Lightwave review

(For ECE 6323) han q le - copyrighted

1. Basic

1.0 A note about wave optics and ray optics (geometrical optics)

- Wave optics is the mathematically rigorous “classical” theory of light (which does not include quantum theory.

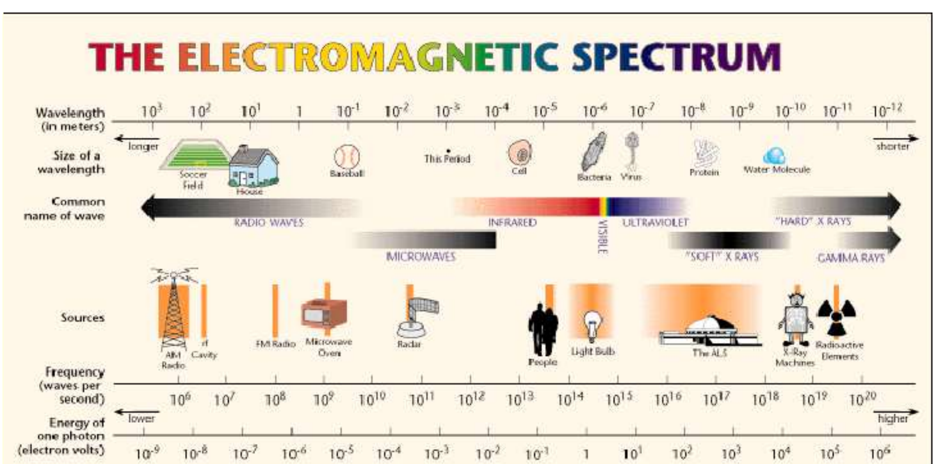
- Ray (or geometrical) optics is a mathematical approximation of wave optics when the wavelength is very small compared with the sizes of the objects or geometrical features of interest. Since this is often true in many optical system applications, it is not unusual to see that ray optics, which is intuitively easier to apply is often used instead of wave optics.

- However, for optical communications, ray optics is largely inadequate to truly understand or do any engineering practice.

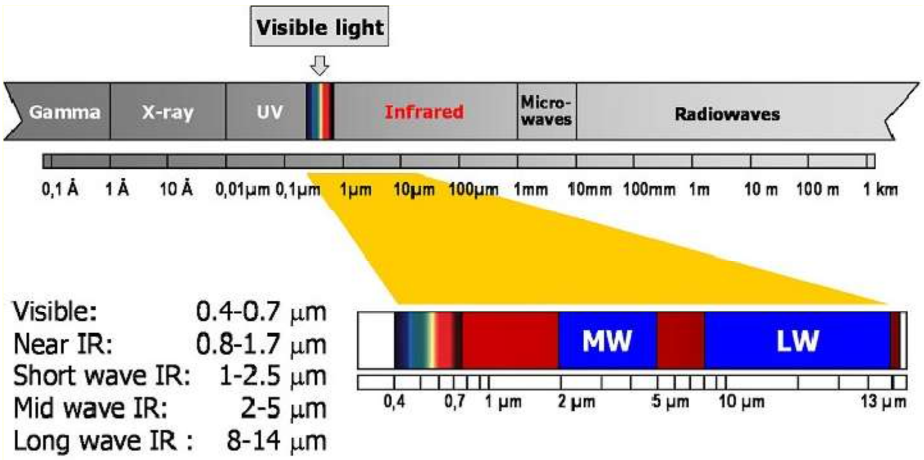
- In addition, even wave optics alone is not sufficient to describe many other important phenomena. Fundamentally, the more complete theories would be - in this order: quantum electrodynamics (QED), quantum electro-weak theory, and GUT - grand -unifying theory,...

We don't really need any of those except for quantum optics, which is a part of QED dealing with photons, the particle of light with low energy - in the electron-Volt (eV) energy range for optical communications.

1.1 Electromagnetic wave theory



Important wavelength range for optical communications: 1.6-1.4 μm, 1.3 μm. For LAN, 0.8 μm.



1.2 Maxwell's equations

The electromagnetic field is represented by 2 vector fields
- **E**: electric field
- **H**: magnetic vector field

In a dielectric medium such as silica (e. g. optical fiber) we need two more vector fields:
- **D**: electric dispacement field: $\mathbf{D} = \mathbf{E} + 4\pi\,\mathbf{P}$
where **P** is the polarization field in the medium
- **B**: magnetic induction field: $\mathbf{B} = \mathbf{H} + 4\pi\,\mathbf{M}$
where **M** is the magnetization field of the medium

The EM fields are generated by sources, which are:
- **J**: electric current density: $\mathbf{J} = \sigma\,.\,\mathbf{E}$, and
- ρ: electric charge density.

For optics, we don't need to include the sources.

Maxwell's equations without sources

We use the CGS-Gaussian system of unit.

Maxwell equations:

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$$

$$\nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = 0$$

$$\nabla\,.\,\mathbf{D} = 0$$

$$\nabla\,.\,\mathbf{B} = 0$$

In a linear medium, polarization **P** and magnetization **M** are linear with respect to **E** and **H**:

$$\mathbf{P} = \chi\,.\,\mathbf{E}\ ,$$

where χ is a second-rank tensor called electric susceptibility, and

$$\mathbf{M} = \eta\,.\,\mathbf{H}$$

where η is the magnetic susceptibility.

Thus:

$\mathbf{D} = \mathbf{E} + 4\pi\,\mathbf{P} = (1 + 4\pi\,\chi)\,\mathbf{E} = \epsilon\,\mathbf{E}$

$\mathbf{B} = \mathbf{H} + 4\pi\,\mathbf{M} = (1 + 4\pi\,\eta)\,\mathbf{H} = \mu\,\mathbf{H}$

In practice for optics, we

1.3 Light quantum

Light carries energy as we know. We learn from modern (quantum) physics that light consists of quantized package energy known as photons. (we will have a review of quantum physics later in the course). Each photon has an energy:

$$E = h\,\nu$$

where h is the Planck's constant.

Thus, the fundamenal description of light is really a harmonic oscillator, which means that light is not described with arbitrary time-dependent function but with harmonic function:

$$\mathbf{E}[\mathbf{r}, t] = \mathbf{E}[\boldsymbol{r}; \omega]\ e^{-i\,\omega\,t}$$

(We will use complex notation throughout. Sin or Cos are obtained from the Im or the Re part of the expression.)

1.4 Wave equations vs. Helmholtz equation

The most important implication of the Maxwell equations is the existence of EM waves. One can derive from the Maxwell's equations:

$$\nabla^2 \mathbf{E} - \frac{\mu\,\epsilon}{c^2}\,\frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

$$\nabla^2 \mathbf{H} - \frac{\mu\,\epsilon}{c^2}\,\frac{\partial^2}{\partial t^2}\mathbf{H} = 0$$

which are known as the wave equations.
We need only one of them (either one), because once we have either field, the other one can be determined via the Maxwell's equations.

Note: in vacuum (for CGS-Gaussian units we use here), μ= 1, ϵ=1 and the wave propagates with the speed of light c.

However, from the above, we know that the fundamental description of light is harmonic wave, hence, the truly operational equation for dealing with light is not the wave equations above but:

$$\nabla^2 \mathbf{F} + \frac{\omega^2}{c^2}\,\mu\,\epsilon\,\mathbf{F} = 0:$$

where F represents either E or H. This is Helmholtz's equation.
It is obtain by simply using $\mathbf{E}[\mathbf{r}, t] = \mathbf{E}[\boldsymbol{r}; \omega]\ e^{-i\,\omega\,t}$ in the wave Eq.

We define: $\qquad k_0 = \frac{\omega}{c}$

which is also referred to as the wave number or the propagation constant.

1.5 Intensity, power, and Energy

Lightwaves carry energy. It is also directional, which means that the energy flows in certain direction. A key parameter describing the energy flow is the Poynting vector, defined as:

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}$$

(*note*: use real representatives of the fields, i. e. $\mathbf{S} = \frac{c}{4\pi}\,\mathrm{Re}[\mathbf{E}] \times \mathrm{Re}[\mathbf{H}]$).

The magnitude of the Poynting vector represents the intensity I or the power density $\left(\frac{dP}{dS}\right)$ of a lightwave. But for light, instantaneous intensity usually does not have much meaning. The relevant quantity is the time-average of the intensity over one cycle (period) of the harmonic wave. It can be shown that we can also write:

$$I \equiv \langle \vec{\mathbf{S}} \rangle = \frac{c}{8\pi}\,\mathrm{Re}\left[\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*\right]$$

The brackets ⟨ ⟩ here denotes time-averaged quantity and the conjugate of $\vec{\mathbf{H}}^*$ removes the time-dependence term.

Power is the integration of intensity over a surface or the cross section (area) of a surface of interest:

$$P = \int_A I_n \, dS$$

where I_n denotes the intensity that is normal to the surface at the integration point of the surface. More generally, we can also write:

$$P = \int_A \langle \vec{\mathbf{S}} \rangle \, . \, \hat{\mathbf{n}} \, dS$$

where $\hat{\mathbf{n}}$ is the normal unit vector of the surface.
If the surface is in a plane with a finite area A:

$$P = I_n \, A$$

Energy is the accumulated power over a certain period of time:

$$E = \int P[t] \, dt$$

The above is the classical concept. From the quantum theory, we actually start with energy as the fundamental concept.

As mentioned above, light consists of photons. Each has a unique frequency with a package of (quantized) energy:

$$E = h\,\nu$$

where h is the Planck's constant. Depending on the system, a photon also has quantum state with respect to space.

The classical description of light wave is simply the statistical ensemble average of a large number of photons.

If we have a spatial density of $N_{\mathrm{ph}}[\nu]$ photons in a medium with the speed $\frac{c}{n}$, the power crossing a surface area S in the direction of light propagation is:

$$P = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(E\,A\,\frac{c}{n}\,\Delta t\right) = \frac{c}{n}\,N_{\mathrm{ph}}[\nu]\,h\,\nu\,S \qquad (6.1.2)$$

In the limit S-> 0, we have power density or intensity:

$$\lim_{\Delta S \to 0} \frac{\Delta P}{\Delta S} = I = \frac{c}{n}\,N_{\mathrm{ph}}[\nu]\,h\,\nu \qquad (6.1.3)$$

Here, we study classical wave optics, where we deal with a large number of photons, not having to worry about the quantum behavior of light.

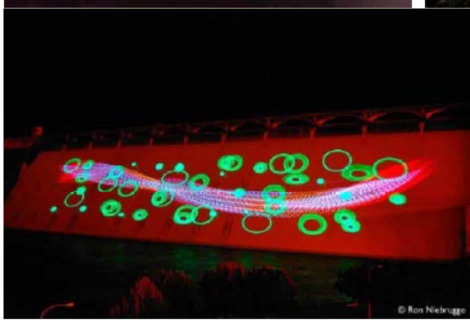
A review test of background knowledge

0.1 Laser beam - Gaussian beam model

Why laser light can be made traveling straight and narrow, but non-laser light seems to always spread?

Light is just ... light: electromagnetic (EM) wave, so why are there "different" behaviors of light?
How do we describe (mathematically) a laser beam that we see often?

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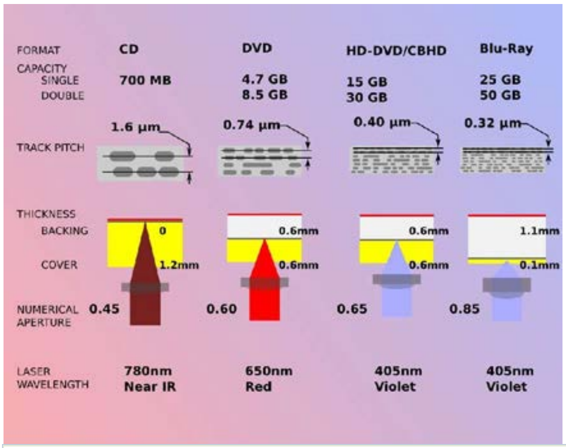
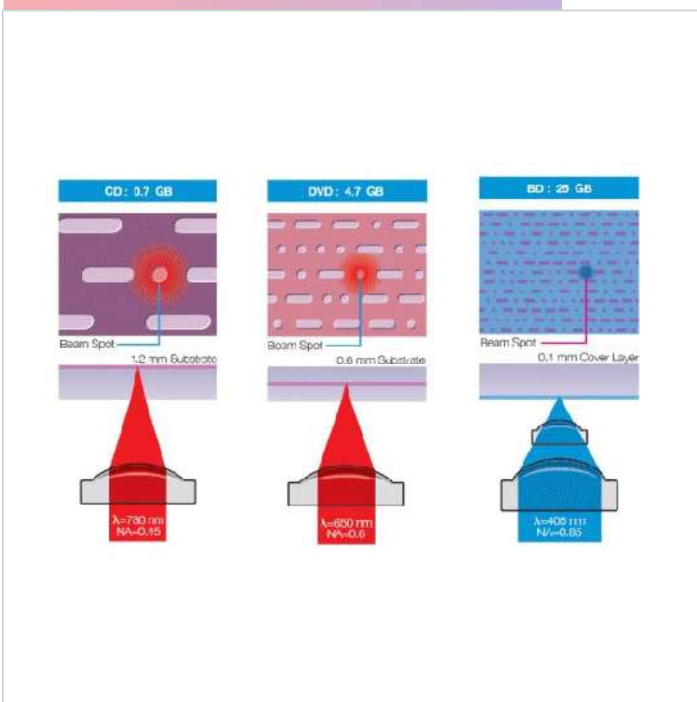
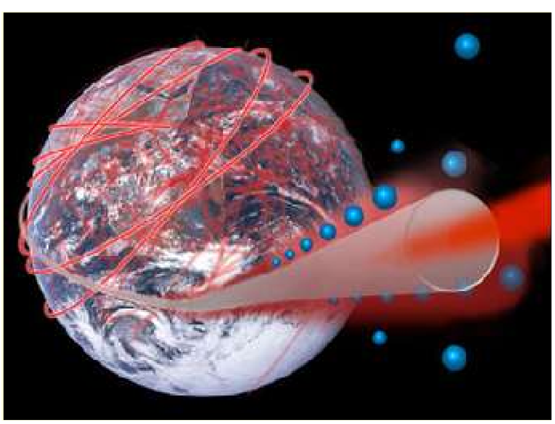
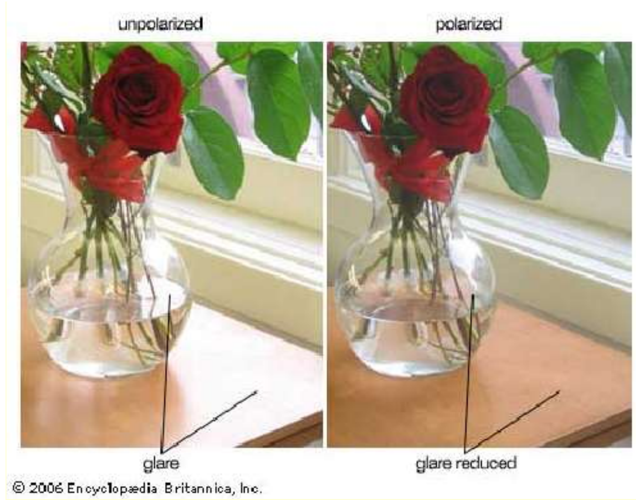
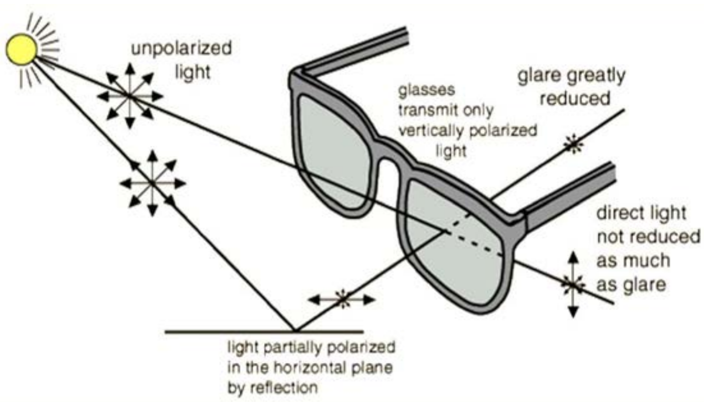




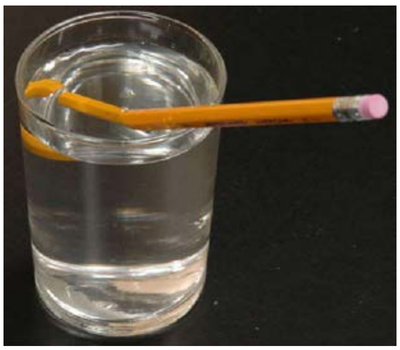


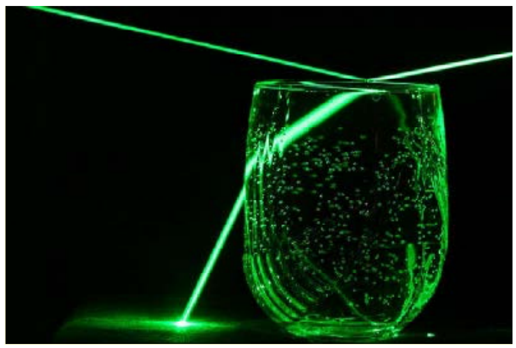
0.2 Light polarization

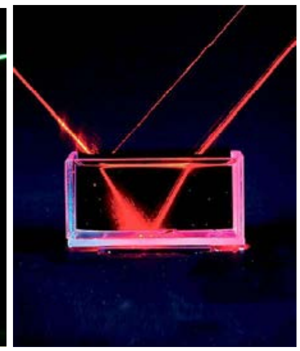




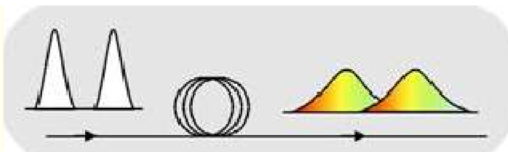
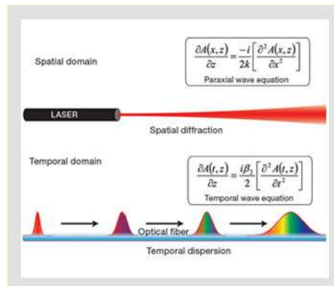
0.3 Reflection, refraction, total internal reflection, evanescence wave





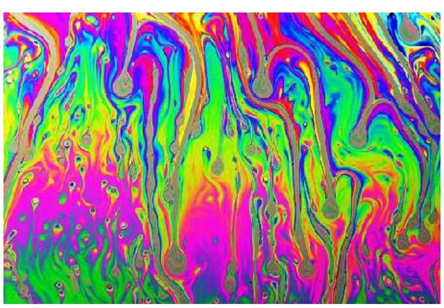


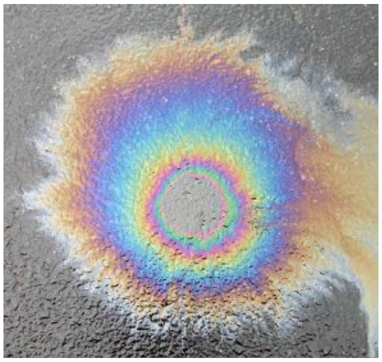
0.4 Dispersion

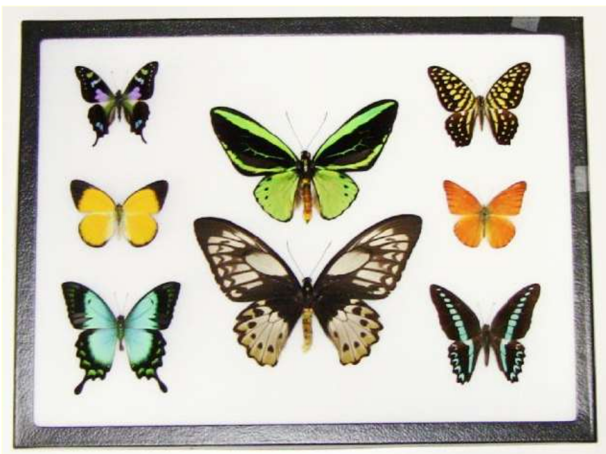
is this the same phenomenon as the rainbow or prism ? (No. we’ll learn later how they are different)
A key thing about learning and knowledge is “discrimation”: the ability to distinguish things that may appear similar superficially but fundamentally different”. An analogy: resolution of a camera.

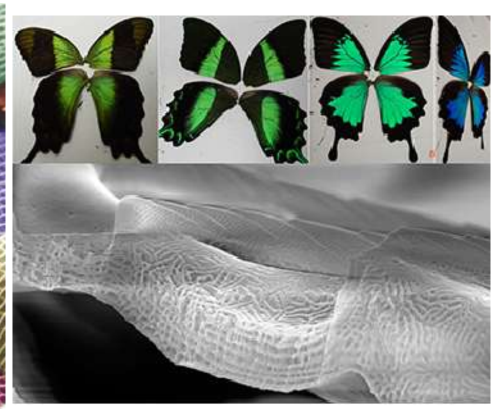
is this the same phenomenon as that of the rainbow or the optical disk? (No. we’ll learn later how they are different)



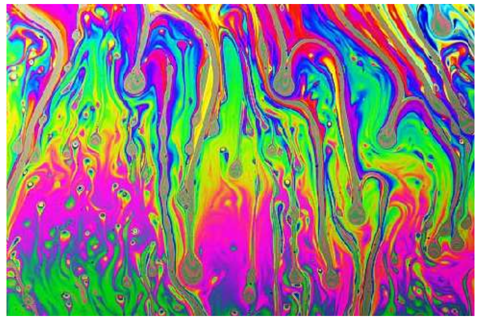
What phenomenon above is most similar to this?

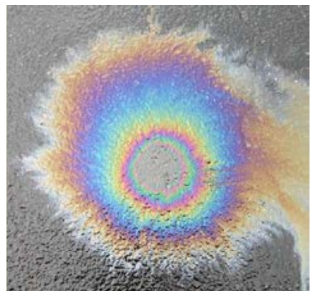




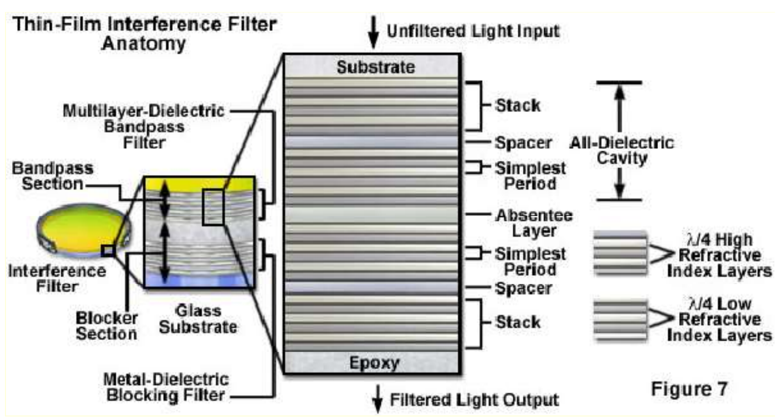
Credit: Opt. Mat. Exp.; and H.L. Tam and K. W. Cheah

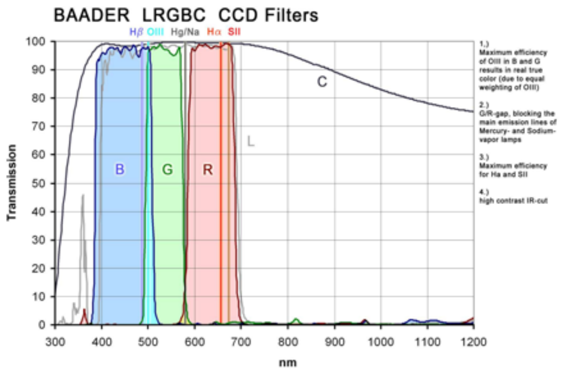
0.5 Interference

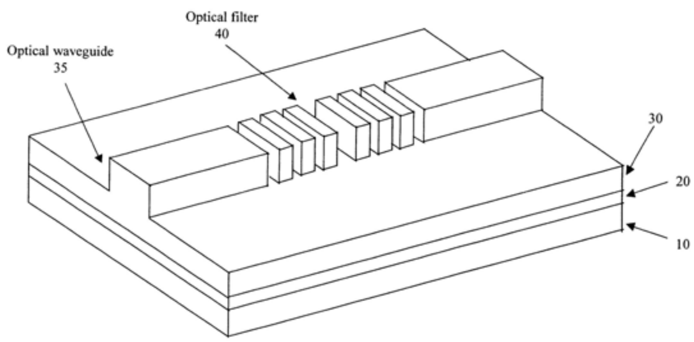



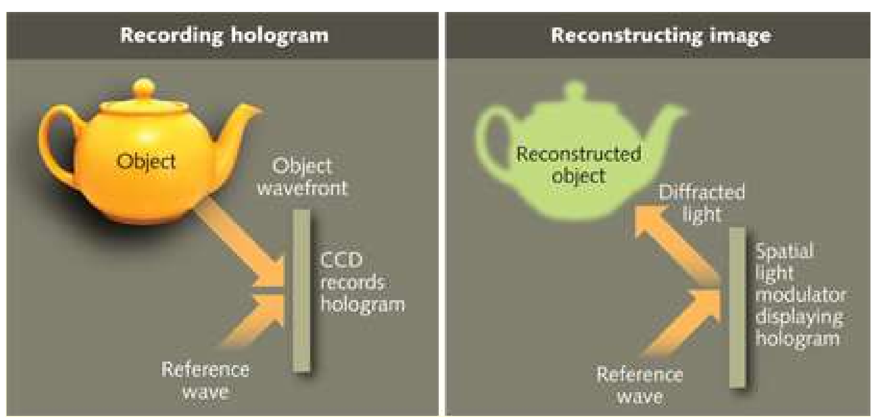


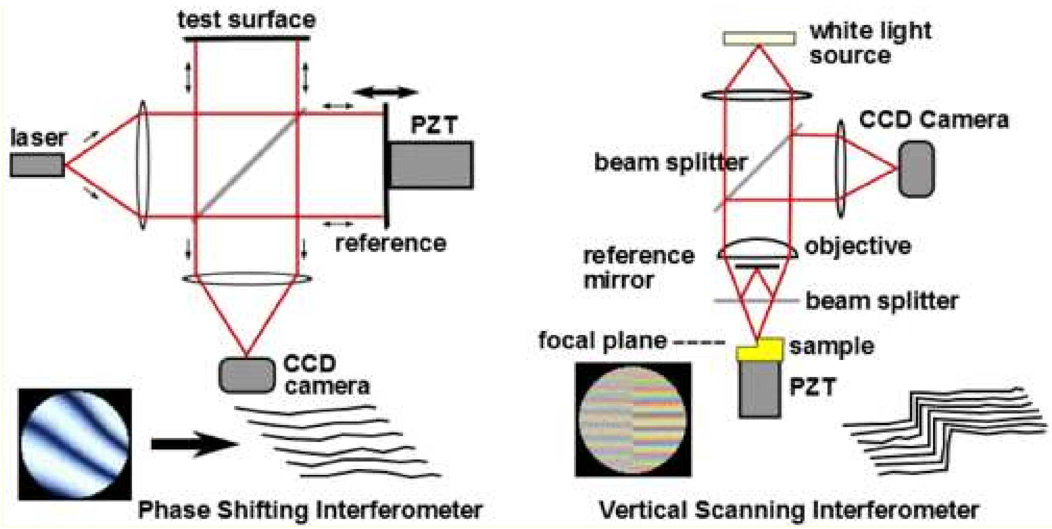

http://www.olympusmicro.com/primer/techniques/fluorescence/interferencefilterintro.html

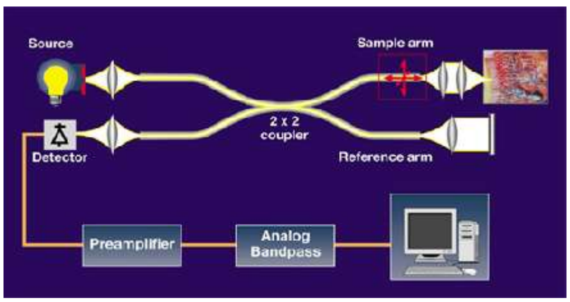


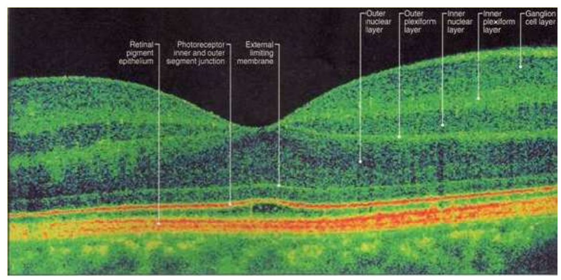

BAADER LRGBC CCD Filters
Hβ OIII Hg/Na Hα SII
Transmission
nm
B
G
R
L
C
1.)
Maximum efficiency of OIII in B and G results in real true color (due to equal weighting of OIII)
2.)
G/R-gap, blocking the main emission lines of Mercury- and Sodium-vapor lamps
3.)
Maximum efficiency for Ha and SII
4.)
high contrast IR-cut

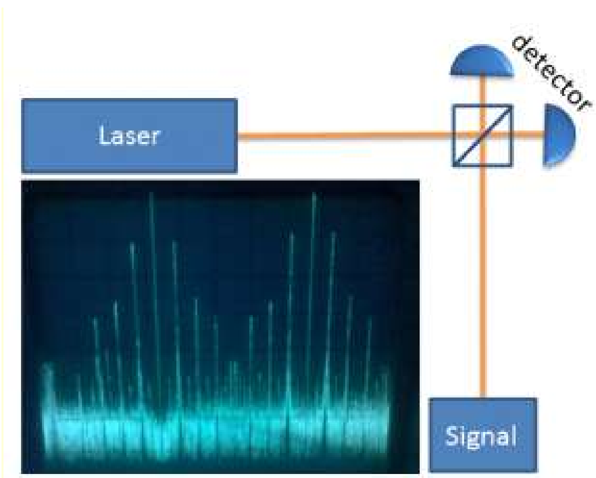

Optical filter
40
Optical waveguide
35
30
20
10


Recording hologram
Object
Object wavefront
CCD records hologram
Reference wave
Reconstructing image
Reconstructed object
Diffracted light
Spatial light modulator displaying hologram
Reference wave





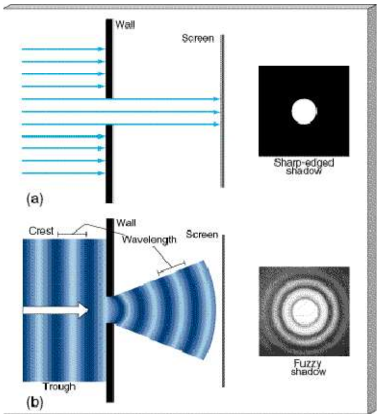


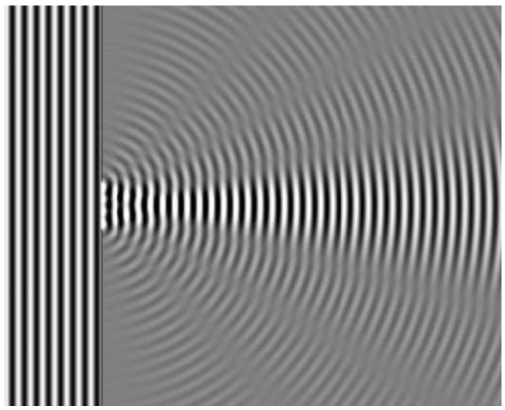


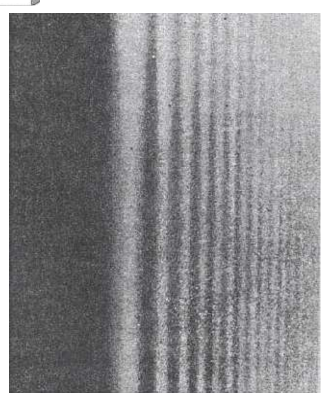
0.6 Diffraction

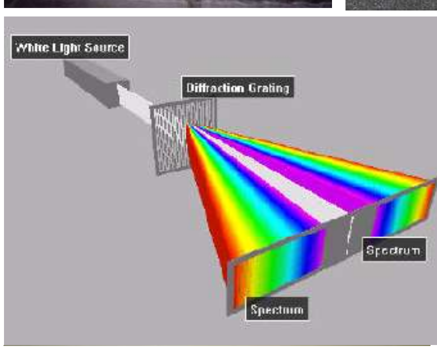
Is this the same phenomenon as the rainbow or prism ? No, it is diffraction

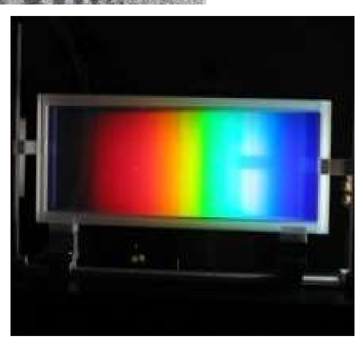
How does light moves, interacts with objects, structures?

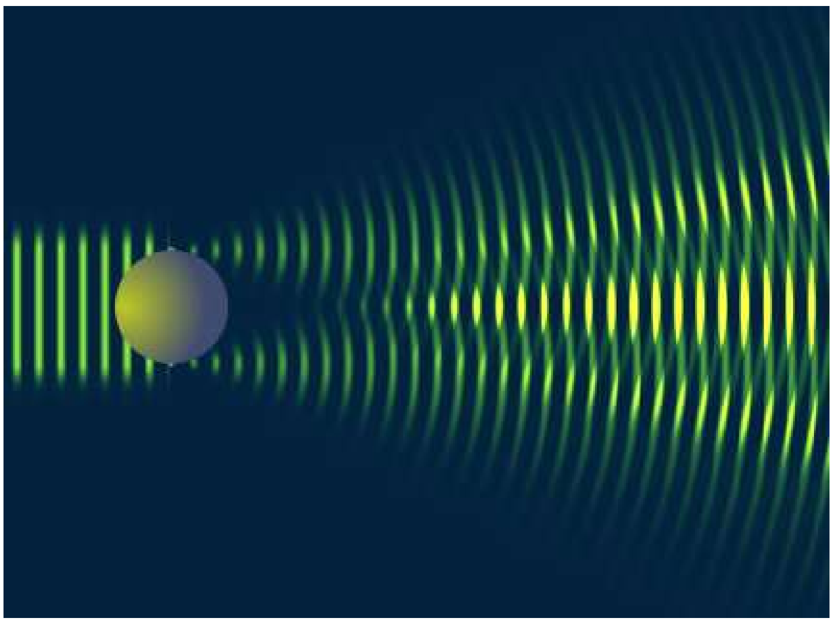




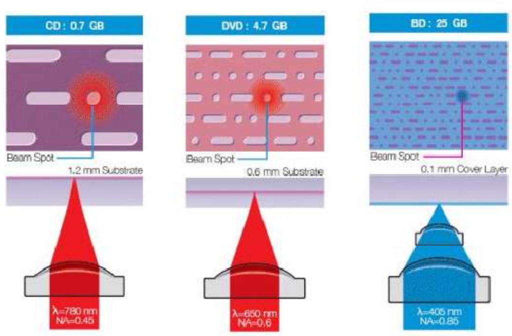








http://www.youtube.com/watch?v=IZgYswtwlT8
http://www.youtube.com/watch?v=4EDr2YY9lyA



0.7 Basic component technology

Light source and lasers. Light detectors. Optics/photonics devices: active/passive. Waveguides, filters, grating, amplifiers, modulators.

2. Illustration: monochromatic harmonic plane waves in dielectric

2.0 A very basic review and illustration

2.1 Vector representation

2.1.1 Vector relationship

Electric field is a vector: $\mathbf{E} = \{E_x, E_y, E_z\}$ (2.1.1a)

same with magnetic: $\mathbf{H} = \{H_x, H_y, H_z\}$ (2.1.1b)

For plane wave, each component has the same phase function:

$$e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \qquad (2.1.2a)$$

where: $k = \frac{\omega}{c}\sqrt{\mu\epsilon} = k_0\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}\sqrt{\mu\epsilon}$ (2.1.2b)

Let's check their relationship with Maxwell's equation

$$\nabla\times\mathbf{E} + \frac{\mu}{c}\frac{\partial\mathbf{H}}{\partial t} = 0 \qquad (2.1.3)$$

- *Calculate* $\nabla\times\mathbf{E}$

$$\nabla\times\vec{\mathbf{E}} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{pmatrix} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x e^{i\mathbf{k}.\mathbf{r}} & E_y e^{i\mathbf{k}.\mathbf{r}} & E_z e^{i\mathbf{k}.\mathbf{r}} \end{pmatrix} e^{-i\omega t} \qquad (2.1.4)$$

$$= i\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \end{pmatrix}\begin{pmatrix} k_y E_z - k_z E_y \\ k_z E_x - k_x E_z \\ k_x E_y - k_y E_x \end{pmatrix} e^{i(\mathbf{k}.\mathbf{r}-\omega t)} = i\,\mathbf{k}\times\vec{\mathbf{E}}$$

This is a basic relation for a harmonic plane wave

$$\nabla\times\vec{\mathbf{E}} = i\,\mathbf{k}\times\vec{\mathbf{E}} \qquad (2.1.5)$$

Here is how to use *Mathematica* to do the above:

```
k =.; x =.; y =.; z =.;
Curl[{F_x e^(i (k1 x+k2 y+k3 z)), F_y e^(i (k1 x+k2 y+k3 z)), F_z e^(i (k1 x+k2 y+k3 z))}, {x, y, z}]
```

$$\{i\, k_2\, F_z\, e^{i(k_1 x+k_2 y+k_3 z)} - i\, k_3\, F_y\, e^{i(k_1 x+k_2 y+k_3 z)},\; i\, k_3\, F_x\, e^{i(k_1 x+k_2 y+k_3 z)} - i\, k_1\, F_z\, e^{i(k_1 x+k_2 y+k_3 z)},\; i\, k_1\, F_y\, e^{i(k_1 x+k_2 y+k_3 z)} - i\, k_2\, F_x\, e^{i(k_1 x+k_2 y+k_3 z)}\}$$

```
FullSimplify[%]
```

$$\{i\,(k_2 F_z - k_3 F_y)\, e^{i(k_1 x+k_2 y+k_3 z)},\; -i\,(k_1 F_z - k_3 F_x)\, e^{i(k_1 x+k_2 y+k_3 z)},\; i\,(k_1 F_y - k_2 F_x)\, e^{i(k_1 x+k_2 y+k_3 z)}\}$$

```
{k1, k2, k3} × {Fx, Fy, Fz}
```

$$\{k_2 F_z - k_3 F_y,\; k_3 F_x - k_1 F_z,\; k_1 F_y - k_2 F_x\}$$

Hence, we obtain the relation in (2.1.5)

- *Calculate* $\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}$

$$\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = -i\,\frac{\mu}{c}\,\omega\,\mathbf{H}$$

- *Apply Maxwell equation for the two terms, express H as function of E*

$$\nabla\times\vec{E} = i\,\mathbf{k}\times\vec{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = i\,\frac{\mu}{c}\,\omega\,\mathbf{H}$$

$$\mathbf{k}\times\vec{E} = \mu\, k_0\, \mathbf{H}$$

$$k_0\sqrt{\mu\,\epsilon}\;\; \hat{\mathbf{k}}\times\vec{\mathbf{E}} = \mu\, k_0\, \mathbf{H}$$

$$\sqrt{\frac{\epsilon}{\mu}}\;\; \hat{\mathbf{k}}\times\vec{\mathbf{E}} = \mathbf{H}$$

- *Do the same for Maxwell equation:* $\nabla\times\boldsymbol{H} - \frac{\epsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t} = 0$

$$\nabla\times\mathbf{H} = i\,\mathbf{k}\times\mathbf{H} = \frac{\epsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t} = -i\,\frac{\epsilon}{c}\,\omega\,\mathbf{E}$$

$$k_0\sqrt{\mu\,\epsilon}\;\; \hat{\mathbf{k}}\times\mathbf{H} = -\epsilon\, k_0\, \mathbf{E}$$

$$\sqrt{\frac{\mu}{\epsilon}}\;\; \hat{\mathbf{k}}\times\mathbf{H} = -\,\mathbf{E}$$

- *What can you conclude about the angles between the three vectors: k, E, and H?*
- *What is the relationship between magnitude of E and H?*

From above: $\sqrt{\frac{\epsilon}{\mu}}\;\; |\mathbf{E}| = |\mathbf{H}|$

Calculate the Poynting vector for a plane wave

$$\mathbf{S} = \frac{c}{4\pi} \mathrm{Re}[\mathbf{E}] \times \mathrm{Re}[\mathbf{H}]$$

We drop the Re[], just keep in mind so.

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \left(\sqrt{\frac{\epsilon}{\mu}}\ \hat{\mathbf{k}} \times \vec{E} \right) = \frac{c}{4\pi} \sqrt{\frac{\epsilon}{\mu}}\ \hat{\mathbf{k}} (|\mathbf{E}|)^2$$

For harmonic wave, the time-average of $\mathrm{Cos}[\omega t]^2$ or $\mathrm{Sin}[\omega t]^2$ is $\frac{1}{2}$, hence:

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \sqrt{\frac{\epsilon}{\mu}} (|\mathbf{E}|)^2$$

Note: we must use real quantities in calculating S, but use the conjugate of H if using the fields in complex form for time-averaged. Example: let $\vec{E} = \hat{x} A \,\mathrm{Cos}[k z - \omega t]$

$$\vec{H} = \hat{y} \sqrt{\frac{\epsilon}{\mu}} A \,\mathrm{Cos}[k z - \omega t]$$

Then, $\vec{S} = \hat{x} \times \hat{y} \frac{c}{4\pi} \sqrt{\frac{\epsilon}{\mu}} A^2 \mathrm{Cos}[k z - \omega t]^2 = \hat{z} \frac{c}{4\pi} \sqrt{\frac{\epsilon}{\mu}} A^2 \mathrm{Cos}[k z - \omega t]^2$

This is "instantaneous intensity," but really not meaningful. What is physically meaningful is the time-averaged intensity, which is what we measure, and what is relevant in optical systems:

$$I = \left|\langle \vec{S} \rangle\right| = \frac{c}{8\pi} \sqrt{\frac{\epsilon}{\mu}} A^2$$

2.1.2 Linear poplarization.

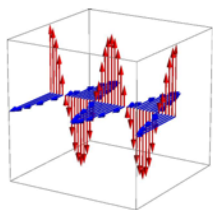
The orientation of the E field vector is generally called "polarization" of the light. However, polarization is also a more general concept than a specific quantity. It is about the description of the properties of the vectorial nature of the E and H field. A question is: why we pick the orientation of E field as the preferred description for polarization but not H field? There are 2 reasons: 1- it's just a convention because E field is more familiar, and 2- H field is really an axial vector, not a true vector as we will see. Hence, using E field orientation as the polarization is more intuitive to understand in terms of vector orientation.

If the orientation of the E field is always in a plane, we define that to be linear polarization.

Because **k**, **E**, and **H** are mutually orthogonal, we can choose coordinate: z= along k, E along x and H along y for the case of linear polarization. We will examine polarization in depth in later chapter.

```
Δz = 0.05 ;
Animate[
  Graphics3D[{Table[{RGBColor[1, 0, 0],
          Arrow[{{ Δz l, 0, 0}, {Δz l,  0 ,  Cos[2 π (Δz l − t)] }} ]}, {l, 0, 40}]
     , Table[{RGBColor[0, 0, 1],
          Arrow[{{ Δz l, 0, 0}, {Δz l, − Cos[2 π (Δz l − t)] , 0.}} ]}, {l, 0, 40}] } ,
    ViewPoint → {5, −8, 3},  PlotRange → {{0, 2}, {−1, 1}, {−1, 1}}],
  {t, 0, 1, 0.02},  AnimationRunning → False,  DisplayAllSteps → True]
```

```
Δz = 0.05 ;
Animate[
  Graphics3D[{Table[{RGBColor[1, 0, 0],
        Arrow[Tube[{{ Δz l, 0, 0}, {Δz l,  0 ,  Cos[2 π (Δz l − t)] }}, 0.01 ] ]}, {l, 0, 40}]
     , Table[{RGBColor[0, 0, 1],
        Arrow[Tube[{{ Δz l, 0, 0}, {Δz l, − Cos[2 π (Δz l − t)] , 0.}} , 0.01]]}, {l, 0, 40}] } ,
    ViewPoint → {5, −8, 3},  PlotRange → {{0, 2}, {−1, 1}, {−1, 1}}],
  {t, 0, 1, 0.02},  AnimationRunning → False,  DisplayAllSteps → True]
```

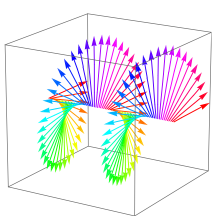


- *Is linear polarization the only possible polarization for plane wave?*

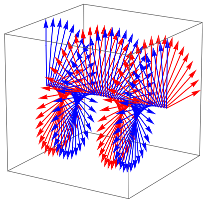
2.1.3 Circular poplarization.

We know that **k**, **E**, and **H** are mutually orthogonal, and we choose above each vector to be always in a plane. But there is nothing forcing it that way.

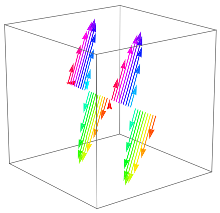
```
Δz = 0.025 ;
Animate[
  Graphics3D[Table[{Hue[Δz l − t, 1, 1],
      Arrow[{{ Δz l, 0, 0}, {Δz l,  Cos[2 π (Δz l − t)] , −Sin[2 π (Δz l − t)]}} ]}, {l, 0, 80}] ,
    ViewPoint → {5, −8, 3},  PlotRange → {{0, 2}, {−1, 1}, {−1, 1}}],
  {t, 0, 1, 0.025},  AnimationRunning → False,  DisplayAllSteps → True]
```



```
Δz = 0.025 ;
Animate[
  Graphics3D[Table[{{Red,
        Arrow[{{ Δz l, 0, 0}, {Δz l,  Cos[2 π (Δz l − t)] , −Sin[2 π (Δz l − t)]}}] }
      , {Blue,
        Arrow[{{ Δz l, 0, 0}, {Δz l,  Sin[2 π (Δz l − t)] , Cos[2 π (Δz l − t)]}}] }}, {l, 0, 80}] ,
    ViewPoint → {5, −8, 3},  PlotRange → {{0, 2}, {−1, 1}, {−1, 1}}],
  {t, 0, 1, 0.025},  AnimationRunning → False,  DisplayAllSteps → True]
```



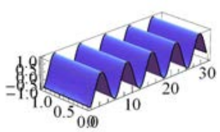
Is the wave shown below possible?



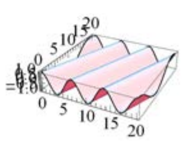
2.2 Single wave traveling

If we deal with ONLY one component of the vector, we can treat it as a scalar field. We can plot the amplitude along the z axis to visualize a wave.

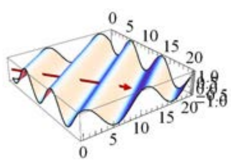
```
Manipulate[ k = {1., 0.} ;   ω = 2 Pi ;
  Plot3D[Re[ⅇ^(ⅈ (-k . {x, y} - ω t)) ], {x, 0, 30}, {y, 0, 1}
, BoxRatios -> {5, 2, 1 }, PlotPoints -> {50, 10},
    ViewPoint -> {-5, -5, 3},  Mesh -> False,  PlotRange -> All
       ,  ImageSize -> {300, 200}] , {t, 0, 1}]
```



```
Manipulate[ k = {Cos[θ],  Sin[θ]} ; ω =  2 Pi ;
  Plot3D[Re[ⅇ^(ⅈ (k . {x, y} - ω t)) ], {x, 0, 20}, {y, 0, 20}
, BoxRatios -> {5, 5, 1 }, PlotPoints -> {50, 50},
    Mesh -> False,  PlotRange → {-1, 1}
       ,  ImageSize -> {300, 200}] , {t, 0, 1},  {θ, 0, 2 Pi}]
```



```
Manipulate[ k = {Cos[θ], Sin[θ]} ; ω = 2 Pi ;
  Show[Plot3D[Re[ⅇ^(i (k.{x, y} - ω t)) ], {x, 0, 20}, {y, 0, 20}
, BoxRatios -> {5, 5, 1 }, PlotPoints -> {50, 50},
      Mesh -> False, PlotRange → {-1, 1}
       , ImageSize -> {300, 200}]
    , Graphics3D[{Red, Arrow[Tube[{{0, 0, 0}, {20 Cos[θ], 20 Sin[θ], 0}} , 0.1]]}]
  ]
  , {t, 0, 1}, {θ, 0, 2 Pi}]
```



1D propagation

In fiber and waveguide, we can actually reduce the wave to 1 dimensional like 1D plane wave because light travels along only one axis.

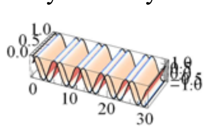
```
β = 1 ; ω = 2 π ;
Manipulate[Plot[ Re[ⅇ^(i (β z - ω t)) ], {z, 0, 50}, ImageSize → {600, 100},
    PlotRange → {{0, 50}, {-1, 1}}, Frame → True, AspectRatio → 0.1, Filling → Axis]
  , {t, 0, 1}, {β, -1, 1, 2} ]
```

A test

```
β = 1 ; ω1 = 2 π ;
Manipulate[β2 = ω2/ω1 β ;
  Plot[ {Re[ⅇ^(i (β z - ω1 t)) ], 2 + Re[ⅇ^(i (β2 z - ω2 t)) ]}, {z, 0, 50}, ImageSize → {600, 100},
    PlotRange → {{0, 50}, {-1, 4}}, Frame → True, AspectRatio → 0.1, Filling → Axis]
  , {t, 0, 1}, {β, -1, 1, 2}, {ω2, 2 π , 10 π} ]
```

Note on real and imaginary: (E and H)

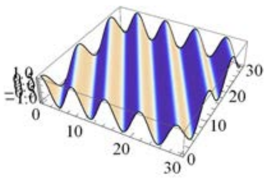
We use a complex function $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ to describe the wave. E and H in nature are real quantity. We obtain them by taking the real part of the complex function. We will see that we can use either the Re or the Im part, but must stick to the choice consistently. NO mix and match between the Re or Im part.
They differ by a 90 degree phase.



2.3 Multiple EM fields of the same frequency

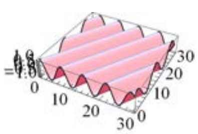
2.3.1 k1 wave

```
ω = 2 Pi ; θ = Pi/6. ;
k = {Cos[θ], Sin[θ]} ; t = 0. ;
a = Plot3D[Re[e^(i (k.{x, y} - ω t))], {x, 0, 30}, {y, 0, 30}
, BoxRatios -> {5, 5, 1 }, PlotPoints -> {50, 50},
    Mesh -> False, PlotRange → {-1, 1}]
```



2.3.2 k2 wave

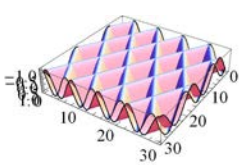
```
ω = 2 Pi ; θ = -Pi/6. ;
k = {Cos[θ], Sin[θ]} ; t = 0. ;
b = Plot3D[Re[e^(i (k.{x, y} - ω t))], {x, 0, 30}, {y, 0, 30}
, BoxRatios -> {5, 5, 1 }, PlotPoints -> {50, 50},
    Mesh -> False, PlotRange → {-1, 1}]
```



```
Show[a, b]
```

2.3.3 Both k1 and k2 wave

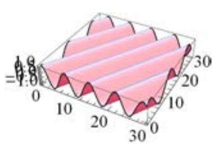
Is this correct description of the presence of 2 waves?

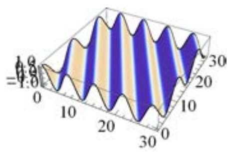


- Graphics3D -

2.3.4 Both k1 and k2 wave: continued

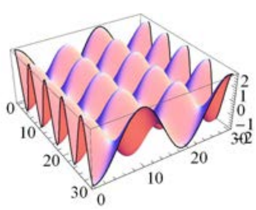
```
Manipulate[
k1 = {Cos[θ], Sin[θ]} ; k2 = {Cos[θ], −Sin[θ]};
Plot3D[Re[Exp[I * ({x , y} . k1 − 2 π t)]
          + Exp[I * ({x , y} . k2 − 2 π t ) ]], {x, 0, 50}, {y, 0, 50}
, BoxRatios −> {5, 5, 1 }, PlotPoints −> {50, 50},
    ViewPoint −> {−5, −5, 3}, Mesh → False, PlotRange → {−2, 2} ]
  , {t, 0, 1}, {θ, 20 * Pi / 190, 45 * Pi / 180} ]
```



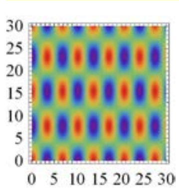


Below is the correct answer: interference effect.

```
Manipulate[
k1 = {Cos[θ], Sin[θ]} ; k2 = {Cos[θ], −Sin[θ]};
Plot3D[Re[Exp[I * ({x , y} . k1 − 2 π t)]
          + Exp[I * ({x , y} . k2 − 2 π t ) ]], {x, 0, 30}, {y, 0, 30}
, PlotPoints −> {50, 50}, Mesh → False, PlotRange → {−2, 2} ]
  , {t, 0, 1}, {θ, 20 * Pi / 190, 45 * Pi / 180} ]
```




```
Manipulate[
k1 = {Cos[θ], Sin[θ]} ; k2 = {Cos[θ], −Sin[θ]};
DensityPlot[Re[Exp[I * ({x , y} . k1 − 2 π t)]
          + Exp[I * ({x , y} . k2 − 2 π t ) ]], {x, 0, 30}, {y, 0, 30}
, PlotPoints −> {50, 50}, Mesh → False, PlotRange → {−2, 2} , ColorFunction → "Rainbow"]
  , {t, 0, 1}, {θ, 10 * Pi / 190, 80 * Pi / 180} ]
```

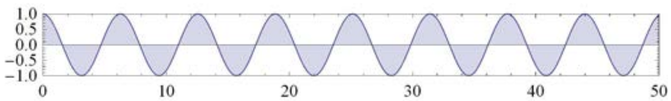


3. Plane waves in complex dielectric medium

3.1 Discussion

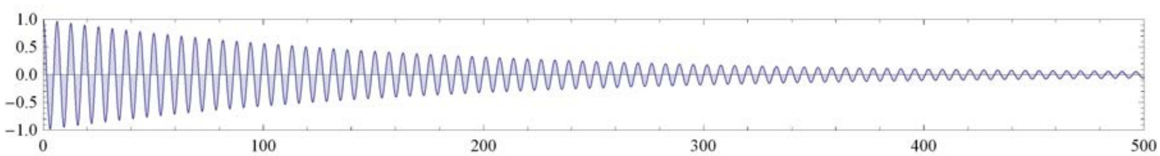
A plane wave has constant intensity and power throughout.

```
β = 1 ; ω = 2 π ;
Manipulate[Plot[ Re[ⅇ^(ⅈ (β z − ω t))], {z, 0, 50}, ImageSize → {600, 100},
    PlotRange → {{0, 50}, {−1, 1}}, Frame → True, AspectRatio → 0.1, Filling → Axis]
  , {t, 0, 1}, {β, −1, 1, 2} ]
```



That's not what happens in reality. Light power is lost gradually in a fiber. Or sunlight is weakened in a sunglass. How do we describe such a phenomenon?

```
β = 1; ω = 2 π ;
Manipulate[Plot[ Re[ⅇ^(ⅈ ((β+ⅈ α) z − ω t))] , {z, 0, 500}, ImageSize → {600, 100},
    PlotRange → {{0, 500}, {−1, 1}}, Frame → True, AspectRatio → 0.1, Filling → Axis]
  , {t, 0, 1}, {α, 0, 0.1} ]
```



The dielectric function ϵ does not have to be real. It can be complex in general.
Thus, we can describe it as:

$$E = e^{i(\mathbf{k}\,.\,\mathbf{r} - \omega t)} = e^{i((\beta + i\alpha/2) z - \omega t)}$$

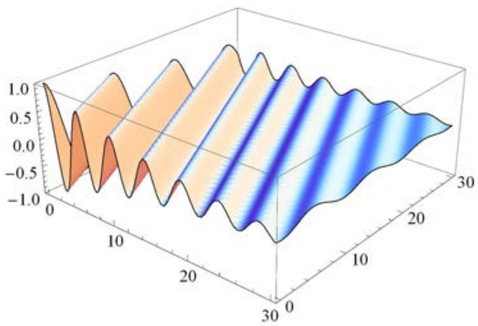
$$k = \frac{\omega}{c}\sqrt{\mu\epsilon} = k_0\sqrt{\mu\epsilon} = k_0(n + i\kappa)$$

where: $(n + i\kappa)^2 = \epsilon$ (we let μ=1)

3.2 Example 1: inhomogenous wave

Let's consider 2-D case: $e^{i k_x x + k_y y}$. We might think that it is just the same as the 1-D case above, but the prop direction is not along z but may be some direction {cos[θ], sin[θ]} in the x-y plane.

```
Manipulate[ϵ = (1.5 + ⅈ κ)^2 ;
k = {Cos[θ] , Sin[θ]} √ϵ ;
Plot3D[Re[Exp[I * ({x , y} .k - 2 * Pi t)]], {x, 0, 30}, {y, 0, 30}
                   , PlotRange -> {-1, 1}
, BoxRatios -> {5, 5, 2 },
   PlotPoints -> {70, 70}, Mesh -> False, PlotRange -> All]
  , {θ, 0, Pi / 2}, {t, 0, 1}, {κ, 0, 0.1}]
```

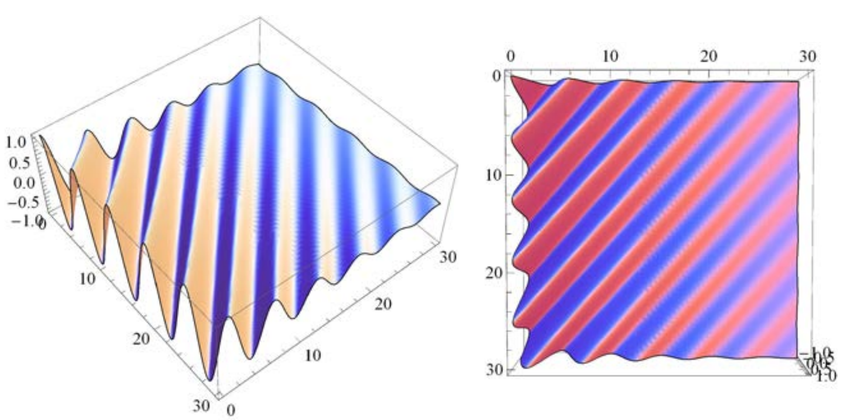


However, recall that $e^{i k_x x + k_y y}$ is a solution of the wave equation as long as:

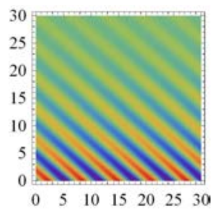
$$k_x{}^2 + k_y{}^2 = \frac{\omega^2}{c^2}\,\epsilon$$

There is no need for any relationship between the real and imaginary of k_x and k_y as above, i. e. $k_x = (k_R + i\, k_I)\,\mathrm{Cos}[\theta]$; $k_y = (k_R + i\, k_I)\,\mathrm{Sin}[\theta]$. If k_x and k_y are not proportional to cos, sin of some angle, what would a solution be like? what type of wave does it describe?

```
Manipulate[ϵ = (1.5 + ⅈ κ)^2 ;
k = {kx , Sqrt[ϵ - kx^2]} ;
Plot3D[Re[Exp[I * ({x , y} .k - 2 * Pi t)]], {x, 0, 30}, {y, 0, 30}
                   , PlotRange -> {-1, 1}
, BoxRatios -> {5, 5, 2 },
   PlotPoints -> {70, 70}, Mesh -> False, PlotRange -> All]
  , {kx, 0, 1}, {t, 0, 1}, {κ, 0, 0.1}]
```



```
Manipulate[ϵ = (1.5 + ⅈ κ)^2 ;
k = {kx , Sqrt[ϵ - kx^2]} ;
DensityPlot[Re[Exp[I * ({x , y} .k - 2 * Pi t)]], {x, 0, 30}, {y, 0, 30}
                    , PlotRange -> {-1, 1}
, BoxRatios -> {5, 5, 2 }, PlotPoints -> {70, 70}, Mesh -> False, PlotRange -> All
   , ColorFunction → "Rainbow"]
  , {kx, 0, 1}, {t, 0, 1}, {κ, 0, 0.1}]
```



This a valid solution, here the wave in the x-direction seems to have constant amplitude, but the y-traveling part has a strong attenuation. Such a wave is called inhomogenous. Where can we use such a solution?
Think about boundary. Say, if we feed a wave into a lossy/gain medium from a non-loss medium, the boundary component can be non-loss. If the sunlight enters your sunglasses at an angle, this is what happens.

3.3 Example 2: evanescence wave

But we can have complex k even if ϵ is NOT complex, because all that is required is:

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2}\epsilon$$

as long as the imaginary term on the left hand side vanishes.

Suppose $k_x^2 > k_0^2 \mu \epsilon$, what is the solution?

Clearly, $k_z^2 = k_0^2 \mu \epsilon - k_x^2 < 0.$

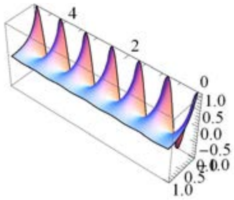
$$k_z = \sqrt{k_0^2 \mu \epsilon - k_x^2} = i\kappa.$$

where $\kappa = \sqrt{k_x^2 - k_0^2 \mu \epsilon}$

How about this solution:

$$e^{i(k_x x + k_z z)} = e^{i k_x x} e^{-\kappa z}$$

```
Animate[λ = 1; k0 = 2π/λ; ω = 2π; k = 2π*1.2/λ;
κ = Sqrt[k^2 - k0^2] ;
  Plot3D[Re[ e^(i (k x - ω t) - κ z)], {x, 0, 5}, {z, 0, 1}
, BoxRatios -> {5, 1, 2}, PlotPoints -> {50, 10},
    ViewPoint -> {-3, 1, 2}, Mesh -> False, PlotRange -> {-1, 1}
        , ImageSize → {300, 200}], {t, 0, 1}, AnimationRunning → False]
```



- *Is it a physically meaningful wave? Where do you think they exist? What happens at z -> - ∞*

The above wave is called evanescent wave in the z dimension. Discussion in class about dielectric waveguide.

3.4 General summary

In more general, we can have a wave $e^{i\, k_R (\mathrm{Cos}[\theta]\, x + \mathrm{Sin}[\theta]\, y)}\; e^{-(k_{I;x}\, x + k_{I;y}\, y)}$ which propagates in the direction of angle θ but with attenuation/gain amplitude that does NOT have to have the same angular relation between x and y. The only requirement is:

$$k_R{}^2\, \mathrm{Cos}[\theta]^2 - k_{I;x}{}^2 + k_R{}^2\, \mathrm{Sin}[\theta]^2 - k_{I;y}{}^2 = k_0{}^2\, \epsilon_R$$

$$2\, k_R\, k_{I;x}\, \mathrm{Cos}[\theta] + 2\, k_R\, k_{I;y}\, \mathrm{Sin}[\theta] = k_0{}^2\, \epsilon_I$$

Or:

$$k_R{}^2 - \left(k_{I;x}{}^2 + k_{I;y}{}^2\right) = k_0{}^2\, \epsilon_R$$

$$2\, k_R \left(k_{I;x}\, \mathrm{Cos}[\theta] + k_{I;y}\, \mathrm{Sin}[\theta] \right) = k_0{}^2\, \epsilon_I$$

The attenuation/gain in each direction can be determined by something else, e. g. a boundary condition.

Suppose, for example, there is another angle involved such that

$$k_{I;x} = k_I\, \mathrm{Cos}[\phi]\, ;\;\; k_{I;y} = k_I\, \mathrm{Sin}[\phi]$$

Then: $\quad 2\, k_R\, k_I\, (\, \mathrm{Cos}[\phi]\, \mathrm{Cos}[\theta] + \mathrm{Sin}[\phi]\, \mathrm{Sin}[\theta]\,) = k_0{}^2\, \epsilon_I$

Or: $\quad 2\, k_R\, k_I\;\, \mathrm{Cos}[\theta - \phi] = k_0{}^2\, \epsilon_I$

The concept of "inhomogeneity" is from the fact that the wave has different phase and amplitude angular direction. We of course always refer to the phase direction as the angle of propagation.

3.5 Power

What is the power of plane wave in a complex dielectric medium?
Consider a wave transmitted from a non-lossy medium such as air into an absorptive dielectric. For simplicity consider TE mode:
The incident field along interface x=0 is: $E_y\, e^{i\, \beta\, z}$
The transmitted field is: $\quad t\, E_y\, e^{i\, \beta\, z}\, e^{i\, x\, (k_R + i\, k_I)}$
where: $\quad \beta^2 + (k_R + i\, k_I)^2 = \epsilon\, k_0{}^2$
Note that β is real because it is along the interface of a lossless medium: see 3.2 example above.

The H field is, as usual: $\quad \mathbf{k} \times \vec{\mathbf{E}} = \mu\, k_0\, \mathbf{H}$
Here, $\mathbf{k}$ is complex. $\mathbf{k} = \beta\, \hat{\mathbf{z}} + (k_R + i\, k_I)\, \hat{\mathbf{x}}$

$$\mathbf{k} \times \vec{\mathbf{E}} = -\beta\, \hat{\mathbf{x}} + (k_R + i\, k_I)\, \hat{\mathbf{z}} = \mu\, k_0\, \mathbf{H}$$

$$\begin{aligned}\mathbf{S} &= \frac{c}{4\pi}\, \mathrm{Re}[\mathbf{E}] \times \mathrm{Re}[\mathbf{H}] \\ &= \frac{c}{4\pi}\, \hat{\mathbf{y}}\, \mathrm{Cos}[\beta\, z + x\, k_R]\; e^{-x\, k_I} \\ &\qquad \times \frac{1}{\mu\, k_0} (-\beta\, \hat{\mathbf{x}} + \hat{\mathbf{z}}\, k_R)\, \mathrm{Cos}[\beta\, z + x\, k_R]\, e^{-x\, k_I} - k_I\; \hat{\mathbf{z}}\; \mathrm{Sin}[\beta\, z + x\, k_R]\, e^{-x\, k_I}\end{aligned}$$

Take time average and let μ=1:

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} \frac{(\beta \hat{\mathbf{z}} + \hat{\mathbf{x}} k_R)}{k_0} e^{-2 x k_I}$$

Hence: $$I = \frac{c}{8\pi} \frac{\sqrt{(\beta^2 + k_R{}^2)}}{k_0} e^{-2 x k_I}$$

4 Wave in other geometries: spherical and cylindrical

4.1 Spherical wave in isotropic media

Let's start from the Helmholtz equation:

$$\nabla^2 F + k_0{}^2 \epsilon \mu F = 0.$$

Earlier we write: $\nabla^2 = \partial_{x,x} + \partial_{y,y} + \partial_{z,z}$. What if we use a different coordinate?

```
Laplacian[F[r, θ, ϕ], {r, θ, ϕ}, "Spherical"]
```

$$\frac{\frac{F^{(0,2,0)}(r,\theta,\phi)}{r} + F^{(1,0,0)}(r,\theta,\phi)}{r} + F^{(2,0,0)}(r,\theta,\phi) + \frac{1}{r}$$
$$\csc(\theta)\left(\sin(\theta) F^{(1,0,0)}(r,\theta,\phi) + \frac{\cos(\theta) F^{(0,1,0)}(r,\theta,\phi)}{r} + \frac{\csc(\theta) F^{(0,0,2)}(r,\theta,\phi)}{r}\right)$$

Let's consider a wave that is purely spherical, it has no dependence on the angles:

```
Laplacian[F[r], {r, θ, ϕ}, "Spherical"]
```

$$F''(r) + \frac{2 F'(r)}{r}$$

$$\mathbf{k0 = .;}$$
$$\mathbf{DSolve\left[\frac{2 F'(r)}{r} + F''(r) + k0^2 \epsilon \mu F[r] == 0, F[r], r\right]}$$

$$\left\{\left\{F(r) \to \frac{c_1 e^{-r\sqrt{-k0^2 \mu \epsilon}}}{r} + \frac{c_2 e^{r\sqrt{-k0^2 \mu \epsilon}}}{2 r \sqrt{-k0^2 \mu \epsilon}}\right\}\right\}$$

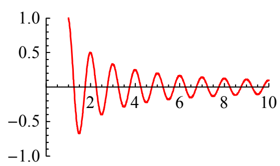
The solution is

$$\frac{e^{-i k r}}{r} c_1 + \frac{e^{i k r}}{r} c_2 \qquad (5.4.1)$$

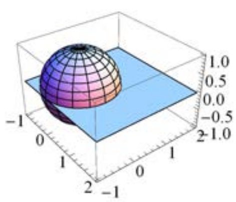
where $$k = k_0 \sqrt{\epsilon \mu}$$

So, we have a spherical wave solution also: $\frac{e^{\pm i k r - i \omega t}}{r}$. What do we do with this solution? What is the physical meaning? what happens at r = 0?

```
λ = 1; k = 2π/λ; ω = 2π; n = 20;
Animate[Plot[ Re[ (1/r) ⅇ^(i (k r - ω t))]
                    , {r, 1, 10}
  , PlotStyle -> {{RGBColor[1., 0., 0], Thickness[0.007]},
                                              {RGBColor[0., 0., 1], Thickness[0.007]}}
                    , PlotRange → {{0, 10}, {-1, 1}}
      , ImageSize -> {500, 200}], {t, 0, 1}, AnimationRunning → False]
```

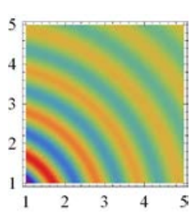



```
Show[Plot3D[0, {x, -1, 2}, {y, -1, 2},  Mesh → False], SphericalPlot3D[1, {θ, 0, Pi}, {ϕ, 0, 2 Pi}],
  PlotRange → {{-1, 2}, {-1, 2}, {-1, 1}}, BoxRatios → {3, 3, 2}]
```

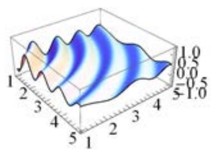


Let’s cut across the equator and plot the wave amplitude in density plot:

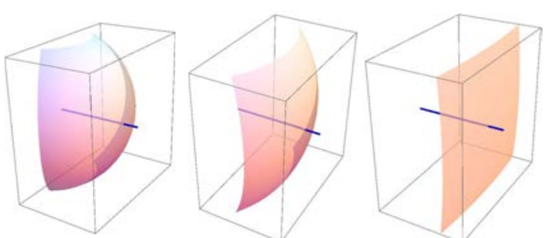
```
ω = 2 π ; k = 2 π ;
Manipulate[
 DensityPlot[Re[ⅇ^(ⅈ (v k Sqrt[x^2+y^2] - ω t)) / Sqrt[x^2 + y^2]], {x, 1, 5}, {y, 1, 5},
  PlotPoints → {40, 40}, PlotRange → {-1, 1}, ColorFunction → "Rainbow"]
 , {{v, 1}, {-1, 1}}, {t, 0, 1}]
```



```
ω = 2 π ; k = 2 π ;
Manipulate[
 Plot3D[Re[ⅇ^(ⅈ (k v Sqrt[x^2+y^2] - ω t)) / Sqrt[x^2 + y^2]], {x, 1, 5}, {y, 1, 5},
  PlotPoints → {40, 40}, Mesh → False, PlotRange → {-1, 1}]
 , {{v, 1}, {-1, 1}}, {t, 0, 1}]
```



```
Manipulate[θm = ArcSin[1/r] ;
 Show[Graphics3D[{Blue, Tube[{{−r, 0, 0}, {0.2, 0, 0}} ] }],
  ParametricPlot3D[ {r (Cos[θ] Cos[ϕ] − 1 ) , r Cos[θ] Sin[ϕ],  r Sin[θ] },
   {θ, −θm, θm}, {ϕ, −θm, θm},  Mesh → False, PlotStyle → Opacity[0.7]],
  PlotRange → {{−1, 0.2}, {−1, 1}, {−1, 1}}, BoxRatios → {1.2, 2, 2} ]
 ,
 {r,
  1,
  10}]
```



- *Where (in which phenomena) do you expect to see this type of wave?*

Remember: like plane wave, an ideal or perfect spherical wave doesn't exist in nature. But they can be very useful approximation to numerous problems. For example, if we consider a very small source of light from a distance much larger than the extent of the source (far away) and many (a large number) wavelengths, it is as good as a spherical wave for any practical purpose calculation. The point source can be just a small aperture, or even a single atom or molecule emitting the photon.

Of course each photon has a finite momentum **k**, hence a specific direction. But if **k** is purely random with equal probability in any direction, then the atom or the molecule is practically a source of spherical wave.

We will see that in dealing with various problems in diffraction, spherical wave is a useful model.

Spherical wave in non-isotropic media can be quite more complicated (mathematically)

Illustration - Link to Dipole Illustration

4.2 Cylindrical geometry wave

We choose a different coord syst and we get an answer not expected from plane wave solutions. What if we choose another one like cylindrical?

```
Laplacian[F[ρ], {ρ, θ, z}, "Cylindrical" ]
```

$$F''(\rho) + \frac{F'(\rho)}{\rho}$$

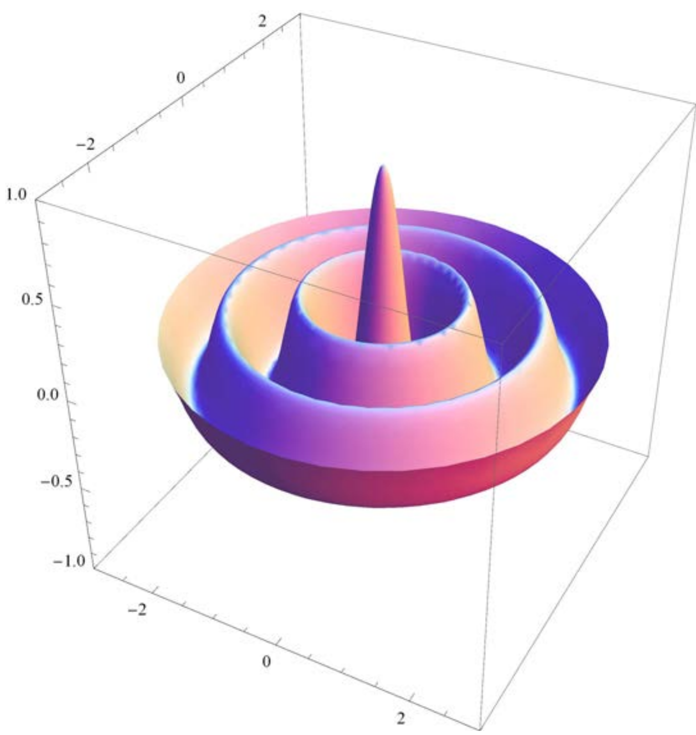
```
DSolve[F''(ρ) + F'(ρ)/ρ + k0^2 ϵ F[ρ] == 0, F[ρ], ρ]
```

$$\left\{\left\{F(\rho) \to c_1 J_0\left(\text{k0} \sqrt{\epsilon}\, \rho\right) + c_2 Y_0\left(\text{k0} \sqrt{\epsilon}\, \rho\right)\right\}\right\}$$

Again, it can be an approximated model for some some waves in some problem.

The solution now is Bessel J and Y function!. What are other coordinates? what are other solutions?

```
k = 2 Pi ;
Animate[ParametricPlot3D[ {ρ Cos[ϕ], ρ Sin[ϕ], Cos[2 π t] J0(k ρ) },
    {ρ, 0, 3}, {ϕ, 0, 2 π}, PlotRange → {{−3, 3}, {−3, 3}, {−1, 1}}, Mesh → False,
    PlotPoints → {20, 20}, BoxRatios → {1, 1, 1}], {t, 0, 1}, AnimationRunning → False]
```



Notice that it behaves like a standing wave! which means that it must be a sum of incoming and outgoing wave. That's the only way to have finiteness at the center. Can we construct a radiating wave?

- *Where (in which phenomena) do you expect to see this type of waves?*

5. Principle of linear superposition

5.1 Discussion - concept

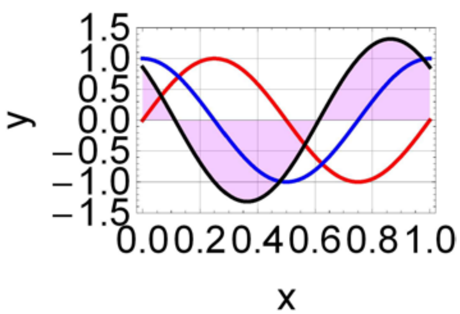
We know that light waves all around us, in various technologies are certainly more complicated than those plane wave, spherical or cylindrical waves above. How do we describe these waves?
Just like in differential equation, we know that there are infinite solutions, but all of them can be constructed from the basis set of solutions, the same is applicable to light waves. The solutions we discuss above are “basis” solution and using the linear superposition principle, we can construct any light wave.

Example:

```
DSolve[y''[x] + k^2 y[x] == 0, y[x], x]
```

$\{\{y(x) \to c_2 \sin(2\,\pi\,x) + c_1 \cos(2\,\pi\,x)\}\}$

```
Manipulate[ Plot[{Sin[2 π x], Cos[2 π x], q . {Sin[2 π x], Cos[2 π x]} }, {x, 0, 1},
    PlotStyle → { {Thick, Red}, {Thick, Blue}, {Thick, Black}}, PlotRange → {−1.5, 1.5}
    , Filling → {{3 → {Axis, Hue[0.8, 1, 1, 0.2]}}}
    , Frame → True, GridLines → Automatic,
    FrameLabel → {Style["x", 20 ], Style["y", 20]} , LabelStyle → {20, FontFamily → "Arial"}]
  , {{q, {0.5, 0.5}}, {−1, −1}, {1, 1}}]
```



The black curve shows that any solution to the differential equation is a linear combination of the red (Sin) and blue (Cos) basis functions.

Coefficients c_1 and c_2 are arbitrary and we can construct infinite solutions. Sin and Cos are basis functions. In many systems, the basis functions are orthogonal, and they can be normalized to become orthonormal.

A basis set may have an infinite number of orthonormal functions, such as Sin[n ω x], Cos[n ω x], n= 0,1, 2,.. An example of linear combination of an infinite number of basis function is the Fourier series theorem, which allows us to construct any arbitrary-shape periodic function.

For discrete basis set, we write a general function as a sum:

$$\Psi[x,\,t] = \sum_{n=1}^{\infty} c_n\,\psi_n[x,\,t] \qquad (5.1.1)$$

For continuous basis set, we use integration:

$$\Psi[x,\,t] = \int_{\square}^{\square} c[k]\,\psi_n[k;\,x,\,t]\,dk \qquad (5.1.2)$$

We know that plane waves form a continuous set, so in principle, we can form a solution like this for a monochromatic wave

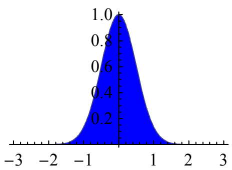
$$\int_{\square}^{\square} c[\mathbf{k}]\, e^{i(\mathbf{x}.\mathbf{k}-\omega t)}\, d\mathbf{k} \;=\; e^{-i\omega t}\int_{\square}^{\square} c[\mathbf{k}]\, e^{i\,\mathbf{x}.\mathbf{k}}\, d\mathbf{k} \qquad (5.1.3)$$

So, this is the principle, let's implement it in an example:

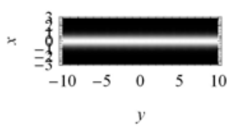
5.2 Example of linear superposition: 2D Gaussian beam

Suppose we have a monochromatic 2D light beam in x and z. The beam is constant along y direction. For example, if we put a screen perpendicular to z (i. e. x, y plane), it looks like this:

```
Plot[ⅇ^(-2 x^2/w^2) /. w -> 1, {x, -3, 3}, Filling → Axis, FillingStyle → Blue]
```




```
DensityPlot[ⅇ^(-2 x^2/w^2) /. w -> 1 , {y, -10, 10}, {x, -3, 3},
 ColorFunction → GrayLevel, FrameLabel → {y, x}, AspectRatio → 0.3 ]
```



This is what we call "Gaussian" profile (along x-direction). How do we describe the E field wave E[x,z,t] of this light beam? (the above give us ONLY a profile in x at z=0, and time-averaged.

Since it is monochromatic, we know that the time factor is simply $e^{-i\omega t}$. It's up to us to find out the z part.

We know that $k^2 = k_x^2 + k_z^2 = \frac{\omega^2}{c^2}\mu\epsilon = k_0^2\,\mu\epsilon$. So only one of the 2 k's can be independent. Let's pick k_x to be independent and the integrating variable. Then, the formula (5.1.3) becomes:

$$e^{-i\omega t}\int_{\square}^{\square} c[k_x]\, e^{i(x k_x + z k_z)}\, dk_x \qquad (5.2.1a)$$

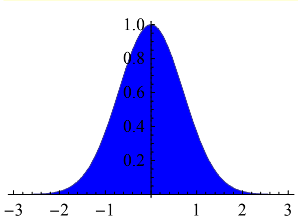
$$e^{-i\omega t}\int_{-\infty}^{\infty} c[k_x]\, e^{i\left(x k_x + z\sqrt{k_0^2\,\mu\epsilon - k_x^2}\right)}\, dk_x \qquad (5.2.1b)$$

Now, what we really need is $c[k_x]$. This is the "arbitrary" coefficient that allows us to construct any solution. How do we choose $c[k_x]$ in this case?

5.2.1 Integral expression for Gaussian profile

For the Gaussian profile above:

```
Plot[ⅇ^(-x^2/w^2) /. w -> 1, {x, -3, 3}, Filling → Axis, FillingStyle → Blue]
```



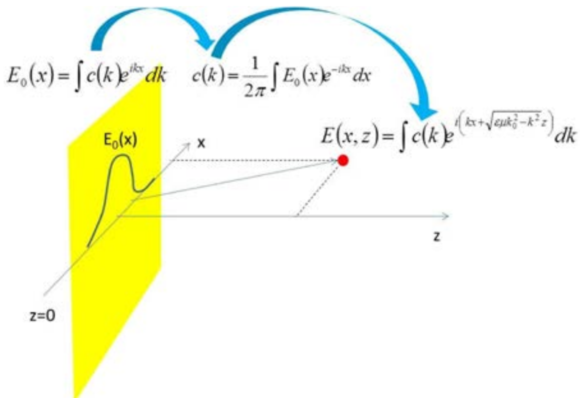
This is what we have: (the solution at z=0) = e^{-x^2/w^2}.

We want: $\int_{-\infty}^{\infty} c[k_x]\, e^{i\left(x\,k_x + z\sqrt{k_0^2\,\mu\,\epsilon - k_x^2}\right)} \,dk_x$ at z = 0 = e^{-x^2/w^2}

Or: $$\sim \int_{-\infty}^{\infty} c[k_x]\, e^{i\,x\,k_x}\, dk_x = e^{-x^2/w^2} \quad (5.2.2)$$

How do we solve for $c[k_x]$? We need inverse Fourier transform:

$$c[k_x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/w^2}\, e^{-i\,x\,k_x}\, dx \quad (5.2.3)$$



k = .;

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,x\,k}\, e^{-x^2/w^2}\, dx$$

$$\text{ConditionalExpression}\left[\frac{e^{-\frac{1}{4}k^2 w^2}}{2\sqrt{\pi}\sqrt{\frac{1}{w^2}}},\ \text{Re}\left[\frac{1}{w^2}\right] > 0\right]$$

So the solution is $$c[k_x] = \frac{w}{2\sqrt{\pi}}\, e^{-\frac{1}{4}k_x^2 w^2} \quad (5.2.4)$$

What is the meaning of $c[k_x]$?

c[k] describes the beam profile in terms of a "package" of several beams with different spatial frequency. Key concept: spatial frequency k. c[k] is the envelop or coefficient of the "package". (we'll study a time wave package later on). It can also be called the "angular spectrum".

Now we have our solution, complete with the time factor:

$$E[x, z, t] = A\, \frac{w}{2\sqrt{\pi}}\, e^{-i\,\omega\,t} \int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2 w^2}\, e^{i\left(x\,k + z\sqrt{k_0^2\,\epsilon - k^2}\right)}\, dk \quad (5.2.5)$$

This cannot be integrated in a closed form, but we can do an approximation.

5.2.2 Paraxial approximation

Let’s find an approximation to do the ingration for the beam. Let's look at the integral:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2 w^2}\, e^{i\,k\,x}\, e^{i\sqrt{k_0^2\,\epsilon - k^2}\,z}\, dk$$

If z is very large (far field) and the beam is paraxial (mostly in the propagation axis), we can think that k_z is the dominant term and k_x is small. Then, we can approximate:

$$\sqrt{k_0{}^2\,\epsilon - k^2} \;=\; \sqrt{k_n{}^2 - k^2} \;=\; k_n \sqrt{1 - \frac{k^2}{k_n{}^2}} \;\approx\; k_n\left(1 - \frac{k^2}{2\,k_n{}^2}\right) \;=\; k_n \;-\; \frac{k^2}{2\,k_n}$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2 w^2}\, e^{i\,k\,x}\, e^{i\,\sqrt{k_0{}^2\,\epsilon - k^2}\;z}\, dk \;\approx\; \int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2 w^2}\, e^{i\,k\,x}\, e^{i\,k_n\,z - i\,\frac{k^2}{2\,k_n}\,z}\, dk$$

$$= e^{i\,k_n\,z} \int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2 w^2}\, e^{i\,k\,x}\, e^{-i\,\frac{k^2}{2\,k_n}\,z}\, dk$$

$$= e^{i\,k_n\,z} \int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2\left(w^2 + i\,\frac{2\,z}{k_n}\right)}\, e^{i\,k\,x}\, dk$$

Now, we can perform integration. Define $P = \left(w^2 \;+\; i\,\frac{2\,z}{k_n}\right)$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}k^2\,P}\, e^{i\,k\,x}\, dk$$

$$\text{ConditionalExpression}\left[\frac{2\,\sqrt{\pi}\;e^{-\frac{x^2}{P}}}{\sqrt{P}},\ \mathrm{Re}(P) > 0\right]$$

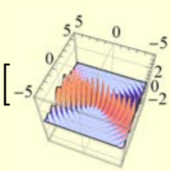
Hence:

$w =.\ ;\ n\ =\ .;\ z =\ .;$

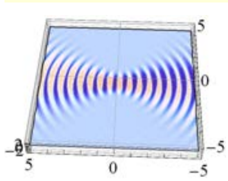
$$e^{i\,k_n\,z}\;\frac{2\,\sqrt{\pi}\;e^{-\frac{x^2}{P}}}{\sqrt{P}}\ \ /.\ P \to \left(w^2 \;+\; i\,\frac{2\,z}{k_n}\right)$$

$$\frac{2\,\sqrt{\pi}\;e^{i\,z\,k_n - \frac{x^2}{w^2 + \frac{2\,i\,z}{k_n}}}}{\sqrt{w^2 + \frac{2\,i\,z}{k_n}}}$$

Thus: $E[x, z, t] \;=\; A\,\frac{w}{2\,\sqrt{\pi}}\;\frac{2\,\sqrt{\pi}\;e^{i\,z\,k_n - \frac{x^2}{w^2 + \frac{2\,i\,z}{k_n}}}}{\sqrt{w^2 + \frac{2\,i\,z}{k_n}}}\;e^{-i\,\omega\,t} \;=\; A\,\frac{1}{\sqrt{1 + \frac{2\,i\,z}{w^2\,k_n}}}\;\;e^{i\,z\,k_n - \frac{x^2}{w^2 + \frac{2\,i\,z}{k_n}}}\;e^{-i\,\omega\,t}$ (5.2.6)

Show[

, **BoxRatios → {1, 1, 0.1}**]



5.3 Illustration of some properties of 2D Gaussian beam

5.3.1 Phase front.

The most obvious thing we see in the Gaussian beam calculation above is the circular “wave front”

The key insight to the wavefront is in this term: $e^{i z k_n - \frac{x^2}{w^2 + \frac{2 i z}{k_n}}}$ (5.3.1)
Let’s do some reduction to separate the amplitude and the phase:

$$i z k_n - \frac{x^2}{w^2+\frac{2 i z}{k_n}} = i\, k_n \left(z - \frac{x^2}{i\, k_n w^2 - 2 z}\right) = i\, k_n \left(z + \frac{x^2 (2 z + i\, k_n w^2)}{(2 z - i\, k_n w^2)(2 z + i\, k_n w^2)}\right)$$

$$= i\, k_n \left(z + \frac{x^2\, 2 z}{4 z^2 + k_n{}^2 w^4}\right) - x^2 \frac{k_n{}^2\, w^2}{4 z^2 + k_n{}^2 w^4}$$

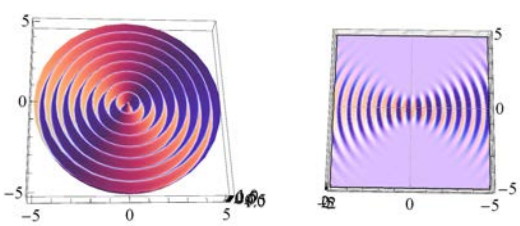
$$= i\, k_n \left(z + \frac{x^2}{2 z + \frac{k_n{}^2 w^4}{2 z}}\right) - \frac{x^2}{w^2} \frac{1}{1+\left(\frac{2 z}{k_n w^2}\right)^2} \approx i\, k_n \left(z + \frac{x^2}{2 z}\right) - \frac{x^2}{w^2} \frac{1}{1+\left(\frac{2 z}{k_n w^2}\right)^2}$$

But: $i\, k_n \left(z + \frac{x^2}{2 z}\right) \approx i\, k_n \sqrt{z^2 + x^2} = i\, k_n\, r$

This is a circular (cylindrical) wavefront indeed with the term $e^{i k_n r}$

```
λ = 0.6328 ; n = 1; w = 0.5 ; kn = 2 π n / λ ;
ParametricPlot3D[{r Cos[θ], r Sin[θ], Re[ⅇ^(ⅈ kn r)]}, {r, 0, 5},
  {θ, 0, 2 π}, Mesh → False, PlotPoints → {40, 15}, BoxRatios → {1, 1, 0.1}]
```

Let’s compare:

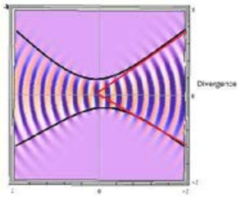


A natural concept in describing wave is "phase front". We commonly think of "wave front" in layman term. It's the same. A phase front is a surface of all points that have the same continuous phase. Mathematically, if we express a wave as:
$e^{i \phi[x,y,z]}$ (where ϕ[x,y,z] is real), then, any surface defined by ϕ[x,y,z]=constant is a phase front. (other name: cophasal surfaces, wave surfaces)

Notice the phasefront, but what else do we need?

5.3.2 Beam divergence

For the Gaussian beam, the amplitude is NOT circularly uniform, but it is rather distribute in the center lobe:



We need a term that make a “lobe” in the center along the propagation and the lobe should have a constant angular divergence.

We now can examine the second term (amplitude in the x-dimension) of the exponent:

$$e^{i z k_n - \frac{x^2}{w^2+\frac{2 i z}{k_n}}} \approx e^{i k_n r}\, e^{-\frac{x^2}{\left(1+\left(\frac{2z}{k_n w^2}\right)^2\right)}} \approx e^{i k_n r}\, e^{-\frac{x^2}{w^2\left(\frac{2z}{k_n w^2}\right)^2\left(\left(\frac{k_n w^2}{2z}\right)^2+1\right)}}$$

$$\approx e^{i k_n r}\, e^{-\frac{(x/z)^2}{\left(\frac{2}{k_n w}\right)^2\left(\left(\frac{k_n w^2}{2z}\right)^2+1\right)}} \approx e^{i k_n r}\, e^{-\frac{\mathrm{Tan}[\theta]^2}{\left(\frac{2}{k_n w}\right)^2\left(\left(\frac{k_n w^2}{2z}\right)^2+1\right)}}$$

$$\approx e^{i k_n r}\, e^{-\frac{\mathrm{Tan}[\theta]^2}{\left(\frac{\lambda_n}{\pi w}\right)^2\left(\left(\frac{k_n w^2}{2z}\right)^2+1\right)}}$$

What does the term $e^{-\frac{\mathrm{Tan}[\theta]^2}{\left(\frac{\lambda_n}{\pi w}\right)^2\left(\left(\frac{k_n w^2}{2z}\right)^2+1\right)}}$ tell us? It is a Gaussian profile with farfield divergence angle θ, and for z-> ∞:

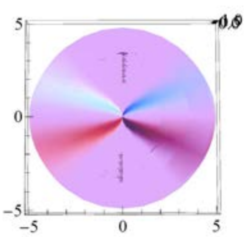
$$e^{-\frac{\mathrm{Tan}[\theta]^2}{\left(\frac{\lambda_n}{\pi w}\right)^2}} \sim e^{-\frac{\mathrm{Tan}[\theta]^2}{\mathrm{Tan}[\theta_0]^2}} \qquad (5.3.2a)$$

where:

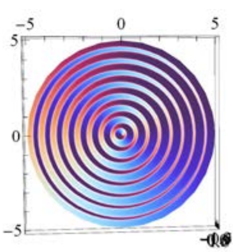
$$\mathrm{Tan}[\theta_0] = \frac{\lambda_n}{\pi w} \qquad (5.3.2b)$$

In deed, this is the term that gives is the Gaussian profile along the propagation direction and with a divergence angle θ_0

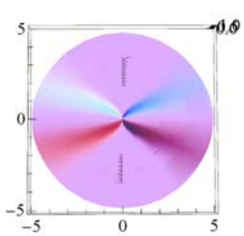
```
λ = 0.6328 ; n = 1; w = 0.5 ; kn = 2 π n / λ ;
ParametricPlot3D[{r Cos[θ], r Sin[θ], ⅇ^(-Tan[θ]^2/((λ/(n π w))^2 ((kn w^2/(2 r Sin[θ]))^2 + 1)))}, {r, 0, 5},
  {θ, 0, 2 π}, Mesh → False, PlotPoints → {100, 15}, BoxRatios → {1, 1, 0.1}]
```



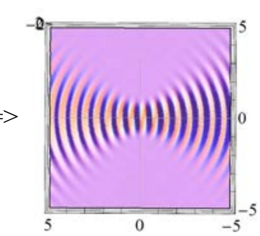
Now, we see that:



x



=>



Hence, we shows that we can describe a beam with a wide range of divergence behavior like a Gaussian by using the plane wave basis set.

5.4 Linear superposition in time (discussion only)

Summary

Summary:
1. Any light packet (spatially as well as temporally) can be decomposed as a sum of many light components.
2. The components can be pure harmonic plane waves (this is Fourier optics) or harmonic waves in other systems
3. The behavior of the beam is the net sum of those of the components
4. Most important to develop an intuitive understand of the behavior of beam: "natural" description of the beam, not just any brute force solution