ECE 6341

Spring 2016

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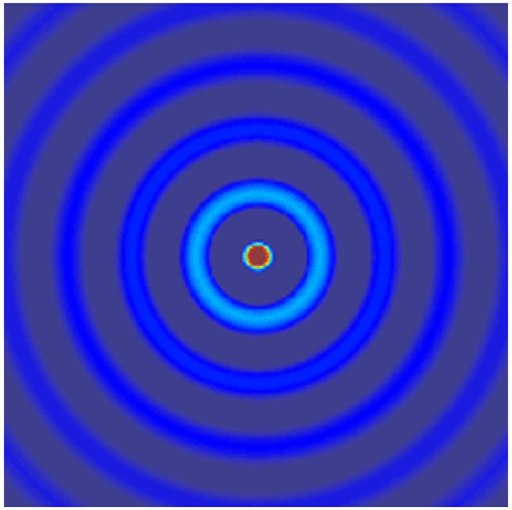
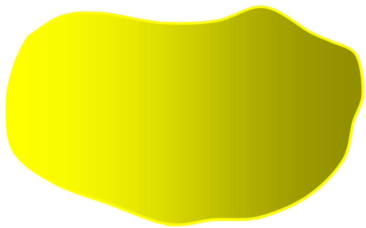


Notes 1

Fields in a Source-Free Region



Sources

Source-free homogeneous region

$$(\varepsilon,\mu)$$

$$(\underline{E},\underline{H})$$

Note: For a lossy region, we replace

$$\varepsilon \to \varepsilon_c$$

$$\begin{aligned}\varepsilon_c &= \varepsilon - j(\sigma/\omega)\\ &= \varepsilon_c' - j\varepsilon_c''\\ &= \varepsilon_c'(1 - j\tan\delta)\\ &= \varepsilon_0\varepsilon_{rc}'(1 - j\tan\delta)\\ &= \varepsilon_0\varepsilon_{rc}\end{aligned}$$

$$\nabla\times\underline{E} = -j\omega\mu\underline{H}$$

$$\nabla\times\underline{H} = j\omega\varepsilon\underline{E}$$

$$\nabla\cdot\underline{E} = 0$$

$$\nabla\cdot\underline{H} = 0$$

Fields in a Source-Free Region (cont.)

(a) $\nabla \cdot \underline{H} = 0$

$$\Rightarrow \quad \underline{H} = \frac{1}{\mu} \nabla \times \underline{A}$$

Ampere's law:

$$\underline{E} = \frac{1}{j\omega\varepsilon} \nabla \times \left(\frac{1}{\mu} \nabla \times \underline{A} \right)$$

(b) $\nabla \cdot \underline{E} = 0$

$$\Rightarrow \quad \underline{E} = -\frac{1}{\varepsilon} \nabla \times \underline{F}$$

Faraday's law:

$$\underline{H} = -\frac{1}{j\omega\mu} \nabla \times \left(-\frac{1}{\varepsilon} \nabla \times \underline{F} \right)$$

The field can be represented using <u>either</u> $\underline{A}$ or $\underline{F}$

Fields in a Source-Free Region (cont.)

Case (a)

Assume we use $\underline{A}$:

$$\underline{H} = \frac{1}{\mu} \nabla \times \underline{A}$$

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$

$$\Downarrow$$

$$\nabla \times \underline{E} = -j\omega\left(\nabla \times \underline{A}\right)$$

$$\Downarrow$$

$$\nabla \times \left(\underline{E} + j\omega\underline{A}\right) = \underline{0}$$

$$\Downarrow$$

$$\underline{E} + j\omega\underline{A} = -\nabla\Phi$$

Hence

$$\underline{E} = -\nabla\Phi - j\omega\underline{A}$$

This is the "mixed potential" form for the electric field.

Fields in a Source-Free Region (cont.)

Next, use $\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$

$$\frac{1}{\mu}\nabla \times \left(\nabla \times \underline{A}\right) = j\omega\varepsilon\left(-\nabla\Phi - j\omega\underline{A}\right)$$

$$\Downarrow$$

$$\nabla \times \left(\nabla \times \underline{A}\right) - k^2\underline{A} = -j\omega\varepsilon\mu\nabla\Phi$$

$$\Downarrow$$

$$\nabla\left(\nabla \cdot \underline{A}\right) - \nabla^2\underline{A} - k^2\underline{A} = -j\omega\varepsilon\mu\nabla\Phi$$

$$\text{Recall}: \ \nabla^2\underline{A} \equiv \nabla\left(\nabla \cdot \underline{A}\right) - \nabla \times \left(\nabla \times \underline{A}\right)$$

Fields in a Source-Free Region (cont.)

$$\nabla\left(\nabla\cdot\underline{A}\right)-\nabla^2\underline{A}-k^2\underline{A}=-j\omega\varepsilon\mu\nabla\Phi$$

Choose $\nabla\cdot\underline{A}=-j\omega\varepsilon\mu\Phi$ (Lorenz Gauge)

Then $\nabla^2\underline{A}+k^2\underline{A}=\underline{0}$

This is the "vector Helmholtz equation" for the magnetic vector potential.

Fields in a Source-Free Region (cont.)

Case (b)

Assume we use $\underline{F}$:

$$\underline{E} = -\frac{1}{\varepsilon}\nabla \times \underline{F}$$

Invoking duality:

$$\underline{H} = -\nabla\Psi - j\omega\underline{F}$$

Ψ = magnetic scalar potential

Choose $$\nabla \cdot \underline{F} = -j\omega\varepsilon\mu\Psi$$

We then have:

$$\nabla^2 \underline{F} + k^2 \underline{F} = \underline{0}$$

Fields in a Source-Free Region (cont.)

To be even more general, let

$$\underline{E} = \underline{E}^{a} + \underline{E}^{f}$$

$$\underline{H} = \underline{H}^{a} + \underline{H}^{f}$$

where ($\underline{E}^a$, $\underline{H}^a$) and ($\underline{E}^f$, $\underline{H}^f$) each satisfy Maxwell's equations.

(This is an arbitrary partition.)

The representation is not unique, since there are many ways to split the field. For example, we could use

$$\underline{E}^{a} = 0.1\underline{E}$$

$$\underline{E}^{f} = 0.9\underline{E}$$

Fields in a Source-Free Region (cont.)

We construct the vector potentials so that

$$\underline{A} \rightarrow (\underline{E}^a, \underline{H}^a)$$

$$\underline{F} \rightarrow (\underline{E}^f, \underline{H}^f)$$

We then have

$$\underline{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{A}\right) - \frac{1}{\varepsilon}\nabla\times\underline{F}$$

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{F}\right)$$

Fields in a Source-Free Region (cont.)

Depending on the type of source (outside the source-free region) that is producing the field within the source-free region, it may be more convenient to represent the field using only $\underline{A}$ or only $\underline{F}$.

EXAMPLES

Electric dipole: choose only $\underline{A}$ ($\underline{A}$ is in the same direction as $\underline{J}$)

Magnetic dipole: choose only $\underline{F}$ ($\underline{F}$ is in the same direction as $\underline{M}$)

This solution was discussed in ECE 6340.

In principle, one could represent the field of the electric dipole using the $\underline{F}$ vector potential, but it would be much more difficult than with the $\underline{A}$ vector potential!

TE_z/TM_z Theorem

In a source-free homogeneous region, we can always represent the field using the following form:

$$\underline{A} = \underline{\hat{z}}\, A_z(x, y, z)$$
$$\underline{F} = \underline{\hat{z}}\, F_z(x, y, z)$$

$\underline{\hat{z}}$ is an arbitrary <u>fixed</u> direction (called here the "pilot vector" direction).

(A proof of this theorem is given later.)

This theorem gives us a systematic way to represent the fields in a source-free region, involving only two scalar field components.

TE_z/TM_z Theorem (cont.)

Note: A simpler solution often results if we choose the best direction for the pilot vector.

EXAMPLE

An infinitesimal unit-amplitude electric dipole pointing in the z direction:

$$\underline{A} = \underline{\hat{z}}\, A_z(x, y, z)$$

The solution is

$$A_z(x, y, z) = \mu \frac{e^{-jkr}}{4\pi r}$$

We only need A_z and not F_z.

If we had picked the pilot vector direction to be something different, such as x, we would need BOTH A_x and F_x.

TE_z/TM_z Theorem (cont.)

TE_z / TM_z property of Fields

$$\boxed{\begin{aligned}\underline{A} &= \underline{\hat{z}}\, A_z(x,y,z) \\ \underline{F} &= \underline{\hat{z}}\, F_z(x,y,z)\end{aligned}}$$

Consider, for example,

$$\underline{H}^a = \frac{1}{\mu}\nabla \times \left(\underline{\hat{z}} A_z\right)$$

$$= \frac{1}{\mu}\left(\cancel{A_z \nabla \times \underline{\hat{z}}} - \underline{\hat{z}} \times \nabla A_z\right) \quad \Rightarrow H_z^a = 0$$

Hence, $H_z^a = 0$ (A_z →TM_z)

Similarly, $E_z^f = 0$ (F_z →TE_z)

TE_z/TM_z Theorem (cont.)

The fields are found as follows:

$$\psi = A_z$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

$$E_x = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial x\partial z}$$

$$E_y = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial y\partial z}$$

$$H_x = \frac{1}{\mu}\frac{\partial\psi}{\partial y}$$

$$H_y = -\frac{1}{\mu}\frac{\partial\psi}{\partial x}$$

$$H_z = 0$$

Note: There is a factor μ difference with the Harrington text.

TE_z/TM_z Theorem (cont.)

$$\psi = F_z$$

$$H_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

$$E_x = -\frac{1}{\varepsilon}\frac{\partial \psi}{\partial y}$$

$$H_x = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 \psi}{\partial x \partial z}$$

$$E_y = \frac{1}{\varepsilon}\frac{\partial \psi}{\partial x}$$

$$H_y = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 \psi}{\partial y \partial z}$$

$$E_z = 0$$

Note: There is a factor ε difference with the Harrington text.

TE_z/TM_z Theorem (cont.)

From the vector Helmholtz equation we have:

$$\nabla^2 \underline{A} + k^2 \underline{A} = \underline{0}$$

Taking the z component gives: $\underline{\hat{z}}\left(\nabla^2 A_z\right) + \underline{\hat{z}}\left(k^2 A_z\right) = \underline{0}$

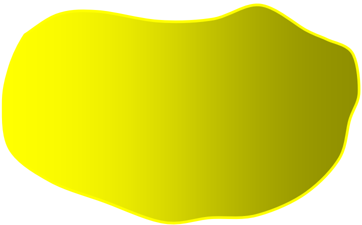
or $\nabla^2 A_z + k^2 A_z = 0$

Similarly, $\nabla^2 F_z + k^2 F_z = 0$

Hence, any EM problem in a source-free region reduces to solving the scalar Helmholtz equation:

$$\nabla^2 \psi + k^2 \psi = 0$$

Summary of TE_z/TM_z Result



Sources

Source-free homogeneous region

$$(\varepsilon, \mu)$$

$$(\underline{E}, \underline{H})$$

$\underline{A}$

$\underline{F}$

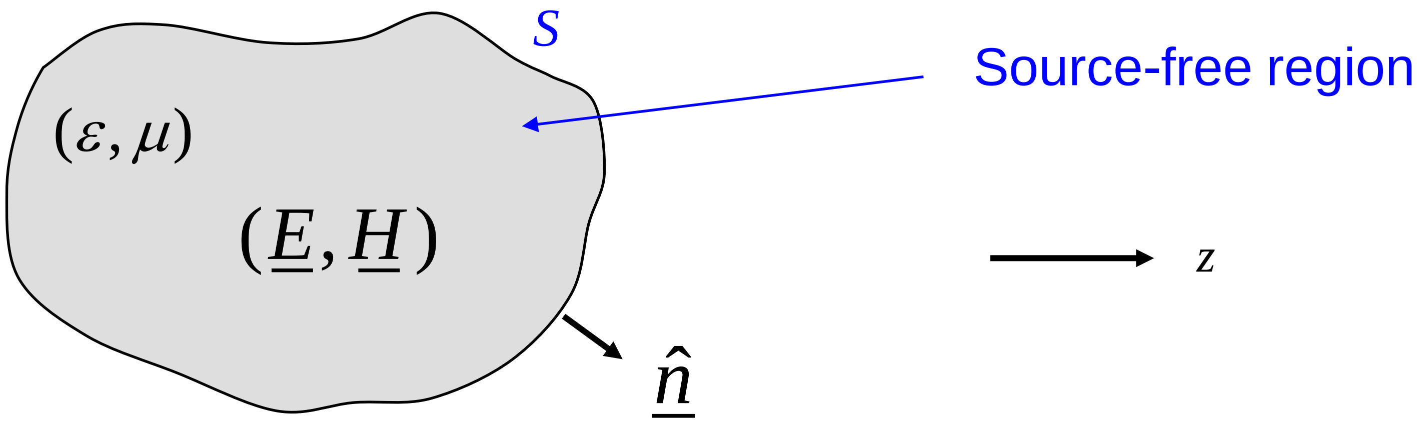
Arbitrary pilot direction "z"

$$\boxed{\begin{aligned} \underline{A} &= \underline{\hat{z}}\, A_z(x,y,z) \\ \underline{F} &= \underline{\hat{z}}\, F_z(x,y,z) \end{aligned}}$$

$$\nabla^2 \psi + k^2 \psi = 0$$

$\psi = A_z$ or F_z

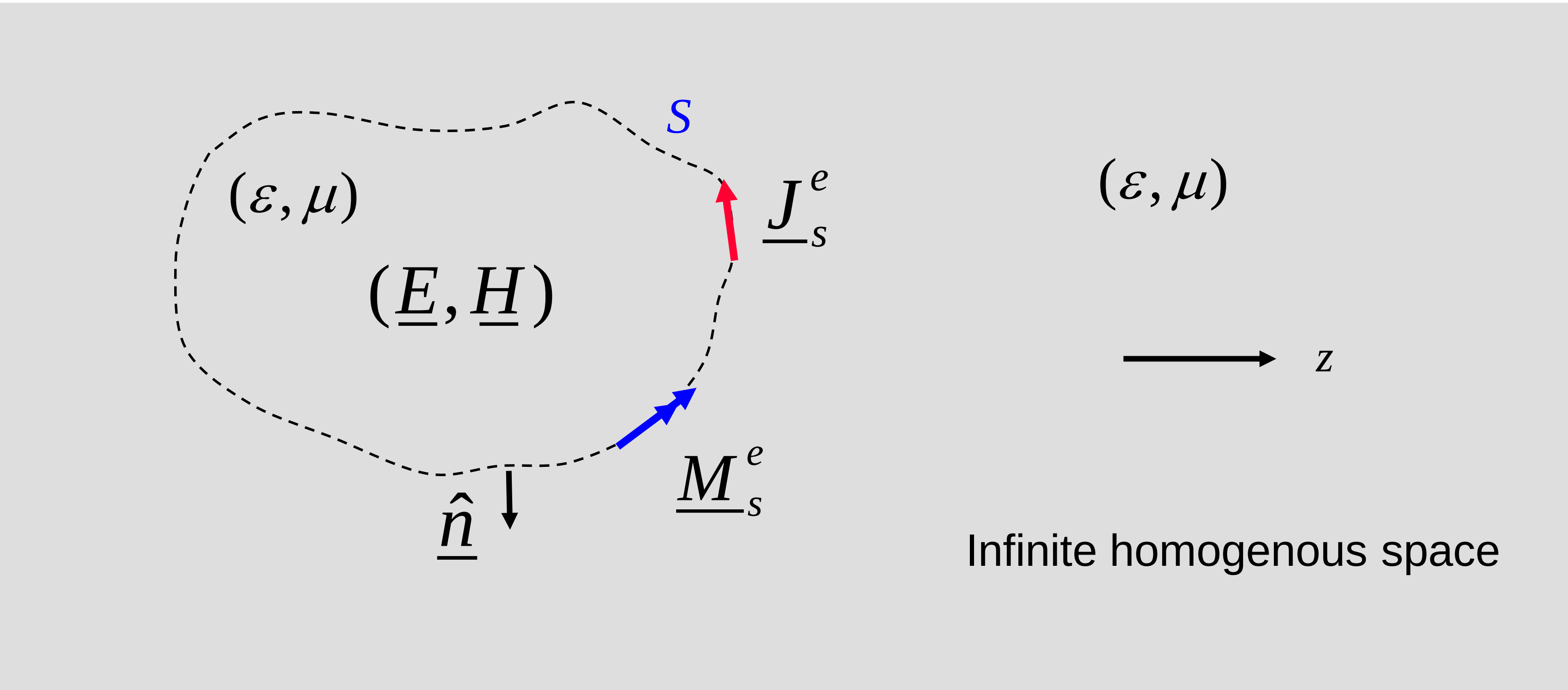
Proof of TE$_z$/TM$_z$ Theorem



Apply equivalence principle:

Keep original material and fields inside S.
Put zero fields outside S.
Put (ε, μ) outside S.

Proof of TE$_z$/TM$_z$ Theorem



$$\underline{J}_s^e = \left(-\underline{\hat{n}}\right) \times \underline{H}$$

$$\underline{M}_s^e = -\left(\left(-\underline{\hat{n}}\right) \times \underline{E}\right)$$

Proof of $\mathrm{TE}_z/\mathrm{TM}_z$ Theorem (cont.)

Consider the z component of the currents:

$$\underline{J}_z^e \rightarrow A_z$$

$$\underline{M}_z^e \rightarrow F_z$$

(from 6340)

Also, from a horizontal dipole source, we can show that:

$$\underline{J}_{x,y}^e \rightarrow A_z + F_z$$

$$\underline{M}_{x,y}^e \rightarrow A_z + F_z$$

This will be established later in the class notes and the homework by solving the problem of a horizontal (x or y directed) dipole source using A_z and F_z.

Hence, the fields in the source-free region due to all of the equivalent currents may be represented with A_z and F_z.

Non-Uniqueness of Potentials

A_z and F_z <u>are not unique</u>.

To illustrate, consider:

$$\begin{cases} A_z = c_1 e^{-jkz}, & k = \omega\sqrt{\mu\varepsilon} \\ F_z = 0 & (\text{TM}_z \text{ field}) \end{cases}$$

$$\left.\begin{aligned} E_z &= \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z = 0 \\ E_x &= \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x \partial z} = 0 \\ E_y &= \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y \partial z} = 0 \end{aligned}\right\} \Rightarrow \quad \underline{H} = -\frac{1}{j\omega\mu}\nabla \times \underline{E} = \underline{0}$$

This set of potentials produces a null field!

Hence, adding this set of potentials to a solution does not change the fields.