Homework 4

Assignment: Please do Probs. 1-3, 5-10, and 13-14 from the set below. Feel free to do the other problems as well for your own benefit, if you wish to have extra practice.

1) Consider a lossless transmission line ($R = 0$ and $G = 0$). Verify by direct substitution into the telegrapher's equations that the following general solution for voltage and current satisfies the telegrapher's equations (the coupled first-order differential equations in the time-domain for the voltage and current):

$$v(z,t) = f\left(t - z/v_p\right) + g\left(t + z/v_p\right)$$

$$i(z,t) = \frac{1}{Z_0}\left[f\left(t - z/v_p\right) - g\left(t + z/v_p\right)\right],$$

where v_p is the phase velocity and f and g are arbitrary functions. What is the physical interpretation of the two functions f and g?

2) Consider a length ℓ of transmission line as shown below. The line may be connected to any arbitrary loads or devices at the input and output ends. Show that the phasor voltage and current at the output end are related to the phasor voltage and current at the input end according to the "ABCD" matrix, as

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_o \\ I_o \end{bmatrix},$$
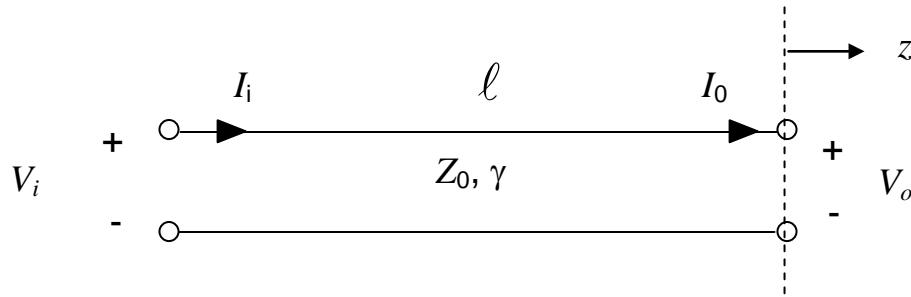
where

$$A = \cosh\left(\gamma\ell\right)$$
$$B = Z_0 \sinh\left(\gamma\ell\right)$$
$$C = \left(1/Z_0\right)\sinh\left(\gamma\ell\right)$$
$$D = \cosh\left(\gamma\ell\right).$$

(Note: Some books define the ABCD matrix to be the inverse of the one above.) Derive the above ABCD coefficients.
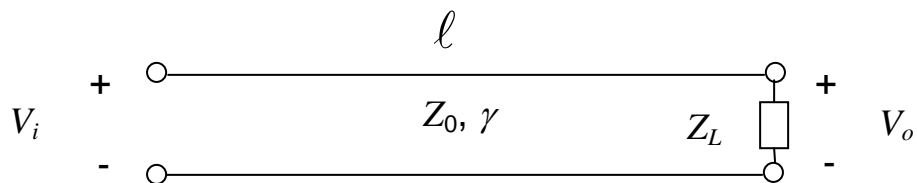
Hint: Start by writing

$$V(z) = V^+ e^{-\gamma z} + V^- e^{+\gamma z}$$

$$I(z) = \frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{+\gamma z}.$$

Then write the unknown coefficients V^+ and V^- in terms of the input voltage and current by applying boundary conditions at the left (input) end ($z = -\ell$) of the transmission line.

3) A generator is connected to a transmission line that has a load at the end, as shown below.

Show that the output voltage is related to the input voltage by

$$V_o = V_i\, e^{-\gamma \ell} \left[\frac{1+\Gamma}{1+\Gamma e^{-2\gamma \ell}} \right],$$

where the reflection coefficient at the load is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}.$$

Start by writing down a general expression for the voltage on the transmission line, and then examine your expression at the input and output ends, using the definition of the load reflection coefficient as the ratio of the backward to the forward propagating voltage waves at the load.

4) Derive exact expressions for α and β for a transmission line in terms of R, L, G, and C. Your final expressions will have square roots in them (this is fine), but they should not have any trigonometric expressions such as $\tan^{-1}$, etc. Be careful with the signs of the square roots in your answer. A good way to check the correctness of the signs is to take the limiting case of a low-loss line. Hint: Start by writing down an expression for γ^2.

5) A transmission line has parameters R, L, G, C, but has only a small amount of loss, so that $R << \omega L$ and $G << \omega C$. Under these conditions, show that

$$v_p \approx \frac{1}{\sqrt{LC}}\left[1+\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right]^{-1}$$

and
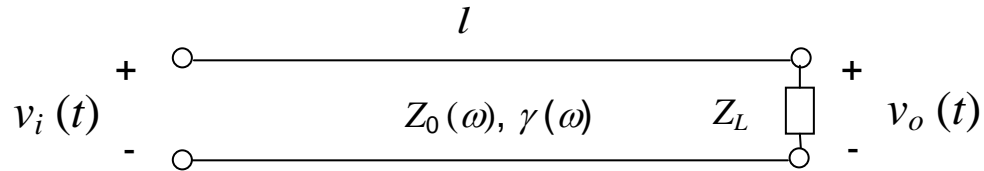
$$v_g \approx \frac{1}{\sqrt{LC}}\left[1-\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right]^{-1}.$$

(Note that the group velocity expression is derived assuming that the R and G parameters are frequency independent, which is a somewhat hypothetical situation, which will not be true for practical conductors and lossy dielectrics.)

Is the phase velocity greater or less than the velocity of light (in the medium of the line)? What about the group velocity?

Hint: Use Taylor series for $(1+z)^{1/2}$ to approximate the formula for γ, and make sure that you keep enough terms in your approximation.
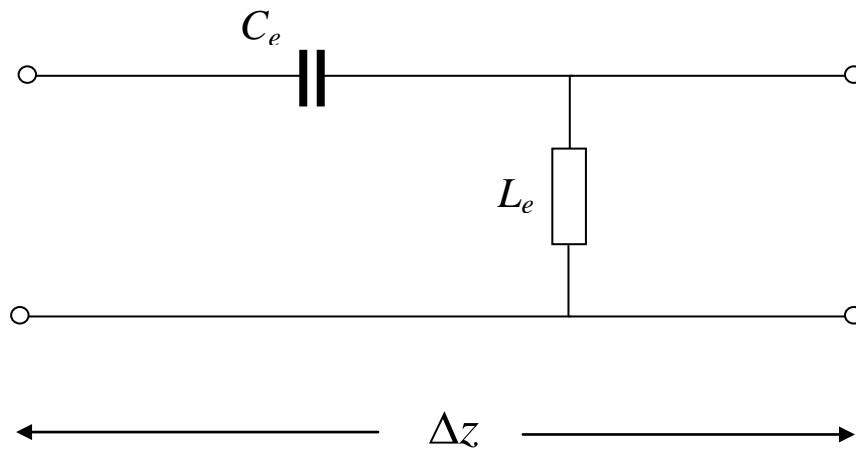
6) An input signal generator $v_i(t)$ at $z = -l$ is connected to a general (possibly lossy) transmission line of length l that has a load Z_L at the end of the line ($z = 0$). For this line the characteristic impedance and the propagation constant are arbitrary functions of frequency. Use Fourier-transform theory to determine the output voltage $v_o(t)$. The answer will not be in closed form -- it will involve an integral in ω from 0 to infinity, just as was done in class for the semi-infinite line. (Hint: Use the result of Prob. 3.)
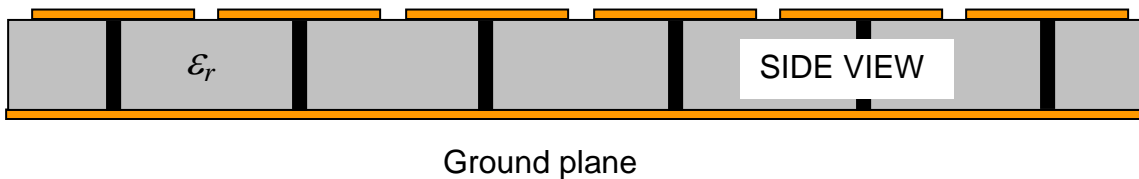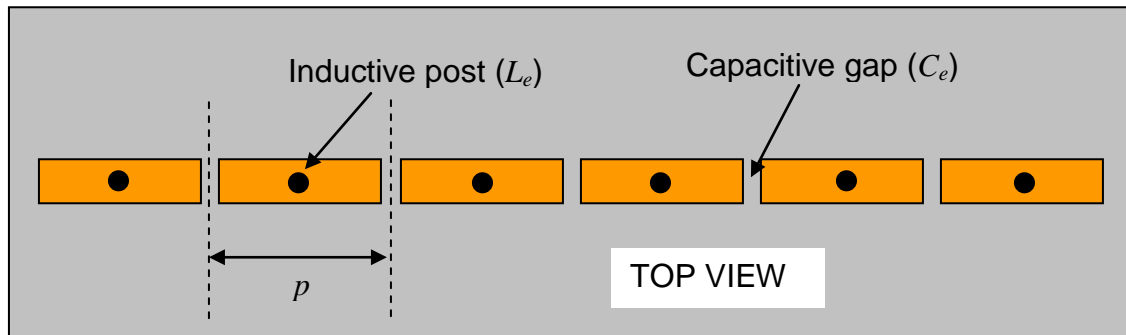
7) An "artificial" transmission line is made by cascading sections of a circuit with discrete components having element values L_e [Henrys] and C_e [Farads] as shown below. Denote $C_0 = C_e \, \Delta z$ and $L_0 = L_e \, \Delta z$. (Note that these quantities do not represent capacitance and inductance per unit length!)

Determine α, β, Z_0, v_p, and v_g in terms of C_0 and L_0. Assume that the group velocity is a positive number, corresponding to power flow in the positive z direction. Be careful with the signs in your answers!

Hint: Note that the general expressions $\gamma = \sqrt{ZY}$ and $Z_0 = Z/\gamma$ hold for both natural and artificial lines.
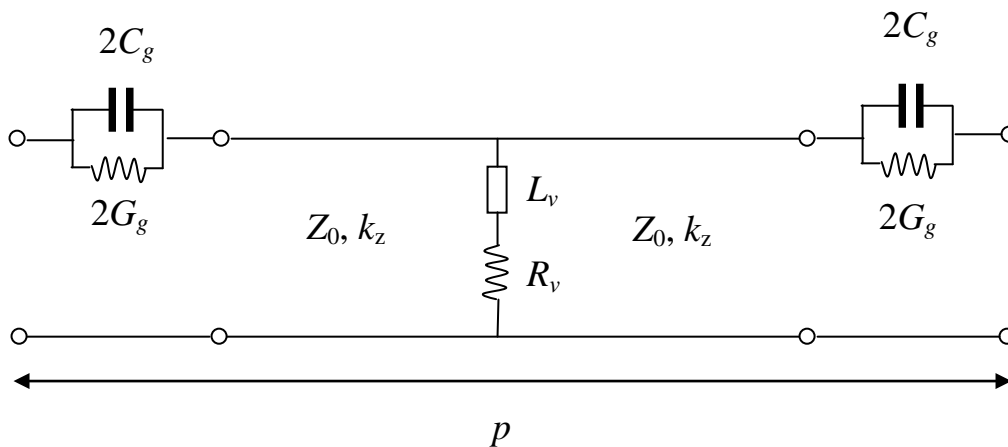

8) A "metamaterial" transmission line is formed from microstrip line by introducing capacitive gaps and vertical inductive posts ("vias") as shown below. A "unit cell" of the line has length p.

Inductive post (L_e) Capacitive gap (C_e)

p

TOP VIEW

ε_r SIDE VIEW

Ground plane

The equivalent circuit for a unit cell of this structure is shown below. It consists of a lossless transmission line of length p, with a parallel inductance L_v in the middle and series capacitors $2C_g$ at the ends to model the gaps. A conductance $2G_g$ is placed in parallel with the capacitors to model radiation from the gaps. The resistance R_v models the resistance of the vertical post (via). The lossless transmission line is assumed to have known values of characteristic impedance Z_0 and wavenumber k_z (from these values you can determine L and C for the line).

Assuming that the period p is very small compared to a wavelength, derive an expression for the complex propagation constant γ of the artificial transmission.

Hint: Replace each of the two sections of microstrip transmission line in the unit cell with a parallel capacitance and a series inductance, determined from the L and C of the line (which in turn may be found from the known values of Z_0 and k_z for the line). Also, note that in the limit as p tends to zero, the order of the series and parallel elements does not matter, so you can switch them if it is convenient to do so.

9) A twin-line transmission line has two perfectly conducting parallel wire conductors of radius a, separated by a distance h (between centers) in free space. The wires may be assumed to be perfectly conducting. An approximate formula for the characteristic impedance (accurate for $h \gg a$) is

$$Z_0 = \frac{\eta_0}{\pi} \ln\left(\frac{h}{a}\right).$$

Assuming that the radius of each wire is 1 [mm], determine the separation h necessary to give a characteristic impedance of 300 [Ω].

10) The above twin-line transmission line is now immersed is seawater. The transmission line carries a VHF signal at a frequency of 100 [MHz]. Assume that the seawater is nonmagnetic and has a relative permittivity of 81 ($\varepsilon_r = 81 - j (0)$) and a conductivity of 4.0 [S/m]. (Note that the imaginary part of ε_r, due to polarization loss, is being neglected at this low frequency.)

   a) Determine the values of the parameters (R, L, G, C) for the line.

   b) Determine the values of the complex propagation constant γ, the phase constant β, the attenuation constant α, and the complex characteristic impedance Z_0.

   c) How long does the line have to be for the signal to decay by 20 [dB]?

   d) If the amplitude of the voltage phasor at a point on the line is 1.0 [V], what is the time-average power flow (in watts) on the line?

11) Consider a lossless transmission line that has an arbitrary cross section. Show that the energy velocity must be equal to the phase velocity (which is also equal to the group velocity for a lossless line). That is, show that

$$v_E = \frac{\langle \mathscr{P}_z \rangle}{\langle \mathscr{H}_l \rangle} = \frac{1}{\sqrt{\mu\varepsilon}}.$$

12) A transmission line normally consists of two parallel conductors. Can a transmission line be made of a single solid conductor (i.e., a wire with an arbitrary cross-sectional shape)? In this case we can think of infinity as the outer conductor. If the mode is TEM, then the 2D field at $z = 0$ will look like that of a static field. Explain why the static field would have to have an infinite voltage drop between the conductor and infinity. Assume that far enough away from the wire the static field is that corresponding to an infinite line charge (arising from the charge density on the wire).

13) A vendor says that the Z_0 of a transmission line is 50 [Ω]. The filling material is Teflon with $\varepsilon_r = 2.1$ and $\tan\delta = 0.001$. Find the L, C, G values at 1 GHz.

14) A transmission line has a characteristic impedance of Z_0^{air} when the line is filled with air. Assume that the line is now filled with a lossy material having ε_{rc}. Show that the characteristic impedance is now

$$Z_0 = \frac{Z_0^{air}}{\sqrt{\varepsilon_{rc}}}.$$

Do this by starting with the formula

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}},$$

and assume that $R = 0$ (perfect conductors). Then determine how each of the parameters (L, C, G) change when the line is filled with the lossy material.

15) A microstrip line is shown below. Assume that the strip of width w is wide enough so that the electric field inside the substrate under the line is approximately constant in the x and y directions, and is given by

$$\underline{E} = -\hat{\underline{y}}\, E_0\, e^{-jkz}.$$

Also assume that the field is approximately zero outside the region under the strip. Finally, assume that the strip and ground plane are perfectly conducting, and that the mode is approximately a TEM mode. The substrate has a complex effective relative permittivity $\varepsilon_{rc} = \varepsilon'_{rc} - j\varepsilon''_{rc}$.

Determine the following properties of the microstrip line in terms of the given dimensions and substrate parameters:

a) The magnetic field inside the substrate below the line.

b) The surface current density $\underline{J}_s$ on the bottom surface of the microstrip line.

c) The voltage $V(z)$ and current $I(z)$. Assume that the line is the "A" conductor and the ground plane is the "B" conductor, and that the voltage is defined as $V = V_{AB}$. The current I is the current flowing on the line in the z direction.

d) The characteristic impedance Z_0. Use the definition $Z_0 = V^+ / I^+$.

e) The parameters G, L, C. (Note that the principle of effective permittivity could be helpful here, as it was on Prob. 7 of HW 2.)

f) The characteristic impedance Z_0. Use the formula for characteristic impedance that gives Z_0 in terms of (R, L, G, C). Show that you get the same result as you found in part (d) above.