Adapted from notes by
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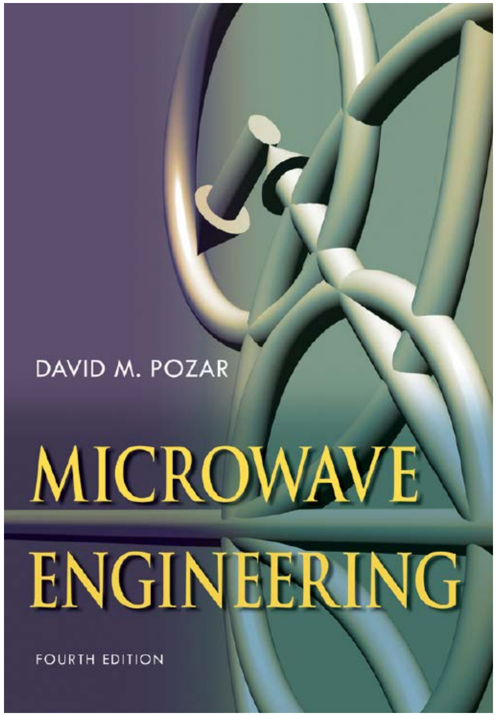
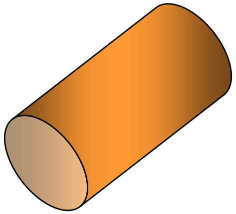
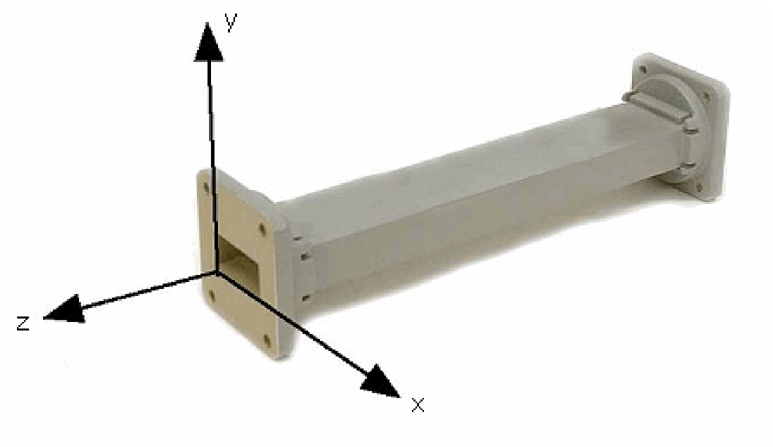


Notes 9
Waveguiding Structures Part 4: Rectangular and Circular Waveguide



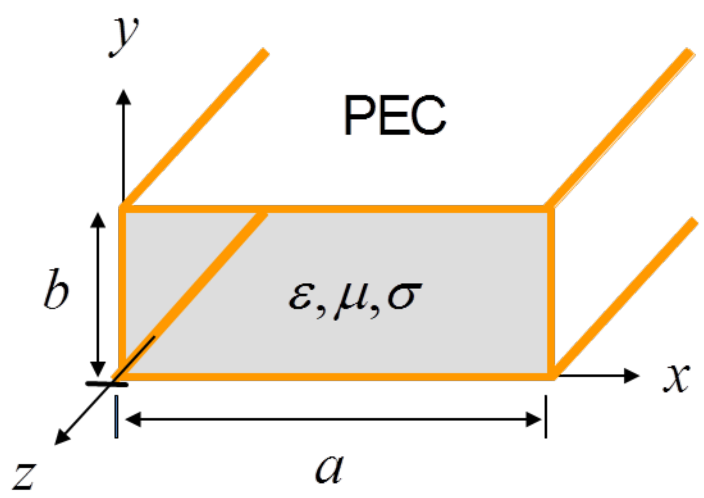
Rectangular Waveguide

- One of the earliest waveguides.
- Still common for high power and low-loss microwave / millimeter-wave applications.



- It is essentially an electromagnetic pipe with a rectangular cross-section.

Single conductor $\Rightarrow$ No TEM mode



For convenience:

- $a \geq b$ (the long dimension lies along x).

Rectangular Waveguide (cont.)

Two types of modes:

TE_z , TM_z

$$k_z = \left(k^2 - k_c^2\right)^{1/2}$$

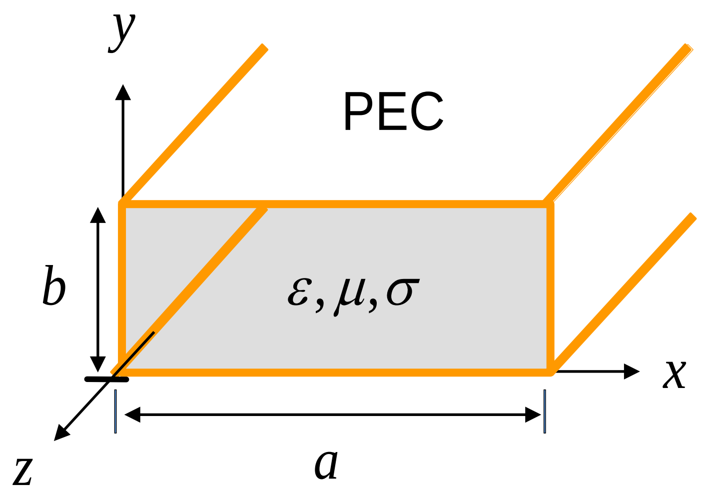
$$k = \omega\sqrt{\mu\varepsilon_c} = k_0\sqrt{\varepsilon_r\left(1 - j\tan\delta_d\right)}$$

The cutoff wavenumber k_c is real.

We need to solve for k_c.

$$f > f_c: \quad k_z = \sqrt{k^2 - k_c^2}$$

$$f < f_c: \quad k_z = -j\sqrt{k_c^2 - k^2}$$



$$\begin{aligned}\varepsilon_c &= \varepsilon - j\frac{\sigma}{\omega} \\ &= \varepsilon' - j\varepsilon'' - j\frac{\sigma}{\omega} \\ &= \varepsilon_c' - j\varepsilon_c'' \\ &= \varepsilon_c'\left(1 - j\frac{\varepsilon_c''}{\varepsilon_c'}\right) \\ &= \varepsilon_c'\left(1 - j\tan\delta_d\right) \\ &= \varepsilon_0\varepsilon_r\left(1 - j\tan\delta_d\right)\end{aligned}$$

TE$_z$ Modes

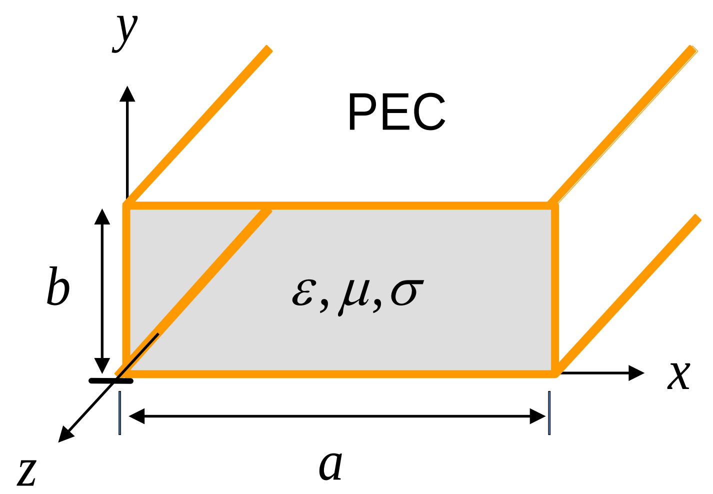
For $+z$ propagation:

$$H_z(x,y,z) = h_z(x,y)\, e^{-jk_z z}$$

where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x,y) = 0$$

$$k_c \equiv \left(k^2 - k_z^2\right)^{1/2}$$



From previous field table:

$$E_x = \frac{-j}{k_c^2}\left(k_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(-k_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x}\right)$$

Subject to B.C.'s:

$$E_x = 0 \;\Rightarrow\; \frac{\partial H_z}{\partial y} = 0 \qquad @\, y = 0,\, b$$

$$E_y = 0 \;\Rightarrow\; \frac{\partial H_z}{\partial x} = 0 \qquad @\, x = 0,\, a$$

TE_z Modes (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2\,h_z(x,y)$$ (eigenvalue problem)

Using separation of variables, let $h_z(x,y)=X(x)Y(y)$

$$\Rightarrow \quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=-k_c^2XY$$

Must be a constant

This is the "separation equation".

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_c^2$$

(If we take one term across the equal sign, we have a function of x equal to a function of y.)

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2$$

where $k_x^2+k_y^2=k_c^2 \longleftarrow$ "separation equation"

TE_z Modes (cont.)

Hence,

$$h_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions:

$$\begin{cases} \dfrac{\partial h_z}{\partial y} = 0 \quad @\, y = 0,\, b \quad \text{(A)} \\ \dfrac{\partial h_z}{\partial x} = 0 \quad @\, x = 0,\, a \quad \text{(B)} \end{cases}$$

(A) $\Rightarrow \quad D = 0$ and $k_y = \dfrac{n\pi}{b} \quad n = 0,1,2,\ldots$

(B) $\Rightarrow \quad B = 0$ and $k_x = \dfrac{m\pi}{a} \quad m = 0,1,2,\ldots$

$\Rightarrow$ $h_z(x,y) = A_{mn}\cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TE_z Modes (cont.)

Therefore,

$$H_z(x,y,z) = A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$k_z = \left(k^2 - k_c^2\right)^{1/2} = \left(k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2}$$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

From the field table, we obtain the following:

$$E_x = \frac{j\omega\mu n\pi}{k_c^2 b}A_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$E_y = -\frac{j\omega\mu m\pi}{k_c^2 a}A_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$H_x = \frac{jk_z m\pi}{k_c^2 a}A_{mn}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$H_y = \frac{jk_z n\pi}{k_c^2 b}A_{mn}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

Note:

$m = 0,1,2,\ldots$

$n = 0,1,2,\ldots$

But $m = n = 0$ is not allowed!

(non-physical solution)

$$\underline{H} = \underline{\hat{z}}\,A_{00}e^{-jkz};\;\; \nabla\cdot\underline{H} \neq 0$$

TE$_z$ Modes (cont.)

Reason for non-physical solution

Start with the vector wave equation:

$$\nabla \times \left(\nabla \times \underline{H} \right) - k^2 \underline{H} = \underline{0}$$

Vector wave equation: from Maxwell's equations.

Take divergence of both sides.

$$\nabla \cdot \left(\nabla \times \left(\nabla \times \underline{H} \right) \right) - k^2 \nabla \cdot \underline{H} = 0$$

The divergence of a curl is zero.

$$\nabla \cdot \underline{H} = 0$$

Magnetic Gauss law

TE_z Modes (cont.)

Reason for non-physical solution

Revisit how we obtained the vector Helmholtz equation:

$$\nabla \times (\nabla \times \underline{H}) - k^2 \underline{H} = \underline{0}$$

Vector wave equation: from Maxwell's equations.

⇓

$$\underbrace{\nabla(\nabla \cdot \underline{H}) - \nabla^2 \underline{H}}_{\nabla \times \nabla \times \underline{H}} - k^2 \underline{H} = \underline{0}$$

From definition of vector Laplacian

Now use:

$$\nabla \cdot \underline{H} = 0$$

Magnetic Gauss law

A needed assumption!

⇓

$$\nabla^2 \underline{H} + k^2 \underline{H} = \underline{0}$$

Vector Helmholtz equation (what we have solved)

TE_z Modes (cont.)

Reason for non-physical solution

Vector wave equation $\Rightarrow$ magnetic Gauss law

Vector Helmholtz equation $\not\Rightarrow$ magnetic Gauss law

The vector Helmholtz equation does <u>not</u> guarantee that the magnetic Gauss law is satisfied. In the mathematical derivation, we need to <u>assume</u> the magnetic Gauss law in order to arrive at the vector Helmholtz equation.

All of the modes that we get by solving the Helmholtz equation should be checked to make sure that they do satisfy the magnetic Gauss law.

Note: The TE_{00} mode is the <u>only</u> one that violates the magnetic Gauss law.

TE_z Modes (cont.)

Lossless case $(\varepsilon_c = \varepsilon = \varepsilon')$

$$k_z^{mn} = \left(k^2 - \left(k_c^{mn}\right)^2\right)^{1/2} = \left(k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2}$$

$\Rightarrow$ TE_{mn} mode is at cutoff when $k = k_c^{mn}$ $\left(k = \omega_c^{mn}\sqrt{\mu\varepsilon}\right)$

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Lowest cutoff frequency is for TE_{10} mode $(a > b)$

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

We will revisit this mode later.

Dominant TE mode (lowest f_c)

TE_z Modes (cont.)

At the cutoff frequency of the TE_{10} mode (lossless waveguide):

$$\lambda_d = \frac{c_d}{f} = \frac{c_d}{f_c^{10}} = \frac{c_d}{\dfrac{1}{2a\sqrt{\mu\varepsilon}}} = 2a$$

so

$$a = \lambda_d / 2 \Big|_{f=f_c}$$

TE_z Modes (cont.)

To have propagation:

$$f > f_c$$

so

$$f > \frac{1}{2a\sqrt{\mu\varepsilon}}$$

or

$$a > \frac{1}{2f\sqrt{\mu\varepsilon}} = \frac{1}{2}\frac{c_d}{f} = \frac{\lambda_d}{2}$$

or

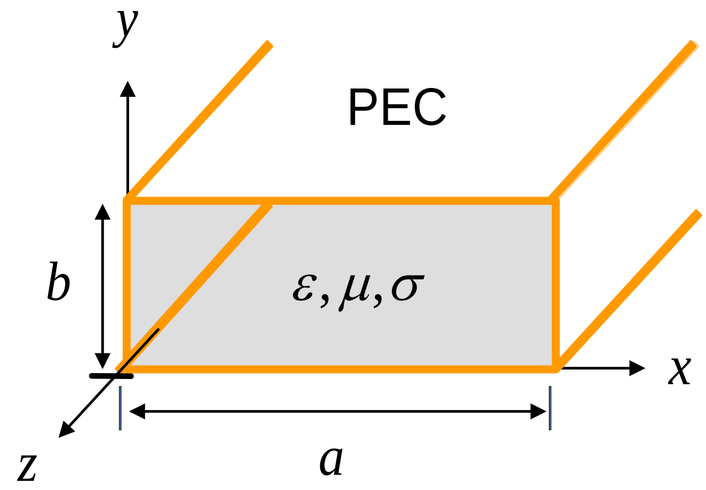
$$a > \frac{\lambda_d}{2}$$

Example: Air-filled waveguide, $f = 10\ \text{GHz}$. We have that $a > 3.0\ \text{cm} / 2 = 1.5\ \text{cm}$.

TM$_z$ Modes

Recall:

$$E_z(x,y,z) = e_z(x,y)\,e^{-jk_z z}$$



where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) e_z(x,y) = -k_c^2 e_z(x,y)$$ (eigenvalue problem)

$$k_c = \left(k^2 - k_z^2\right)^{1/2}$$

Subject to B.C.'s: $E_z = 0$ @ $x = 0,\ a$

@ $y = 0,\ b$

Thus, following same procedure as before, we have the following result:

TM_z Modes (cont.)

$$e_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions: $e_z = 0 \quad @ \; y = 0, b$ (A)

$@ \; x = 0, a$ (B)

(A) $\Rightarrow \; C = 0$ and $k_y = \dfrac{n\pi}{b} \quad n = 0,1,2,\ldots$

(B) $\Rightarrow \; A = 0$ and $k_x = \dfrac{m\pi}{a} \quad m = 0,1,2,\ldots$

$\Rightarrow$ $e_z = B_{mn}\sin\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TM$_z$ Modes (cont.)

Therefore

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$k_z = \left(k^2 - k_c^2\right)^{1/2} = \left(k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2}$$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

From the field table, we obtain the following:

$$H_x = \frac{j\omega\varepsilon_c n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$H_y = -\frac{j\omega\varepsilon_c m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$E_x = -\frac{jk_z m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$E_y = \frac{jk_z n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$m = 1, 2, 3 \ldots$$

$$n = 1, 2, 3 \ldots$$

Note:
If either m or n is zero, the field becomes a trivial one in the TM$_z$ case.

TM_z Modes (cont.)

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$$k_z^{mn} = \sqrt{k^2 - \left(k_c^{mn}\right)^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

(same as for TE modes)

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

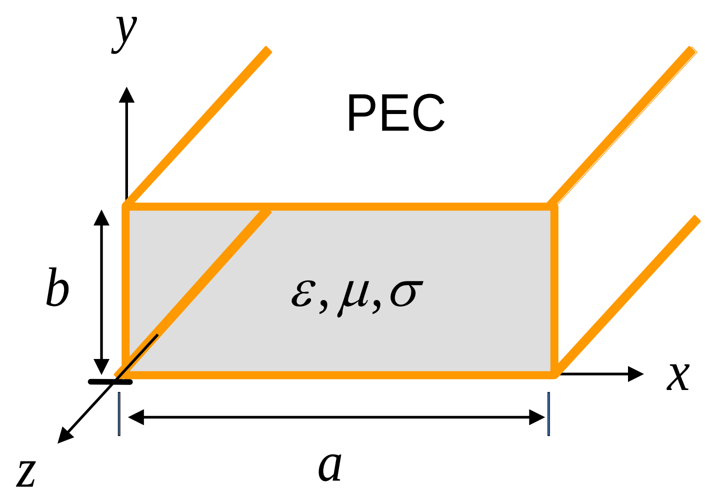
The lowest cutoff frequency is obtained for the TM_{11} mode

$$f_c^{11} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

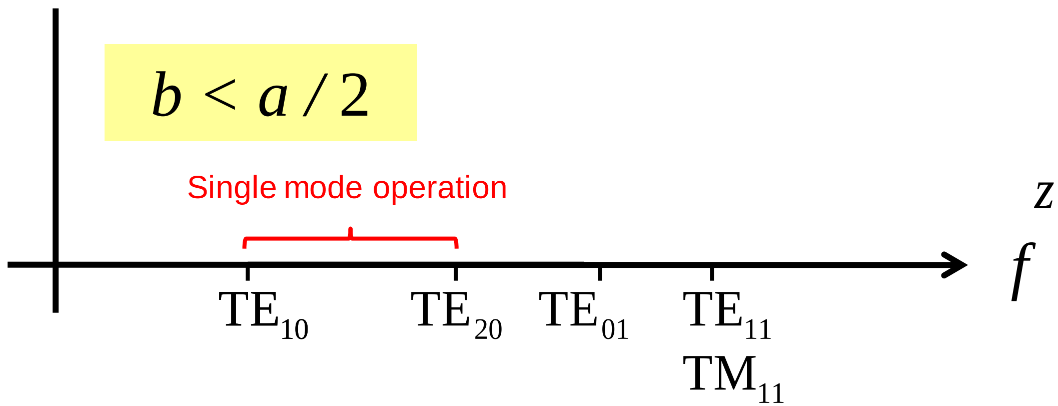
Dominant TM mode (lowest f_c)

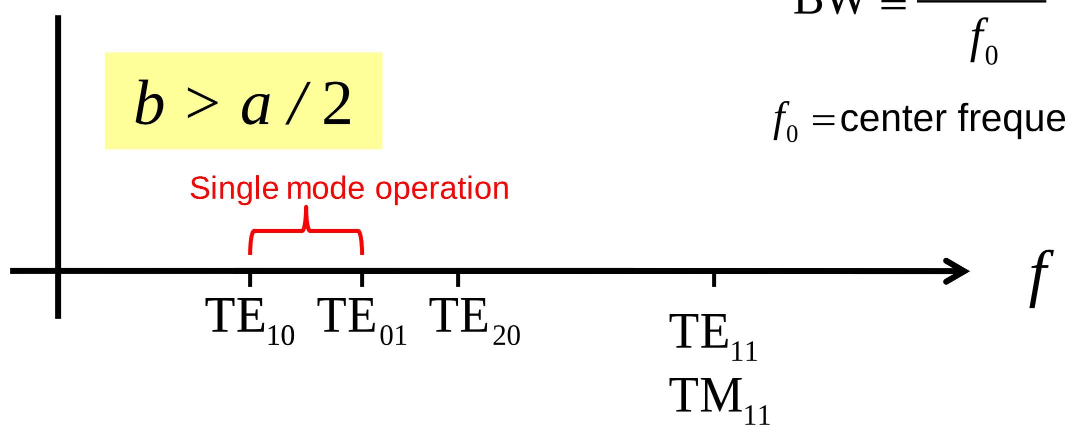
Mode Chart

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$



Two cases are considered:





$$\text{BW} \equiv \frac{f_2 - f_1}{f_0}$$

$f_0 =$ center frequency

The maximum bandwidth for single-mode operation is 67%.

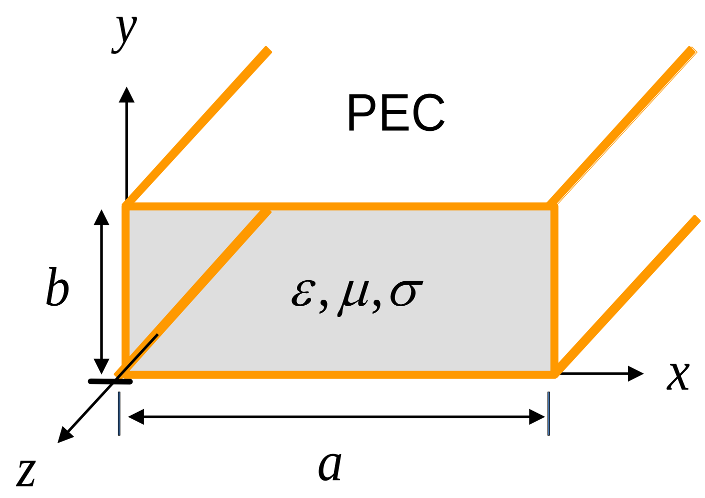
$$\left(b \le a/2\right)$$

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Dominant Mode: TE$_{10}$ Mode

For this mode we have

$$m = 1,\ \ n = 0,\ \ k_c^{10} = \frac{\pi}{a}$$



Hence we have

$$H_z = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$H_x = j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$E_y = \underbrace{-\frac{j\omega\mu a}{\pi} A_{10}}_{E_{10}} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$k_z = k_z^{10} = \left(k^2 - \left(\frac{\pi}{a}\right)^2 \right)^{1/2}$$

$$E_y = E_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$A_{10} \equiv \frac{-\pi}{j\omega\mu a} E_{10}$$

$$E_x = E_z = H_y = 0$$

Dominant Mode: TE_{10} Mode (cont.)

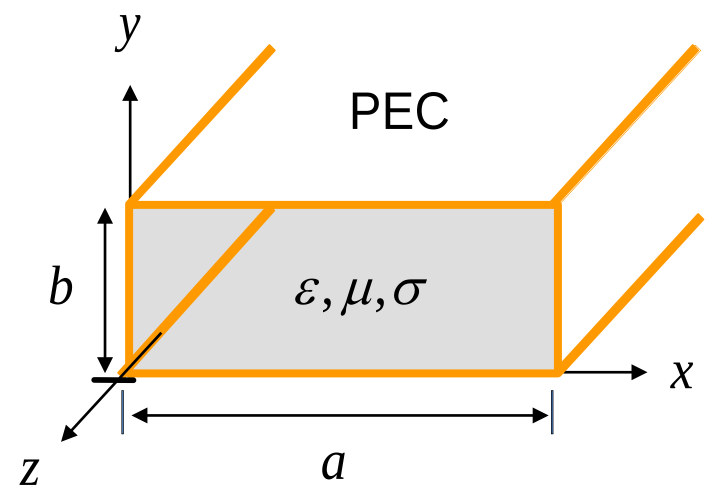
The fields can be put in terms of E_{10}:

$$E_y = E_{10} \sin\left(\frac{\pi}{a}x\right) e^{-jk_z z}$$

$$H_x = -\frac{1}{Z_{TE}} E_{10} \sin\left(\frac{\pi}{a}x\right) e^{-jk_z z}$$

$$H_z = \left(\frac{-\pi}{j\omega\mu a}\right) E_{10} \cos\left(\frac{\pi}{a}x\right) e^{-jk_z z}$$

$$E_x = E_z = H_y = 0$$



$$k_z = k_z^{10} = \left(k^2 - \left(\frac{\pi}{a}\right)^2\right)^{1/2}$$

$$Z_{TE} = \frac{\omega\mu}{k_z}$$

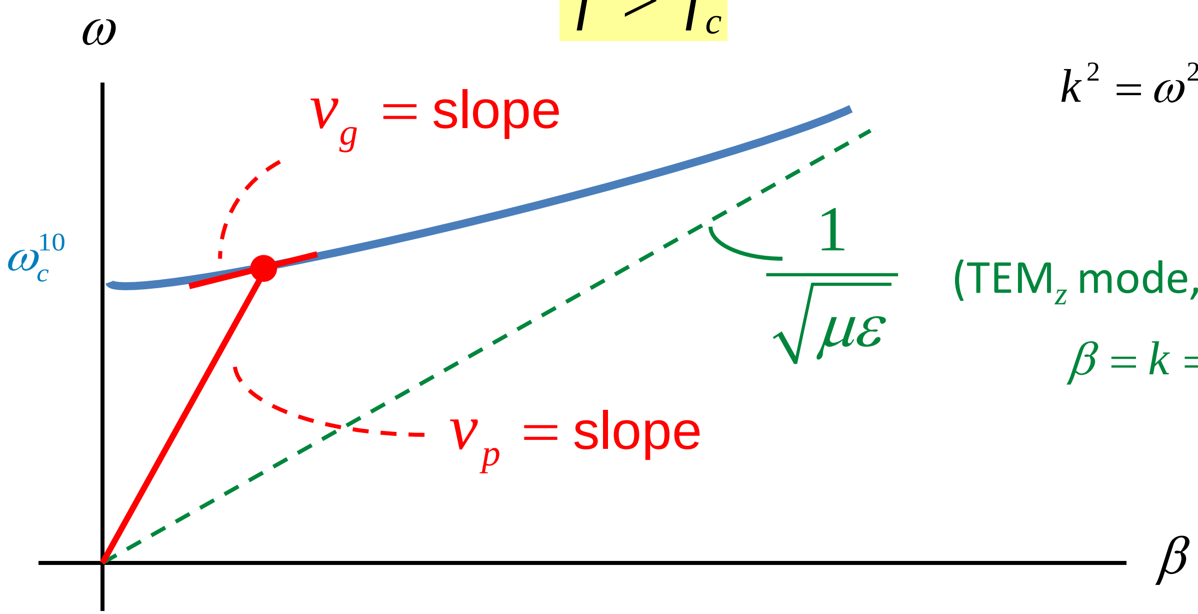
Dispersion Diagram for TE_{10} Mode

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$f > f_c$

$$k_z = \beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$k^2 = \omega^2 \mu \varepsilon$$



(TEM$_z$ mode, or "Light line")

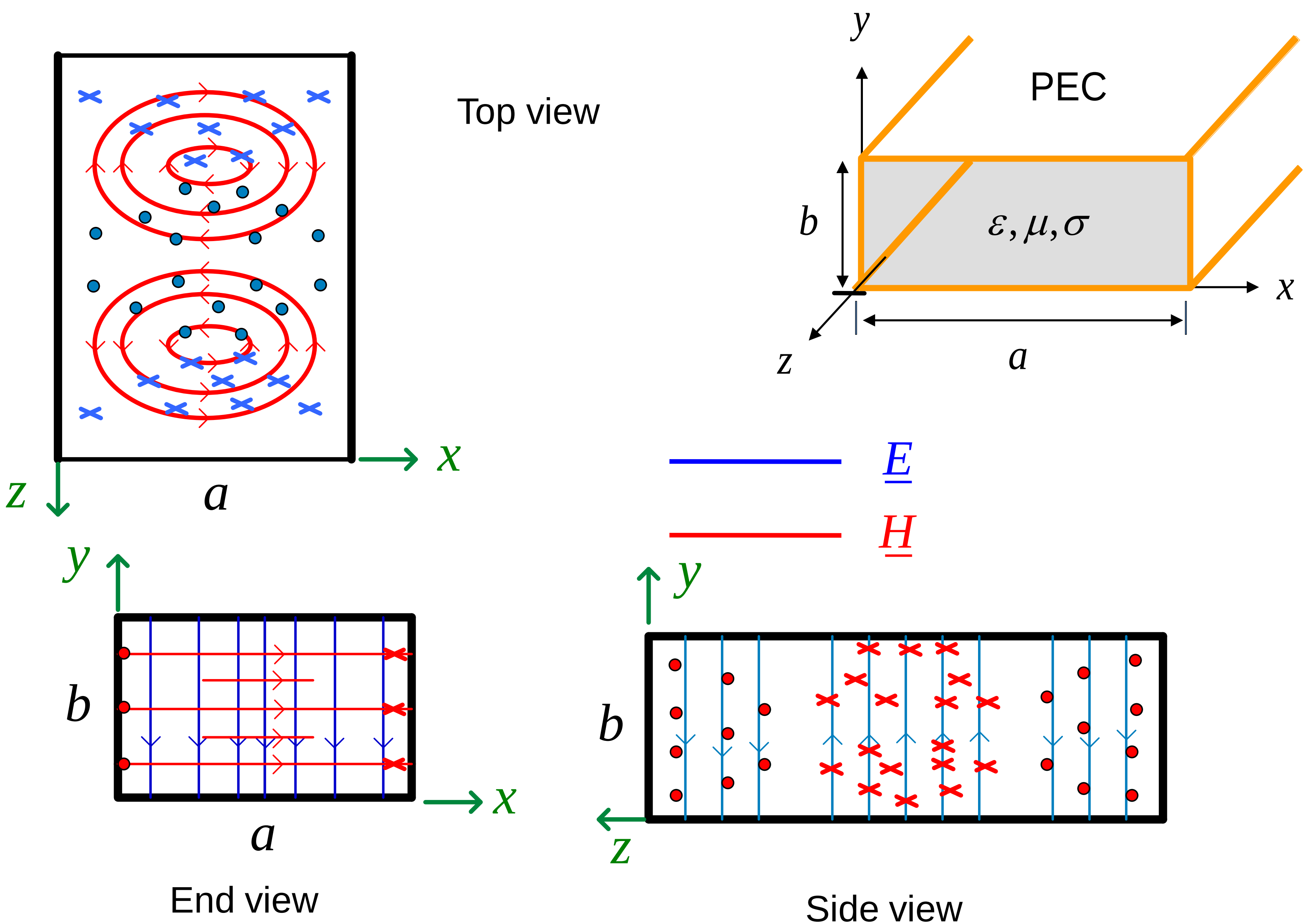
$$\beta = k = \omega\sqrt{\mu\varepsilon}$$

Phase velocity: $v_p = \dfrac{\omega}{\beta}$

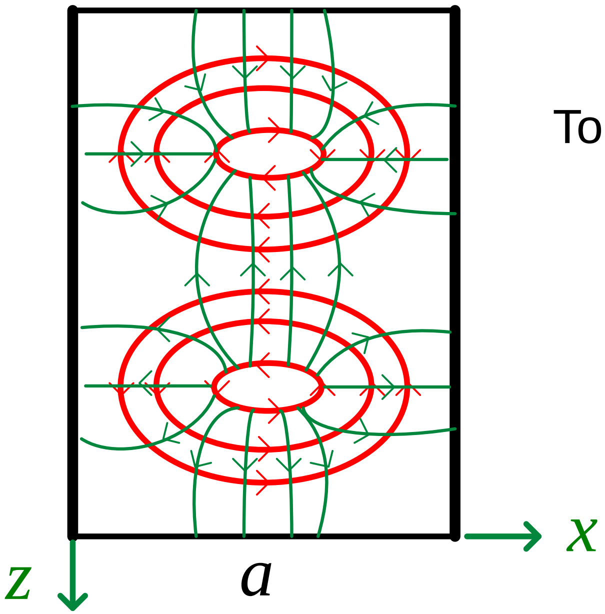
Group velocity: $v_g = \dfrac{d\omega}{d\beta}$

Velocities are slopes on the dispersion plot.

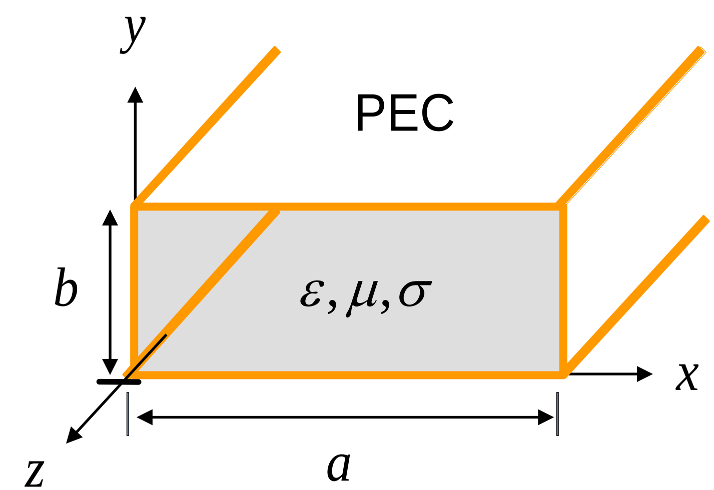
Field Plots for TE_{10} Mode

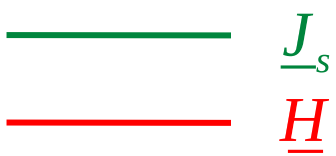


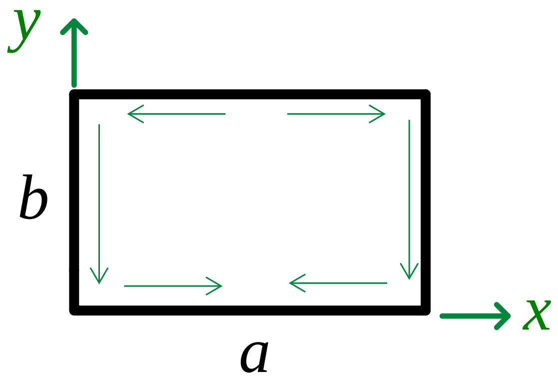
Field Plots for TE_{10} Mode (cont.)



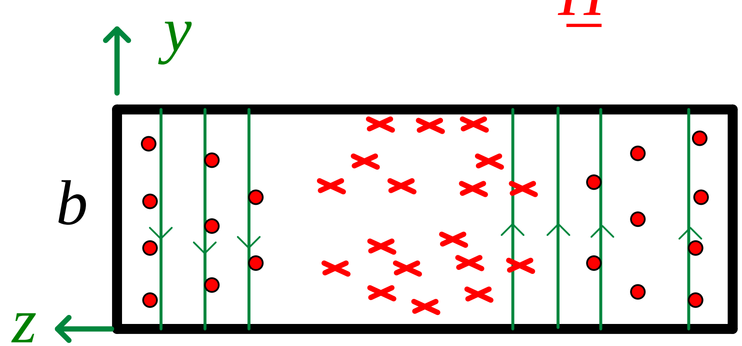
Top view







End view



Side view

Note: One can cut a narrow z-directed slot in the center of the top wall without disturbing the current.

Power Flow for TE_{10} Mode

Time-average power flow in the z direction for $+z$ mode:

$$P_{10}^{+} = \frac{1}{2}\operatorname{Re}\left\{\int_0^a\int_0^b \left(\underline{E}\times\underline{H}^*\right)\cdot\underline{\hat{z}}\,dy\,dx\right\}$$

$$= \frac{1}{2}\operatorname{Re}\left\{\int_0^a\int_0^b -E_y H_x^*\,dy\,dx\right\}$$

$$= \frac{1}{2}\operatorname{Re}\left(\left(\frac{ab}{2}\right)\left|E_{10}\right|^2\left(\frac{k_z}{\omega\mu}\right)e^{-2\alpha z}\right)$$

Note:

$$\int_0^a\int_0^b \sin^2\left(\frac{\pi x}{a}\right)dy\,dx = \frac{ab}{2}$$

$$E_y = E_{10}\sin\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

$$H_x = -\frac{1}{Z_{TE}}E_{10}\sin\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

$$Z_{TE} = \frac{\omega\mu}{k_z}$$

Simplifying, we have

$$P_{10}^{+} = \left(\frac{ab}{4\omega\mu}\right)\operatorname{Re}\left\{k_z\right\}\left|E_{10}\right|^2 e^{-2\alpha z}$$

At breakdown:

$$E_{10} = E_c$$

Note:
For a given maximum electric field level (e.g., the breakdown field), the power is increased by increasing the cross-sectional area (ab).

Dielectric Attenuation for TE_{10} Mode

From Notes 7 we have:

$$f > f_c$$

$$k_z = \beta - j\alpha_d = \sqrt{k^2 - k_c^2}$$

$$\beta = \mathrm{Re}\sqrt{k^2 - k_c^2}$$

$$\alpha_d = -\mathrm{Im}\sqrt{k^2 - k_c^2}$$

$$\beta \approx \sqrt{k_0^2 \mu_r \varepsilon_r - k_c^2}$$

$$\alpha_d \approx \frac{k_0^2 \mu_r \varepsilon_r \tan\delta_d}{2\beta}$$

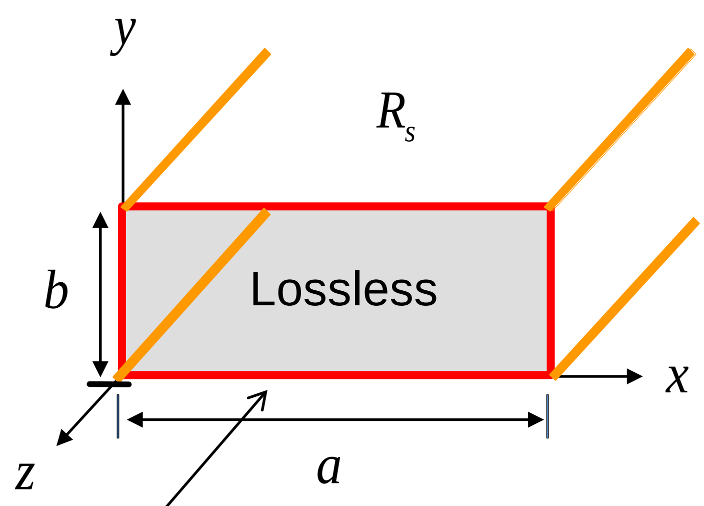
$$k = k' - jk'' = k_0 \sqrt{\mu_r \varepsilon_r}\sqrt{1 - j\tan\delta_d}$$

$$k_c = \frac{\pi}{a}$$

Conductor Attenuation for TE_{10} Mode

Recall $\alpha_c = \dfrac{P_l(0)}{2P_0}$

$P_0 = P_{10}^{+}\big|_{z=0}$ (calculated on previous slide)



$$P_l(0) = \frac{R_s}{2}\int_C \left|\underline{J}_s\right|^2 d\ell$$

$$\underline{J}_s = \underline{\hat{n}} \times \underline{H}$$ on conductor

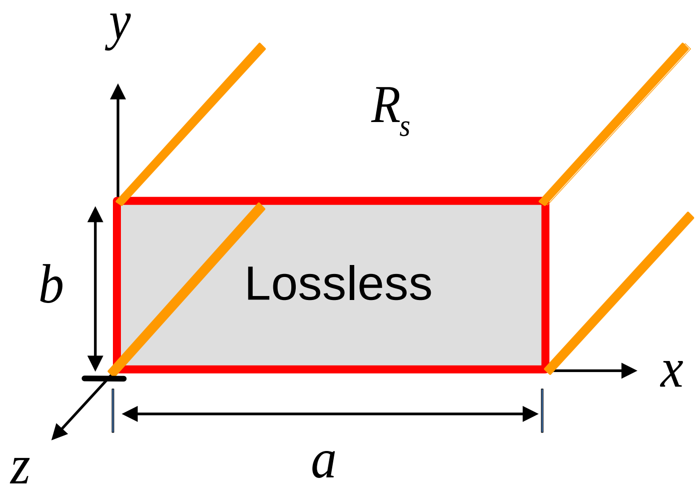
$$C = C^{\text{left}} + C^{\text{right}} + C^{\text{bot}} + C^{\text{top}}$$

Conductor Attenuation for TE_{10} Mode

Side walls

$$@\quad x=0:\quad \underline{J}_s^{\text{left}} = \hat{\underline{x}} \times \underline{H}\Big|_{x=0} = -\hat{\underline{y}} H_z = -\hat{\underline{y}} A_{10}\, e^{-jk_z z}$$

$$@\quad x=a:\quad \underline{J}_s^{\text{right}} = -\hat{\underline{x}} \times \underline{H}\Big|_{x=a} = \hat{\underline{y}} H_z = -\hat{\underline{y}} A_{10}\, e^{-jk_z z}$$



Hence:

$$J_{sy}^{\text{left}} = J_{sy}^{\text{right}} = -A_{10}\, e^{-jk_z z}$$

$$H_z = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$H_x = j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

Conductor Attenuation for TE_{10} Mode (cont.)

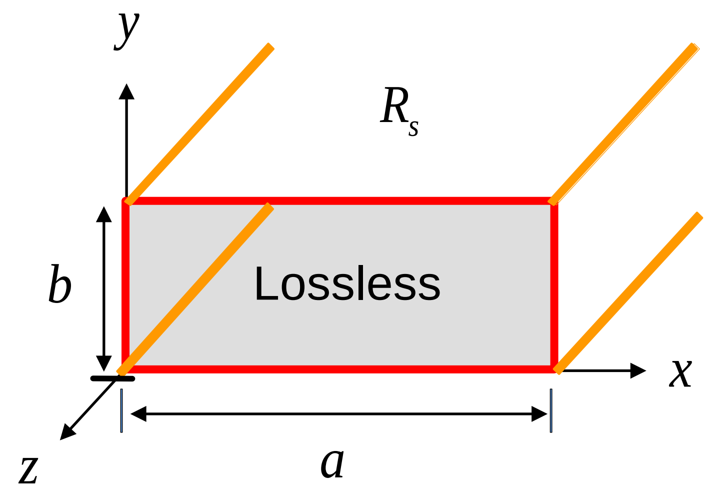
Top and bottom walls

$$@ \quad y=0: \quad \underline{J}_s^{\text{bot}} = \underline{\hat{y}} \times \underline{H}\Big|_{y=0}$$

$$@ \quad y=b: \quad \underline{J}_s^{\text{top}} = -\underline{\hat{y}} \times \underline{H}\Big|_{y=b}$$

$$\underline{J}_s^{\text{top}} = -\underline{J}_s^{\text{bot}}$$

(The fields of this mode are independent of y.)



Hence:

$$J_{sx}^{\text{bot}} = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

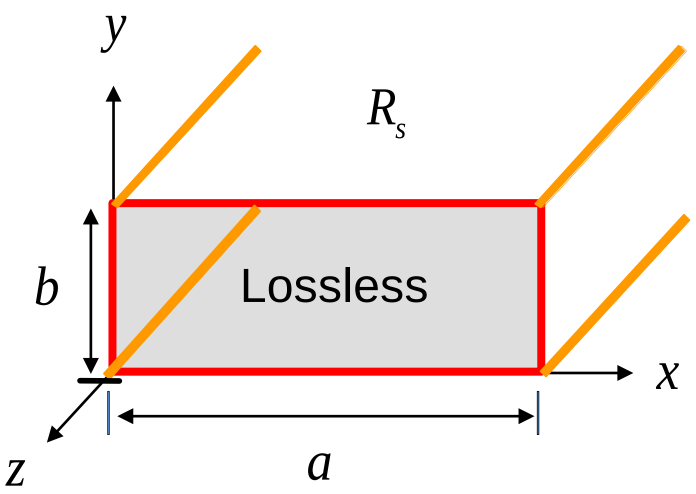
$$J_{sz}^{\text{bot}} = -j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$H_z = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

$$H_x = j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{-jk_z z}$$

Conductor Attenuation for TE_{10} Mode (cont.)

We then have:



$$P_l(0) = 2\left(\frac{R_s}{2}\int_0^b \left|\left|\underline{J}_s^{\text{left}}\right|^2\right| dy + \frac{R_s}{2}\int_0^a \left|\left|\underline{J}_s^{\text{bot}}\right|^2\right| dx\right)$$

$$= R_s \int_0^b \left|\left|J_{sy}^{\text{left}}\right|^2\right| dy + R_s \int_0^a \left(\left|J_{sx}^{\text{bot}}\right|^2 + \left|J_{sz}^{\text{bot}}\right|^2\right) dx$$

$$= R_s \int_0^b \left|-A_{10}\right|^2 dy + R_s \int_0^a \left(\left|A_{10}\cos\left(\frac{\pi}{a}x\right)\right|^2 + \left|-j\frac{k_z a}{\pi}A_{10}\sin\left(\frac{\pi}{a}x\right)\right|^2\right) dx$$

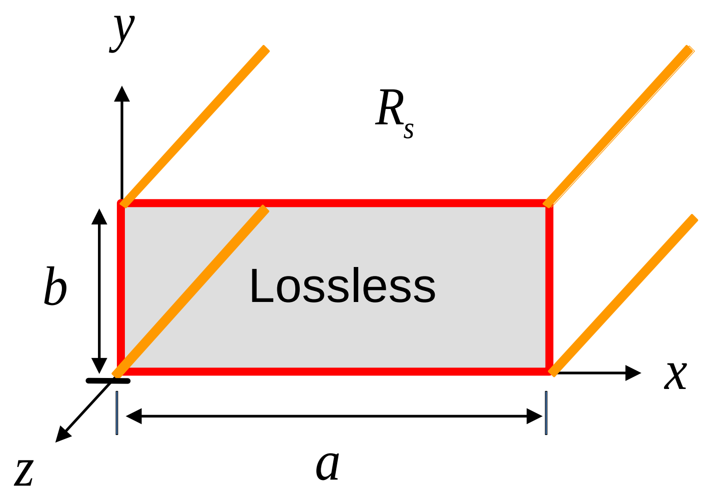
$$= R_s \left|A_{10}\right|^2 \left(\int_0^b dy + \int_0^a \cos^2\left(\frac{\pi}{a}x\right) dx + \left(\frac{|k_z| a}{\pi}\right)^2 \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx\right)$$

$$= R_s \left|A_{10}\right|^2 \left(b + \frac{a}{2} + \frac{|k_z|^2 a^2}{\pi^2}\frac{a}{2}\right)$$

Attenuation for TE_{10} Mode (cont.)

Assume $f > f_c$

$k_z \approx \beta$ (The wavenumber is taken as that of a guide with perfect walls.)



$$P_l(0) = R_s \left|A_{10}\right|^2 \left(b + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2} \right)$$

$$P_0 = \left(\frac{ab}{4\omega\mu} \right) \beta \left|E_{10}\right|^2$$

$$A_{10} = \frac{-\pi}{j\omega\mu a} E_{10}$$

Simplify, using $\beta^2 = k^2 - k_c^2$ $\quad k_c^{10} = \frac{\pi}{a}$

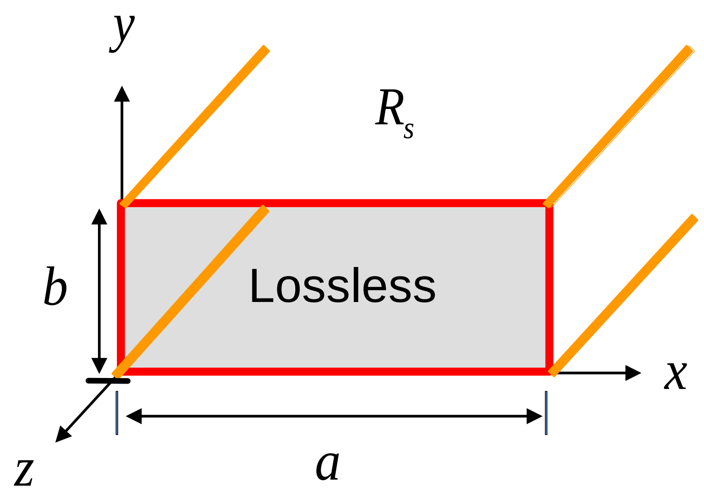
$$\alpha_c = \frac{P_l(0)}{2P_0}$$

Final result:

$$\alpha_c = \frac{R_s}{a^3 b \beta (k\eta)} \left(2b\pi^2 + a^3 k^2 \right) \quad [\text{np/m}]$$

Attenuation for TE_{10} Mode (cont.)

Final Formulas



Two alternative forms for the final result:

$$\alpha_c = \frac{R_s}{a^3 b \beta (k\eta)} \left(2b\pi^2 + a^3 k^2 \right) \quad [\text{np/m}]$$

$$\alpha_c = \frac{R_s}{b\eta} \frac{1}{\sqrt{1 - \left(f_c / f \right)^2}} \left(1 + \frac{2b}{a} \left(\frac{f_c}{f} \right)^2 \right) \quad [\text{np/m}]$$

Attenuation for TE_{10} Mode (cont.)

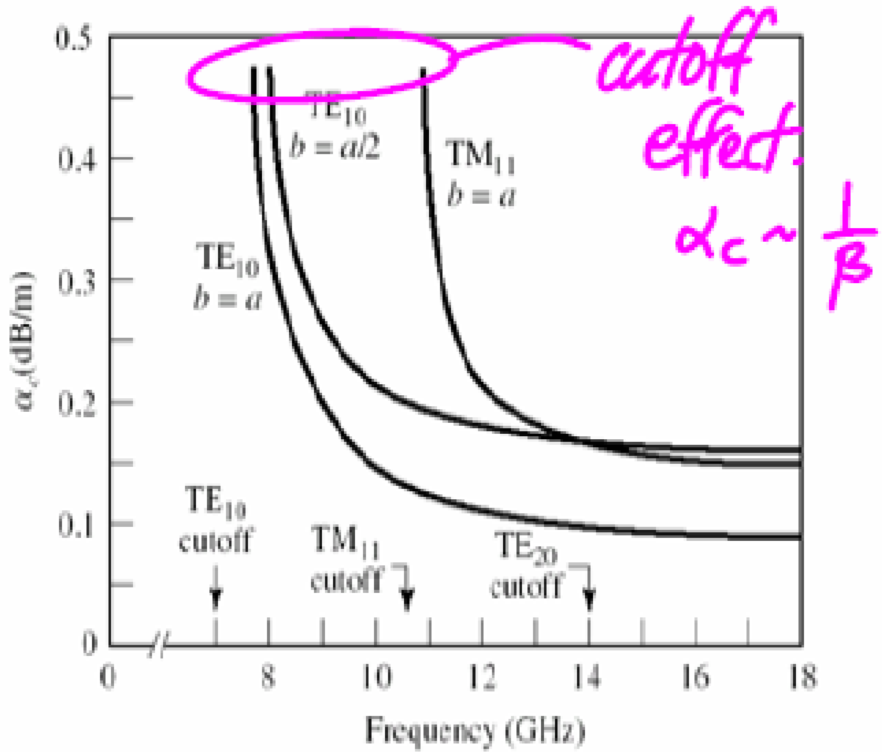
Brass X-band air-filled waveguide

$$\left(\sigma \approx 2.6\times 10^{7}\ [\text{S/m}]\right)$$

$$\text{X band: } 8-12\ [\text{GHz}]$$

(See the table on the next slide.)

$$a = 2.0 \text{ cm}$$



(from the Pozar book)

Attenuation for TE_{10} Mode (cont.)

Microwave Frequency Bands	
Letter Designation	**Frequency range**
L band	1 to 2 GHz
S band	2 to 4 GHz
C band	4 to 8 GHz
X band	8 to 12 GHz
Ku band	12 to 18 GHz
K band	18 to 26.5 GHz
Ka band	26.5 to 40 GHz
Q band	33 to 50 GHz
U band	40 to 60 GHz
V band	50 to 75 GHz
E band	60 to 90 GHz
W band	75 to 110 GHz
F band	90 to 140 GHz
D band	110 to 170 GHz

(from Wikipedia)

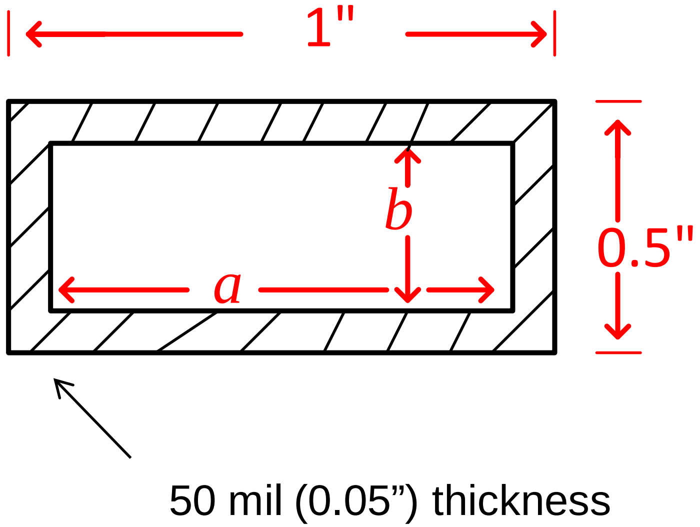
Modes in an X-Band Waveguide

$$a = 2.29\,\text{cm}\ (0.90\,\text{in})$$
$$b = 1.02\,\text{cm}\ (0.40\,\text{in})$$

"Standard X-band waveguide" (WR90)

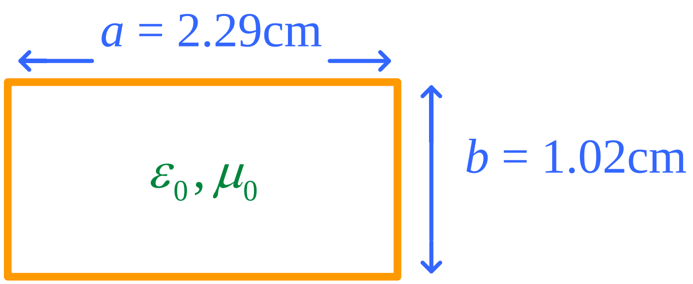
Mode	f_c [GHz]
TE_{10}	6.55
TE_{20}	13.10
TE_{01}	14.71
TE_{11}	16.10
TM_{11}	16.10
TE_{30}	19.65
TE_{21}	19.69
TM_{21}	19.69

X band: $8-12$ [GHz]



Example: X-Band Waveguide

Determine β, α, and λ_g (as appropriate) at 10 GHz and 6 GHz for the TE_{10} mode in a lossless air-filled X-band waveguide.



@ 10 GHz

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\pi}{a}\right)^2} = \sqrt{\left(\frac{2\pi 10^{10}}{2.99792458 \times 10^8}\right)^2 - \left(\frac{\pi}{0.0229}\right)^2}$$

$$\beta = 158.25 \text{ [rad/m]}$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{158.25} = 0.0397$$

$$\lambda_g = 3.97 \text{ [cm]}$$

Lossless: $k = \frac{2\pi}{\lambda_d} = \frac{2\pi}{c_d / f} = \frac{2\pi f}{c_d} = \frac{\omega}{c_d} = \omega\sqrt{\mu\varepsilon} = 2\pi f \sqrt{\mu\varepsilon}$

Example: X-Band Waveguide (cont.)

@ 6 GHz

$$k_z = \left(\omega^2 \mu \varepsilon - \left(\frac{\pi}{a} \right)^2 \right)^{1/2} = \left(\left(\frac{2\pi 6 \times 10^9}{2.99792458 \times 10^8} \right)^2 - \left(\frac{\pi}{0.0229} \right)^2 \right)^{1/2}$$

$$= -j \sqrt{\left(\frac{\pi}{0.0229} \right)^2 - \left(\frac{2\pi 6 \times 10^9}{2.99792458 \times 10^8} \right)^2}$$

$$= -j\, 55.04 \;\; [1/\text{m}]$$

$$\alpha = 55.04 \;\; [\text{np/m}]$$

$$= 478.08 \, [\text{dB/m}]$$

$$\lambda_g = \frac{2\pi}{\beta}$$

Evanescent mode: $\beta = 0$; λ_g is not defined!

Fields of a Guided Wave

Fields Equations in Cylindrical Coordinates

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon_c}{\rho}\frac{\partial E_z}{\partial \phi} \mp k_z \frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi = \frac{-j}{k_c^2}\left(\omega\varepsilon_c \frac{\partial E_z}{\partial \rho} \pm \frac{k_z}{\rho}\frac{\partial H_z}{\partial y}\right)$$

$$E_\rho = \frac{-j}{k_c^2}\left(\pm k_z \frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi = \frac{j}{k_c^2}\left(\mp \frac{k_z}{\rho}\frac{\partial E_z}{\partial \phi} + \omega\mu \frac{\partial H_z}{\partial \rho}\right)$$

These are useful for a circular waveguide.

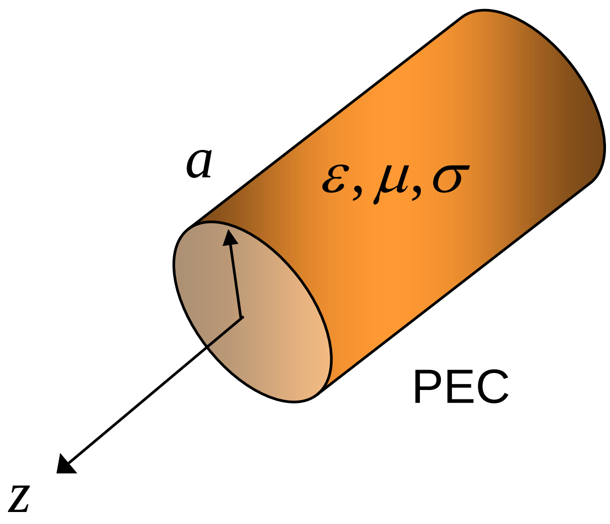
These equations give the transverse field components in terms of the longitudinal components, E_z and H_z.

$$F(z) = e^{\mp j k_z z}$$

$$k^2 = \omega^2 \mu \varepsilon_c$$

$$k_c = \sqrt{k^2 - k_z^2}$$

Circular Waveguide



TM$_z$ mode:

$$\nabla^2 e_z(\rho,\phi) = -k_c^2 e_z(\rho,\phi)$$

(eigenvalue problem)

$$k_z^2 = k^2 - k_c^2$$

The solution in cylindrical coordinates is:

This means any combination of these two functions.

$$e_z(\rho,\phi) = \begin{Bmatrix} J_n(k_c\rho) \\ Y_n(k_c\rho) \end{Bmatrix} \begin{Bmatrix} \sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}$$

Note: The value n must be an integer to have unique fields.

References for Bessel Functions

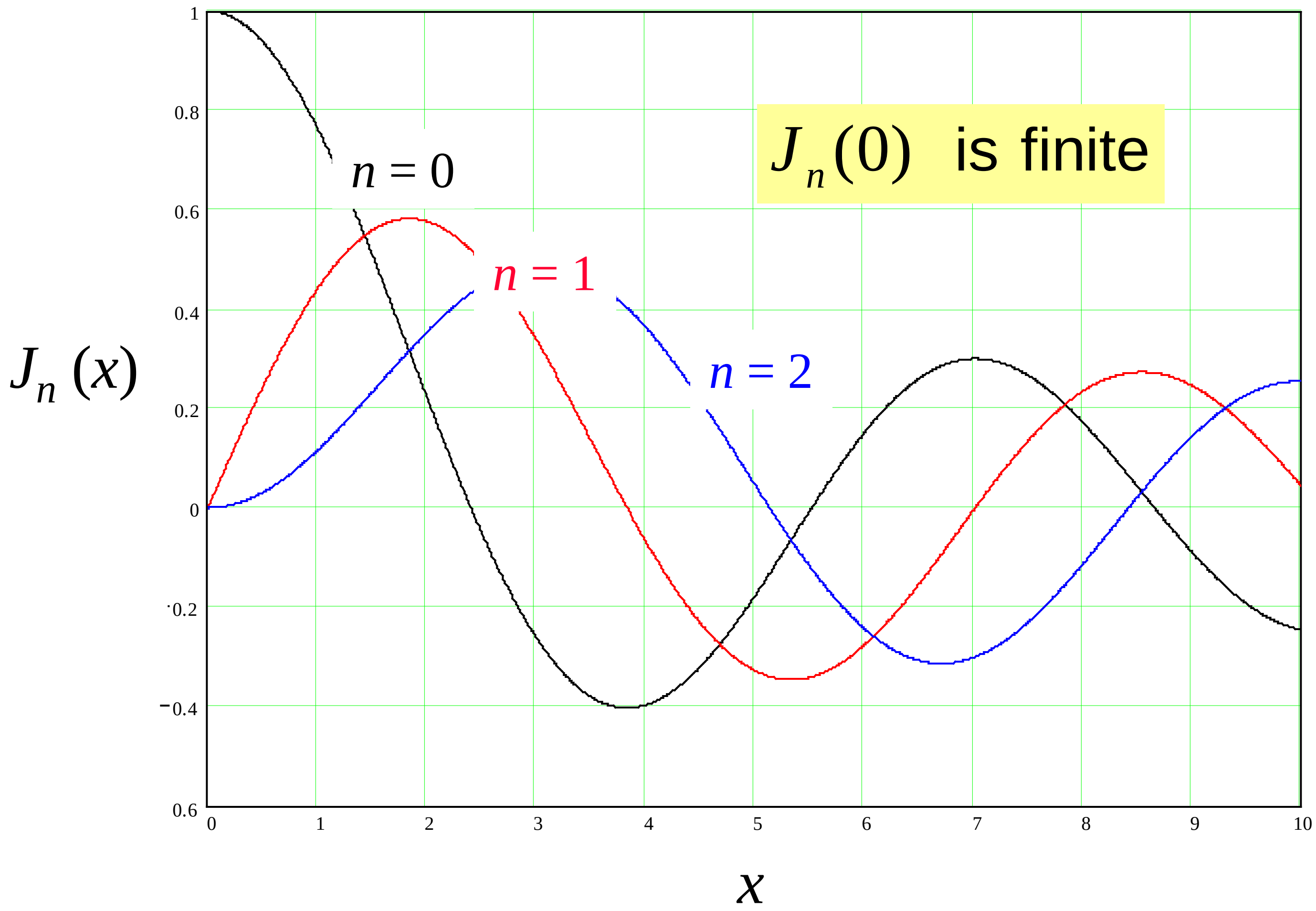
$J_n(x) =$ Bessel function of the first kind of order n

$Y_n(x) =$ Bessel function of the second kind of order n

References:

- M. R. Spiegel, *Schaum's Outline Mathematical Handbook*, McGraw-Hill, 1968.
- M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Government Printing Office, Tenth Printing, 1972.
- N. N. Lebedev, *Special Functions & Their Applications*, Dover Publications, New York, 1972.

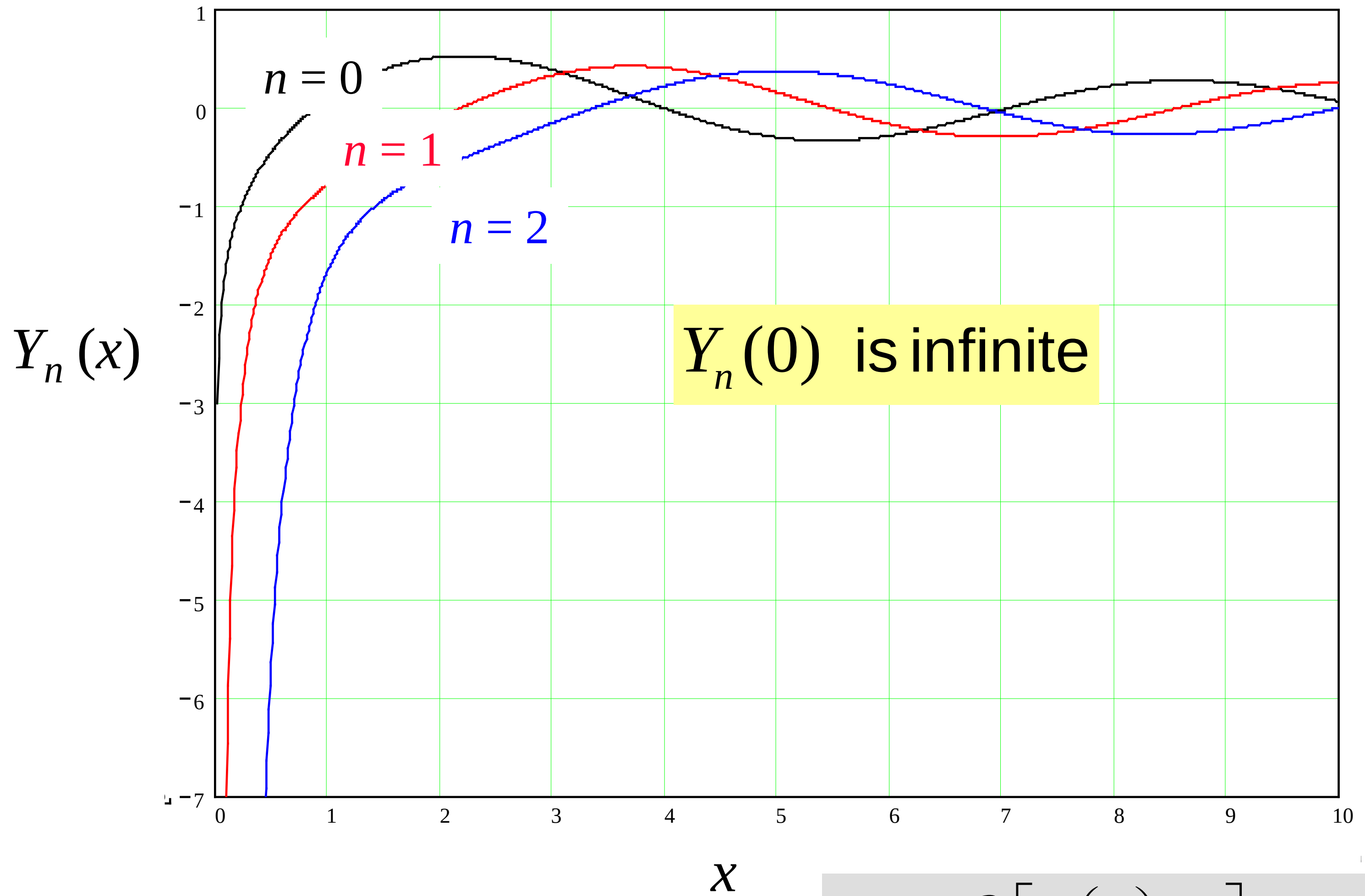
Plot of Bessel Functions



$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \qquad x \to \infty$$

$$J_n(x) \sim x^n\left(\frac{1}{2^n n!}\right) \quad n = 0, 1, 2, \ldots., \quad x \to 0$$

Plot of Bessel Functions (cont.)



$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma\right],\ \gamma = 0.5772156,\ x \to 0$$

$$Y_n(x) \sim -\frac{1}{\pi}(n-1)!\left(\frac{2}{x}\right)^n,\ n = 1,2,3,\ldots\ldots,\ x \to 0$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),\quad x \to \infty$$

Circular Waveguide (cont.)

Choose (somewhat arbitrarily) $\cos(n\phi)$

$$e_z(\rho,\phi) = \begin{Bmatrix} J_n(k_c\rho) \\ Y_n(k_c\rho) \end{Bmatrix} \cos(n\phi)$$

The field should be <u>finite</u> on the z axis.

⇨ $Y_n(k_c\rho)$ is not allowed

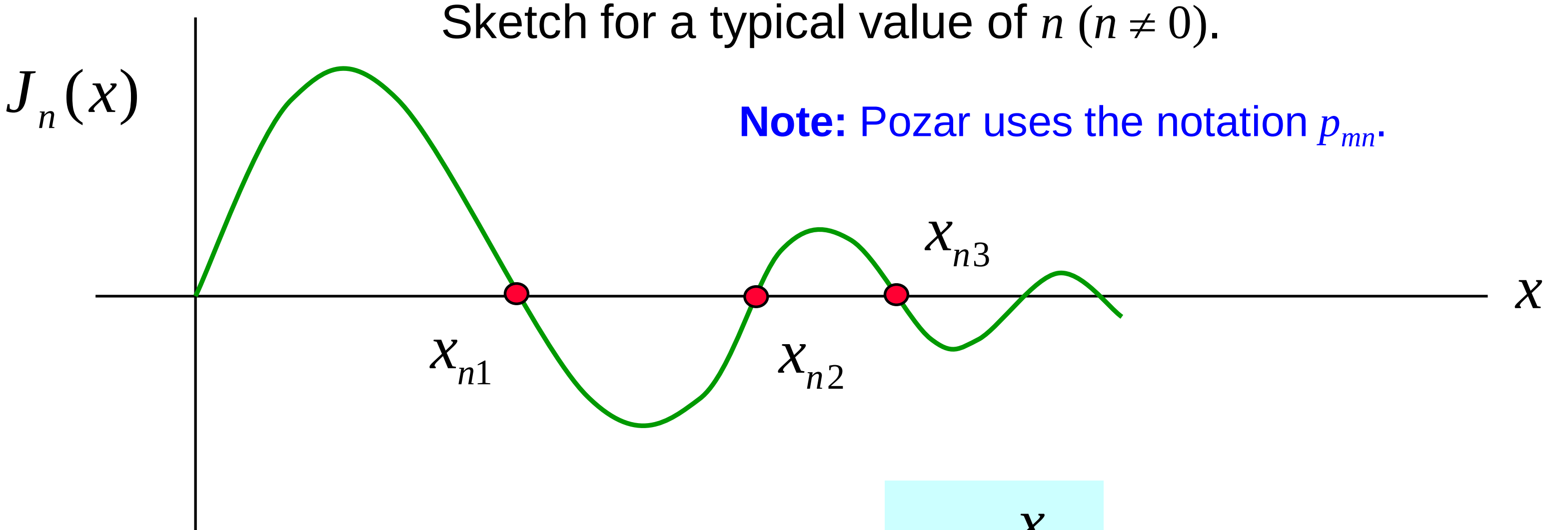
Hence, we have $e_z(\rho,\phi) = \cos(n\phi)\, J_n(k_c\rho)$

$$E_z(\rho,\phi,z) = \cos(n\phi)\, J_n(k_c\rho)\, e^{-jk_z z}$$

Circular Waveguide (cont.)

B.C.'s: $E_z(a,\phi,z)=0$ Hence $J_n(k_c a)=0$

Sketch for a typical value of n $(n \neq 0)$.



Note: Pozar uses the notation p_{mn}.

$$k_c a = x_{np} \Rightarrow k_c = \frac{x_{np}}{a}$$

Note: The value $x_{n0}=0$ is not included since this would yield a trivial solution:

$$J_n\left(x_{n0}\frac{\rho}{a}\right)=J_n(0)=0$$

(This is true unless $n=0$, in which case we cannot have $p=0$.)

Circular Waveguide (cont.)

TM$_{np}$ mode:

$$E_z(\rho,\phi,z) = \cos(n\phi)\, J_n\left(x_{np}\frac{\rho}{a}\right)e^{-jk_z z} \qquad n = 0,1,2\ldots$$

$$k_z = \sqrt{k^2 - \left(\frac{x_{np}}{a}\right)^2} \qquad p = 1,2,3,\ldots\ldots\ldots$$

Cutoff Frequency: TM$_z$

$$k_z^2 = k^2 - k_c^2$$

Assume k is real here.

At $f = f_c$: $k_z = 0$ ⇒ $k = k_c = \dfrac{x_{np}}{a}$

$$2\pi f_c^{\mathrm{TM}} \sqrt{\mu\varepsilon} = \frac{x_{np}}{a}$$

$$f_c^{\mathrm{TM}} = \left(\frac{c_d}{2\pi a}\right) x_{np}$$

$$c_d \equiv \frac{c}{\sqrt{\varepsilon_r}}$$

Cutoff Frequency: TM_z (cont.)

x_{np} values

$p \setminus n$	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

TM_{01}, TM_{11}, TM_{21}, TM_{02}, ……..

TE$_z$ Modes

Proceeding as before, we now have that

$$H_z(\rho,\phi,z) = \cos(n\phi)\, J_n(k_c\rho)\, e^{-jk_z z}$$

Set $\quad E_\phi(a,\phi,z) = 0$

$$\left(H_\rho = 0\Big|_{\rho=a}\right)$$

$$E_\phi = \frac{1}{j\omega\varepsilon_c}\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)$$

(From Ampere's law)

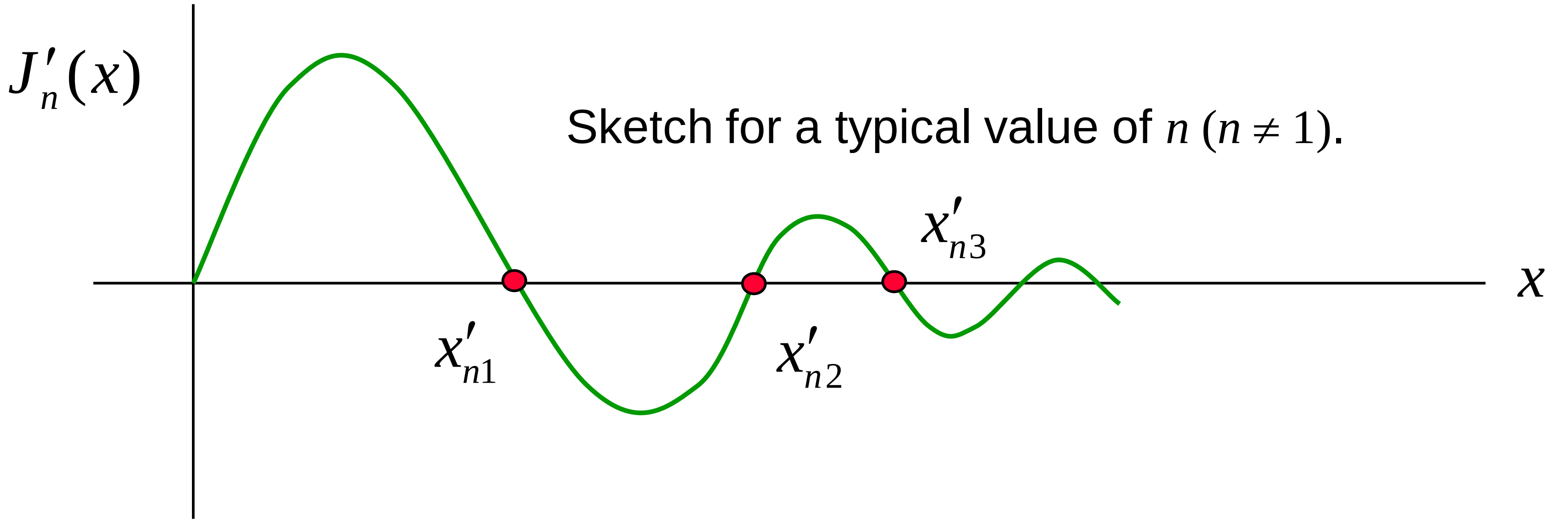
$$\Rightarrow \frac{\partial H_z}{\partial \rho} = 0\Bigg|_{\rho=a}$$

Hence $\quad J_n'(k_c a) = 0$

The prime denotes derivative with respect to the argument.

TE_z Modes (cont.)

$$J_n'(k_c a) = 0$$



Sketch for a typical value of n $(n \neq 1)$.

$$k_c a = x_{np}'$$

$$k_c = \frac{x_{np}'}{a} \qquad p = 1, 2, 3, ...$$

We don't need to consider $p = 0$; this is explained on the next slide.

TE_z Modes (cont.)

$$H_z(\rho,\phi,z) = \cos(n\phi)\, J_n\left(x'_{np}\frac{\rho}{a}\right) e^{-jk_z z} \qquad p = 1, 2, \ldots$$

Note: If $p = 0$, then $x'_{np} = 0$

We then have, for $p = 0$:

$$n \neq 0 \qquad J_n\left(x'_{np}\frac{\rho}{a}\right) = J_n(0) = 0 \qquad \text{(trivial solution)}$$

$$n = 0 \qquad J_0\left(x'_{np}\frac{\rho}{a}\right) = J_0(0) = 1$$

$$\Rightarrow H_z = e^{-jk_z z} \Rightarrow \underline{H} = \underline{\hat{z}}\, e^{-jk_z z} \Rightarrow \underline{H} = \underline{\hat{z}}\, e^{-jkz} \qquad \text{(nonphysical solution)}$$

(violates the magnetic Gauss law)

The TE_{00} mode is not physical.

Circular Waveguide (cont.)

TE$_{np}$ mode:

$$H_z(\rho,\phi,z) = \cos(n\phi)\, J_n\left(x'_{np}\frac{\rho}{a}\right)e^{-jk_z z} \qquad n = 0,1,2\ldots$$

$$k_z = \sqrt{k^2 - \left(\frac{x'_{np}}{a}\right)^2} \qquad p = 1,2,3,\ldots\ldots\ldots$$

Cutoff Frequency: TE_z

$$k_z^2 = k^2 - k_c^2$$

Assume k is real here.

$$k_z = 0 \quad \Rightarrow \quad k_c = k = \frac{x'_{np}}{a}$$

$$2\pi f_c^{TE} \sqrt{\mu\varepsilon} = \frac{x'_{np}}{a}$$

Hence

$$f_c^{TE} = \left(\frac{c_d}{2\pi a}\right) x'_{np}$$

$$c_d = \frac{c}{\sqrt{\varepsilon_r}}$$

Cutoff Frequency: TE_z

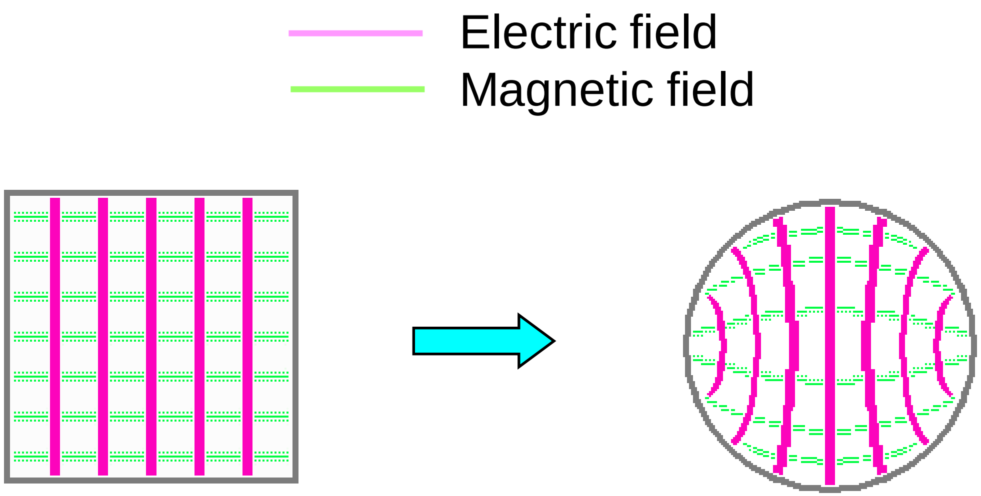
x'_{np} values

$p \backslash n$	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	5.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

TE_{11}, TE_{21}, TE_{01}, TE_{31}, ……..

TE_{11} Mode

The <u>dominant mode</u> of circular waveguide is the TE_{11} mode.



(From Wikipedia)

TE_{10} mode of rectangular waveguide

TE_{11} mode of circular waveguide

The TE_{11} mode can be thought of as an evolution of the TE_{10} mode of rectangular waveguide as the boundary changes shape.

TE_{11} Mode (cont.)

The attenuation due to conductor loss for the TE_{11} mode is:

$$\alpha_c = \frac{R_s}{a\eta} \frac{1}{\sqrt{1-\left(f_c / f\right)^2}} \left(\left(\frac{f_c}{f} \right)^2 + \frac{1}{x_{11}'^2 - 1} \right)$$

$$x_{11}' = 1.841$$

$$k_c = \frac{x_{11}'}{a}$$

The derivation is in the Pozar book (see Eq. 3.133).

TE_{01} Mode

The TE_{01} mode of circular waveguide has the unusual property that the conductor attenuation decreases with frequency. (With most waveguide modes, the conductor attenuation increases with frequency.)

$$\alpha_c^{TE_{01}} = \frac{R_s}{a\eta} \frac{\left(f_c / f\right)^2}{\sqrt{1-\left(f_c / f\right)^2}}$$

Reason: This mode has current only in the ϕ direction, and this component of current (corresponding to H_z) decreases as the frequency increases (for a fixed power flow down the guide, i.e., a fixed E_ϕ). (Please see the equations on the next slide.)

The TE_{01} mode was studied extensively as a candidate for long-range communications – but eventually fiber-optic cables became available with even lower loss. It is still useful for some high-power applications.

Note: This mode is not the dominant mode!

TE_{01} Mode (cont.)

The fields of the TE_{01} mode are:

$$H_z = J_0\left(x'_{01}\frac{\rho}{a}\right)e^{-jk_z z}$$

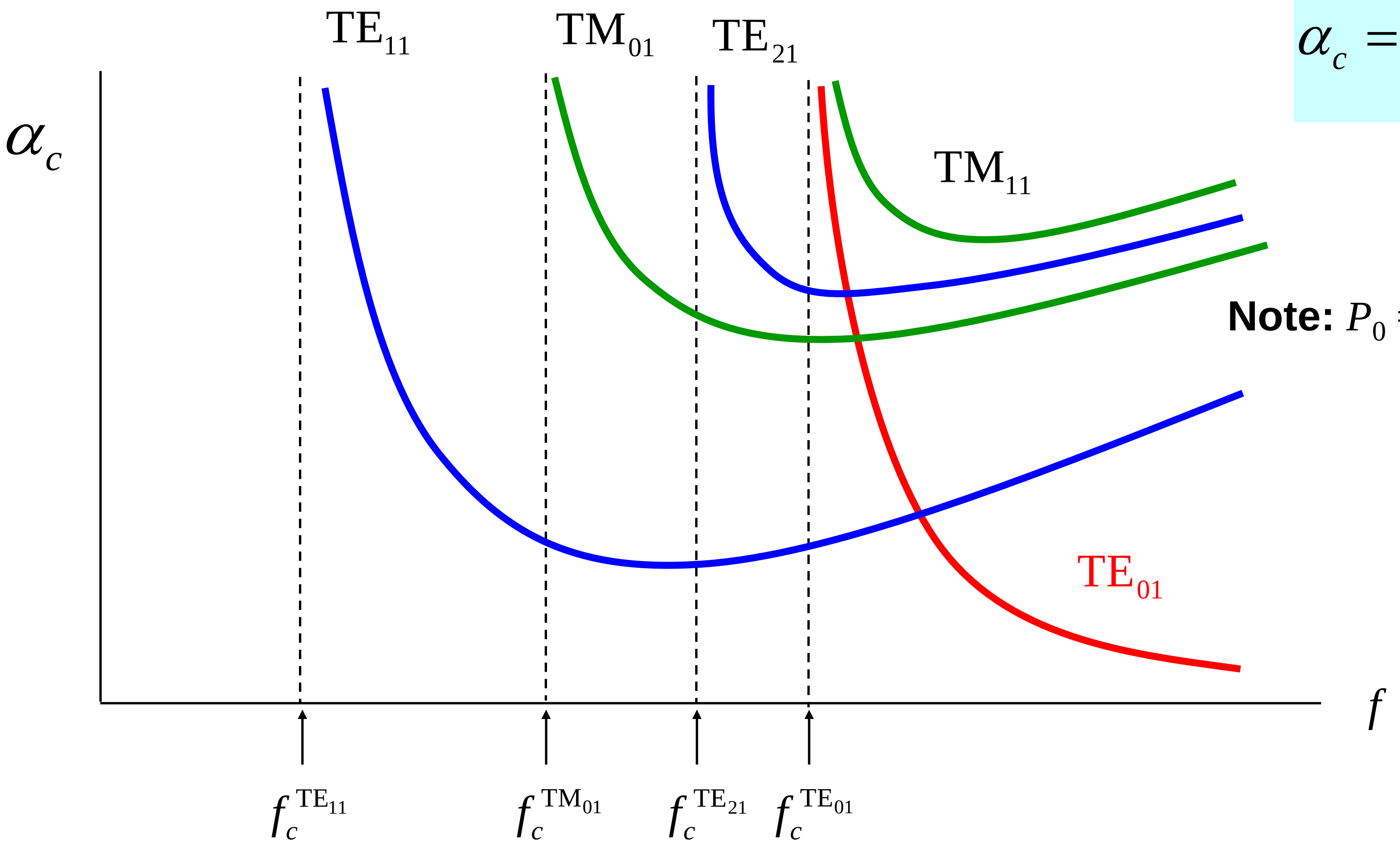
$$E_\phi = j\omega\mu\left(\frac{x'_{01}}{a}\right)\frac{1}{k_c^2}J'_0\left(x'_{01}\frac{\rho}{a}\right)e^{-jk_z z}$$

$$H_\rho = -E_\phi / Z_{TE}^{(0,1)}$$

$$Z_{TE}^{(0,1)} = \frac{\omega\mu}{k_z^{(0,1)}}$$

TE_{01} Mode (cont.)

$$\alpha_c = \frac{P_l(0)}{2P_0}$$



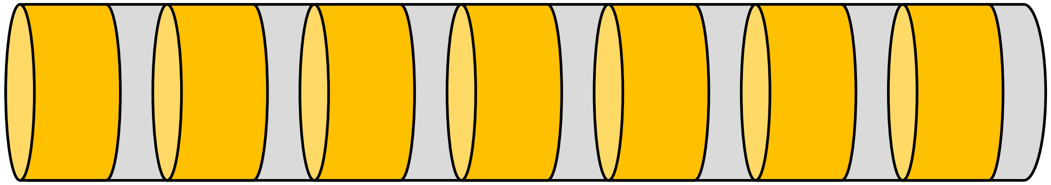
Note: $P_0 = 0$ at cutoff

Note: The attenuation increases at high frequency for all <u>other</u> modes, due to R_s.

TE_{01} Mode (cont.)

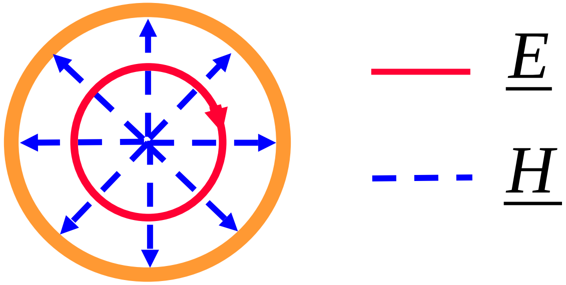
Practical Note:

The TE_{01} mode has only an azimuthal (ϕ - directed) surface current on the wall of the waveguide. Therefore, it can be supported by a set of conducting rings, while the lower modes (TE_{11} ,TM_{01}, TE_{21}, TM_{11}) will not propagate on such a structure.



(A helical spring will also work fine.)

TE_{01} mode:



TE_{01} Mode (cont.)

VertexRSI's Torrance Facility is a leading supplier of antenna feed components for the various commercial and military bands. A patented circular polarized 4-port diplexer meeting all Intelsat specifications leads a full array of products.

Products include:
4-Port Diplexers, CP or Linear;
3-Port Diplexers, 2xRx & 1xTx;
2-Port Diplexers, RxTx, X-Pol or Co-Pol, CP or Linear;
TE21 Monopulse Tracking Couplers;
TE01 Mode Components; Transitions;
Filters; Flex Waveguides;
Waveguide Bends; Twists; Runs; etc.

Many of the items are "off the shelf products".

Products can be custom tailored to a customer's application.

Many of the products can be supplied with standard feed horns for prime or offset antennas.

TE_{01} Mode (cont.)

From the beginning, the most obvious application of waveguides had been as a communications medium. It had been determined by both Schelkunoff and Mead, independently, in July 1933, that an axially symmetric electric wave (TE_{01}) in circular waveguide would have an attenuation factor that decreased with increasing frequency [44]. This unique characteristic was believed to offer a great potential for wide-band, multichannel systems, and for many years to come the development of such a system was a major focus of work within the waveguide group at BTL. It is important to note, however, that the use of waveguide as a long transmission line never did prove to be practical, and Southworth eventually began to realize that the role of waveguide would be somewhat different than originally expected. In a memorandum dated October 23, 1939, he concluded that microwave radio with highly directive antennas was to be preferred to long transmission lines. "Thus," he wrote, “we come to the conclusion that the hollow, cylindrical conductor is to be valued primarily as a new circuit element, but not yet as a new type of toll cable” [45]. It was as a circuit element in military radar that waveguide technology was to find its first major application and to receive an enormous stimulus to both practical and theoretical advance.

K. S. Packard, “The origins of waveguide: A case of multiple rediscovery,” *IEEE Trans. Microwave Theory and Techniques,* pp. 961-969, Sept. 1984.

TE_{01} Mode (cont.)

"In a memorandum dated October 23, 1939, he concluded that microwave radio with highly directive antennas was to be preferred to long transmission lines."

Recall the comparison of dB attenuation:

Waveguiding system: $$\text{dB} = 8.686(\alpha z)$$

Wireless system: $$\text{dB} = -10\log_{10}(G_t G_r) - 20\log_{10}\left(\frac{\lambda_0}{4\pi}\right) + 20\log_{10}(r)$$