

**ECE 6340**  
**Fall 2016**

**Homework 3**

Please do the following problems (you may do the others for practice if you wish):  
Probs. 1-7 , 9, 10, 12

- 1) Assume that we have a plane wave in free space that has

$$\underline{E} = \hat{x} e^{-jk_0 z}$$

$$\underline{H} = \hat{y} \left( \frac{1}{\eta_0} \right) e^{-jk_0 z}$$

a) Determine the following three vectors:  $\underline{S}$ ,  $\underline{\mathcal{L}}(t)$ ,  $\langle \underline{\mathcal{L}}(t) \rangle$ .

b) Is it true that  $\underline{\mathcal{L}}(t) = \operatorname{Re}(\underline{S} e^{j\omega t})$ ?

If not, explain why one would not expect such a relationship to hold. For example, why is it true that

$$\underline{\mathcal{E}}(t) = \operatorname{Re}(\underline{E} e^{j\omega t}),$$

but the same type of relation is not true for the Poynting vector? How must the time-varying function on the left-hand side of such an equation vary with time in order for such a relation to hold?

- 2) Start with the complex Poynting theorem (which assumes sinusoidal steady-state fields) and then take the real part of both sides, and show that this leads to the result

$$\langle \underline{\mathcal{P}}_s \rangle = \langle \underline{\mathcal{P}}_d \rangle + \langle \underline{\mathcal{P}}_f \rangle.$$

Then take the time-average of both sides of the instantaneous (time-domain) Poynting theorem, and show that this leads to the result

$$\langle \underline{\mathcal{P}}_s \rangle = \langle \underline{\mathcal{P}}_d \rangle + \langle \underline{\mathcal{P}}_f \rangle + \left\langle \frac{\partial}{\partial t} (\mathcal{H}_e + \mathcal{H}_m) \right\rangle.$$

Note that the term

$$\left\langle \frac{\partial}{\partial t} (\mathcal{H}_m + \mathcal{H}_e) \right\rangle$$

is missing from the first equation. Explain why this term must be zero, if the fields are sinusoidal.

Hint: Use the definition of time-average of a periodic function and note that the stored energy terms are periodic in time.

- 3) Give a proof that the normal component of the complex Poynting vector must be zero at the surface of a perfect electric conductor (PEC). Do this by using the boundary condition for a PEC,  $\hat{n} \times \underline{E} = 0$ , and also the vector identity (from the back of the book)  $\underline{A} \cdot (\underline{B} \times \underline{C}) = \underline{B} \cdot (\underline{C} \times \underline{A}) = \underline{C} \cdot (\underline{A} \times \underline{B})$ .
- 4) Assume an ideal cavity resonator is composed of an arbitrarily shaped closed surface  $S$  made of perfectly conducting material. Inside the resonator is an arbitrary medium, which is lossless. There are no sources (impressed currents) inside the resonator. Because of the ideal cavity assumption, nonzero fields can only exist inside the cavity at certain discrete frequencies (the resonance frequencies of the cavity). At a given resonance frequency, show that the time-average energy stored in the electric field inside the cavity is equal to the time-average energy stored in the magnetic field.

The same result will apply if the boundary of the cavity is a “perfect magnetic conductor” (PMC), or, to be more general, if part of the boundary is PEC and the rest of it is PMC. Extend your derivation to prove this more general result.

Note: A PMC boundary is a very useful approximation in many practical problems. For example, a PMC boundary approximates the condition on the edges of a microstrip patch antenna.

- 5) Consider a resonant circuit composed of a lossless ideal capacitor connected in series with a lossless ideal inductor.
  - a) Use circuit theory to calculate the time-average energies stored inside the inductor and capacitor in terms of the phasor current  $I$  going through the circuit. Verify that the time-average energies stored in the electric and magnetic fields (of the capacitor and inductor, respectively) are equal at the resonance frequency (the frequency where the input impedance of the circuit is zero).
  - b) Then note that at the resonance frequency the input reactance of the circuit is zero. Use this observation along with the concept of reactive power flowing into a region of

space, and its relation to stored energies according to the complex Poynting theorem, to explain why the two time-average stored energies must be equal at the resonance frequency.

- 6) A one-meter long tube of  $z$ -directed time-harmonic (sinusoidal steady-state) electric current having a total current  $I = (5+j0)$  [A] flows inside of a cylindrical tube of radius  $a$ , with the axis of the tube lying along the  $z$  axis from  $z = 0$  to  $z = 1$  meter. Assume that the total electric field inside the tube is

$$\underline{E} = -\hat{z}(2 + j) \text{ [V/m].}$$

Determine the complex power and the time-average power supplied by this source by using the formula for complex source power generated by a current density.

Next, calculate the voltage  $V_{AB}$ , where  $A$  is the origin, and  $B$  is the point on the  $z$  axis one meter above the origin, by integrating the electric field. Then show that your result for the complex power supplied by the source is consistent with circuit theory, by calculating the complex source power from the following circuit formula:

$$P_s = -\frac{1}{2} V_{AB} I^*.$$

- 7) As derived in class, the input reactance of an arbitrary antenna may be written as

$$X_{in} = \frac{4\omega}{|I_{in}|^2} [\langle \mathcal{W}_m \rangle - \langle \mathcal{W}_e \rangle].$$

Consider a wire antenna operating at low frequency, with an input current that is fixed. Assume that the current distribution on the arms of the antenna remains the same as the frequency changes, since the input current is fixed. Because of this, and the low-frequency assumption, the magnetic field in space surrounding the antenna remains the same as the frequency changes. However, the magnitude of the electric field varies proportional to  $1/\omega$ . That is, the electric field behaves at low frequency as

$$\underline{E}(x, y, z, \omega) \approx \frac{\underline{E}_0(x, y, z)}{\omega}.$$

This can be justified from the continuity equation,

$$\frac{dI(z)}{dz} = -j\omega\rho_l(z),$$

where  $\rho_l(z)$  is the line charge density on the antenna. From this information, show that at low frequency the input reactance of the antenna can be written as

$$X_{in} = \frac{-A}{\omega} + B\omega,$$

where  $A$  and  $B$  are positive constants. Assuming that this expression holds up through the resonance frequency of the antenna, show that the input reactance may be written in a general way as

$$X_{in} = -\frac{A}{\omega} \left[ 1 - \left( \frac{\omega}{\omega_0} \right)^2 \right].$$

where  $\omega_0$  is the resonance frequency [rad/s] (the frequency where the input reactance is zero).

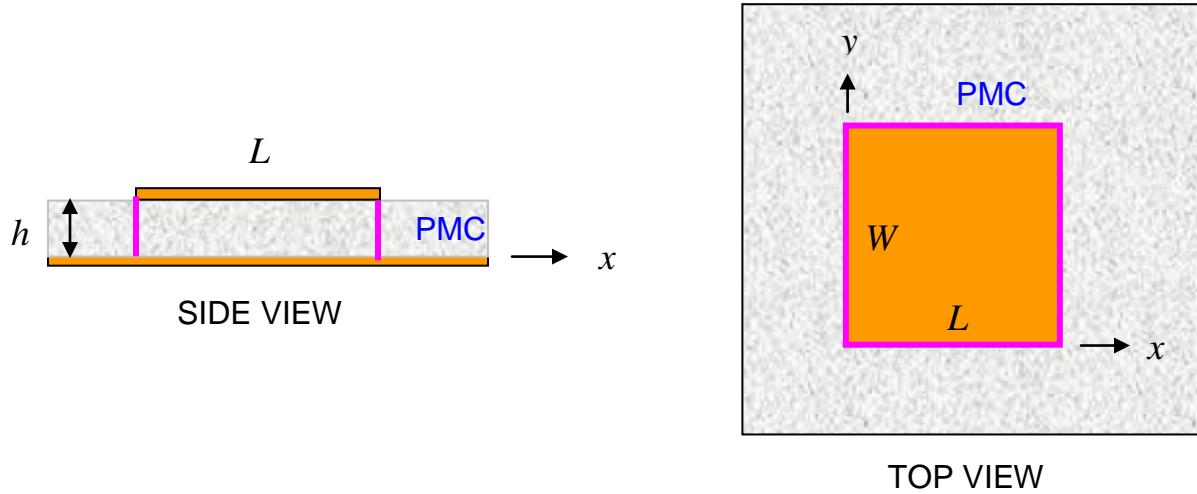
- 8) A microstrip patch antenna is shown below. It consists of a rectangular patch of metal on top of a grounded dielectric substrate. The substrate has a thickness  $h$ , a relative permittivity  $\epsilon_r$ . For analysis purposes, the substrate material inside the patch cavity is assumed to have an effective complex permittivity  $\epsilon_c^{eff}$  and a corresponding effective loss tangent  $\tan\delta_{eff}$ , where

$$\epsilon_c^{eff} = \epsilon_0 \epsilon_r (1 - j \tan\delta_{eff}).$$

(An “effective” loss tangent  $\tan\delta_{eff}$  is often used in microstrip analysis to account for radiation and conductor losses as well as the actual dielectric losses, but you don’t need to know about this to work the problem.)

The patch and ground plane are then assumed to be perfect electric conductors (PECs). The sides of the patch are assumed to be perfect magnetic conductor (PMC), which means that the tangential magnetic field is zero there.

A side view and top view, for a rectangular microstrip patch antenna, is shown in the figure below. (The patch is shown in rectangular shape, but this assumption is not necessary for the problem.)



The quality factor ( $Q$ ) of the patch cavity for a particular resonant mode (with a resonance frequency  $\omega_0$ ) is defined as

$$Q = \omega_0 \frac{\langle \mathcal{W} \rangle}{\langle \mathcal{P}_d \rangle} = \omega_0 \frac{\langle \mathcal{W}_e \rangle + \langle \mathcal{W}_m \rangle}{\langle \mathcal{P}_d \rangle}.$$

- a) Assuming a low-loss cavity, explain using Prob. 4 why this can be written approximately as

$$Q = 2\omega_0 \frac{\langle \mathcal{W}_e \rangle}{\langle \mathcal{P}_d \rangle}.$$

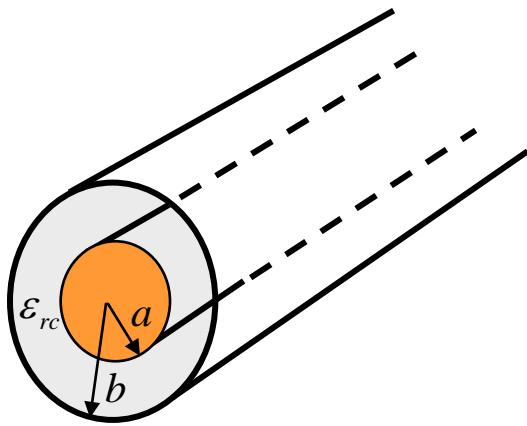
- b) Next, assume that the electric field inside the patch cavity is  $z$  directed and independent of  $z$  (this is a good approximation for thin substrates). Show that the  $Q$  is then given by the simple final formula

$$Q = \frac{1}{\tan \delta_{eff}}.$$

Practical note: The percent bandwidth (SWR < 2 definition) of a microstrip antenna can be related to the  $Q$  of the antenna as

$$BW\% = 100 \frac{1}{\sqrt{2} Q}.$$

- 9) A coaxial cable is shown below. The inner radius is  $a$  and the outer radius is  $b$ . Inside the coaxial cable is a nonmagnetic filling material with a (complex) relative permittivity  $\epsilon_{rc}$ .



Assume that the fields inside the coaxial cable are

$$\underline{E} = \hat{\rho} \left( \frac{1}{\rho} \right) \frac{V_0}{\ln \left( \frac{b}{a} \right)} e^{-jkz}$$

$$\underline{H} = \hat{\phi} \left( \frac{1}{\eta\rho} \right) \frac{V_0}{\ln \left( \frac{b}{a} \right)} e^{-jkz},$$

where  $V_0$  is the voltage drop between the inner and outer conductors and  $\eta$  is the (complex) intrinsic impedance of the lossy dielectric material.

Calculate the Poynting vector inside the cable, and then integrate over the cross section of the cable to find the complex power  $P_f$  flowing down the cable. Then use the expression

$$P_f = \frac{1}{2} \frac{|V_0|^2}{Z_0^*}$$

to find the (complex) characteristic impedance  $Z_0$  of the cable.

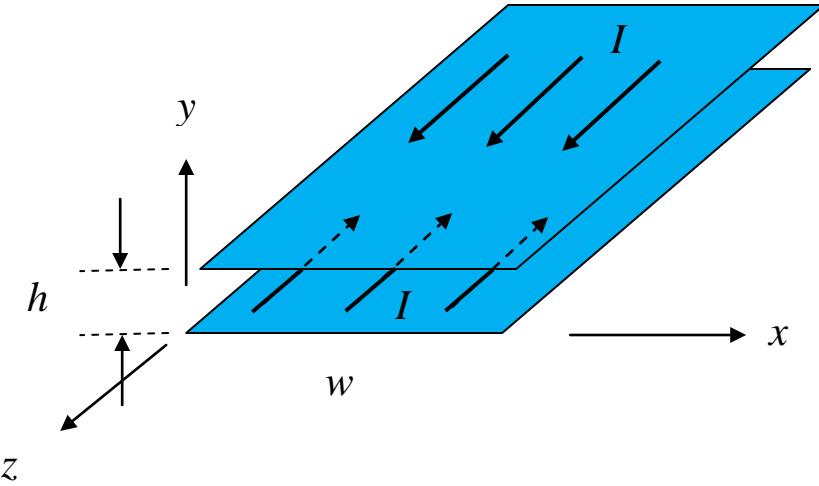
- 10) Assume that sunlight has a power density of  $1 \text{ [kW/m}^2]$  and has a frequency of  $5 \times 10^{14} \text{ [Hz]}$  (we are assuming the light to be at a single frequency (monochromatic) for simplicity). Calculate the density of the photons (number per cubic meter). Assuming that the photons are arranged on a cubical lattice, determine the spacing between them.

- 11) Consider two equal and opposite uniform DC surface currents on a parallel-plate transmission line as shown below, which are infinite in the  $z$  direction. On the top plate the surface current density is given by  $J_{sz} = I / w$ , where  $w$  is the width of the currents and  $I$  is the total current (in amps) flowing on the top plate. Assume that the width  $w$  of the currents is large compared to the separation  $h$ , so we can neglect fringing. Hence, there is a magnetic field only in the region between the two plates, and between the two plates we have a uniform magnetic field of the form

$$\underline{H} = \hat{x} \left( \frac{I}{w} \right).$$

(You should show this by using Ampere's law.) Assume that there is no electric field present.

Use the Maxwell stress tensor to find the total force per meter (i.e., per meter in the  $z$  direction) in the  $y$  direction on the top plate.



- 12) A rectangular metal plate (modeled as a perfect electric conductor) has an area of one square meter. The plate lies in the  $z = 0$  plane (see the figure below). An incident plane wave that impinges on the plate has the following field:

$$\begin{aligned}\underline{E}^{inc} &= \hat{y} e^{+j(k_x x + k_z z)} \\ \underline{H}^{inc} &= \left( \frac{1}{\eta_0} \right) \left[ \hat{x} e^{+j(k_x x + k_z z)} \cos \theta_0 - \hat{z} e^{+j(k_x x + k_z z)} \sin \theta_0 \right]\end{aligned}$$

Note that at any point on the plate (on either the top or bottom surface), the current density is related to the total magnetic field as

$$\underline{J}_s = \hat{\underline{n}} \times \underline{H} ,$$

where the unit normal points outward from the metal surface.

- a) Find the time-average force vector on the plate, using the Maxwell stress tensor. Assume that “physical optics” holds, so that (approximately)

$$\underline{J}_s = \begin{cases} 2\hat{\underline{z}} \times \underline{H}^{inc}, & \text{top surface } (z = 0^+) \\ \underline{0}, & \text{bottom surface } (z = 0^-). \end{cases}$$

- b) Find the time-average force vector on the plate, using the change in momentum of the photons hitting the plate. Verify that your answer is the same as in part (a).

