Jackson, M. Y. Koledintseva, S. Hinaga, J. L. Drewniak, and J. Chen, “An Analysis of Conductor Surface Roughness Effects on Signal Propagation for Stripline Interconnects,” *IEEE Trans. Electromagnetic Compatibility*, Vol. 56, No. 3, pp. 707–714, June 2014.

Conductor Attenuation on Parallel Plates (cont.)

Waveguide Modes



Results for TM/TE Modes (above cutoff): (derivation omitted)

TM$_n$ modes of PPW: $$\alpha_{cn}^{\mathrm{TM}} = \frac{2kR_s}{\beta\eta d}, \quad n > 0$$

TE$_n$ modes of PPW: $$\alpha_{cn}^{\mathrm{TE}} = \frac{2k_c^2 R_s}{k\beta\eta d}, \quad n > 0$$

Note: Below cutoff, we usually do not worry about conductor loss.