# ECE 6341

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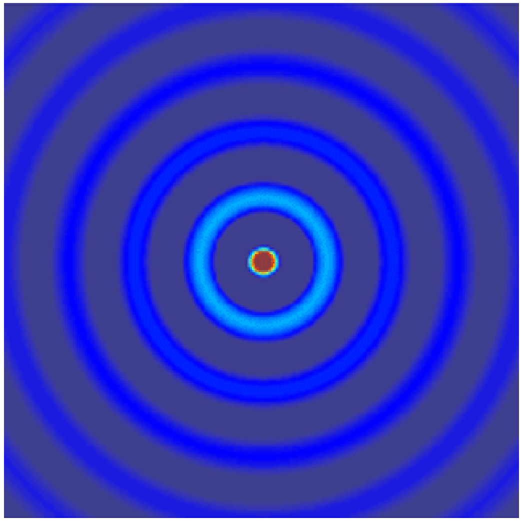
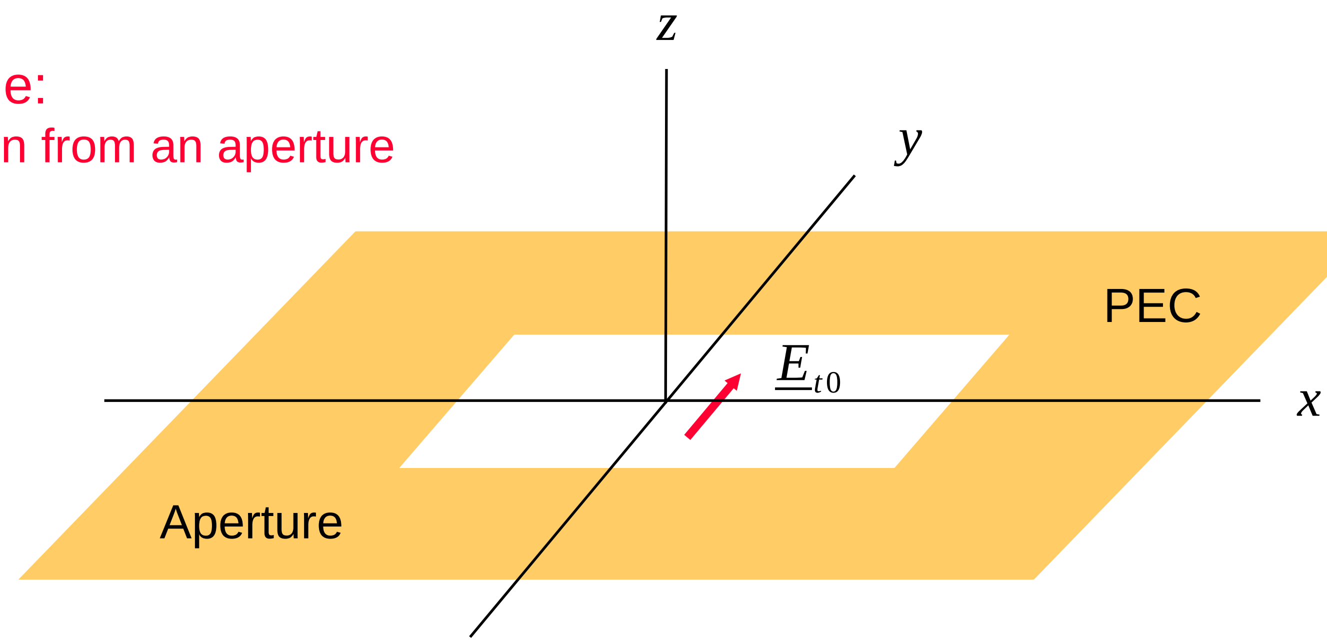
Prof. David R. Jackson
ECE Dept.

## Notes 2



# Fields in a Source-Free Region

Example:
Radiation from an aperture




Assume the following choice of vector potentials:

$$\underline{A} = \underline{\hat{z}}\, A_z(x,y,z) \qquad \nabla^2 A_z + k_0^2 A_z = 0$$

$$\underline{F} = \underline{\hat{z}}\, F_z(x,y,z) \qquad \nabla^2 F_z + k_0^2 F_z = 0$$

# Example (cont.)

Introduce the Fourier transform:

$$\tilde{A}_z(k_x, k_y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_z(x, y, z)\, e^{j(k_x x + k_y y)}\, dx\, dy$$

$$A_z(x, y, z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}_z(k_x, k_y, z)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

$$\nabla^2 A_z + k^2 A_z = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(-k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} + k^2\right)$$

$$\tilde{A}_z(k_x, k_y, z)\, e^{-j(k_x x + k_y y)}\, dk_x\, dk_y$$

$$= 0$$

# Example (cont.)

Hence $$\frac{\partial^2 \tilde{A}_z}{\partial z^2} + (k_0^2 - k_x^2 - k_y^2)\tilde{A}_z = 0$$

Define $$k_z \equiv (k_0^2 - k_x^2 - k_y^2)^{1/2}$$

**Correct choice:** $$k_z = \begin{cases} \sqrt{k_0^2 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 < k_0^2 \\ -j\sqrt{k_x^2 + k_y^2 - k_0^2}, & k_x^2 + k_y^2 > k_0^2 \end{cases}$$

Then we have

$$\frac{\partial^2 \tilde{A}_z}{\partial z^2} + k_z^2 \tilde{A}_z = 0$$

# Example (cont.)

Solution:

$$\tilde{A}_z(k_x, k_y, z) = \tilde{A}_z(k_x, k_y, 0)\, e^{-jk_z z}$$

Similarly,

$$\tilde{F}_z(k_x, k_y, z) = \tilde{F}_z(k_x, k_y, 0)\, e^{-jk_z z}$$

# Example (cont.)

Hence

$$A_z(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{A}_z(k_x,k_y,0)\,e^{-j(k_x x+k_y y+k_z z)}dk_x dk_y$$

$$F_z(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{F}_z(k_x,k_y,0)\,e^{-j(k_x x+k_y y+k_z z)}dk_x dk_y$$

This is a representation of the potentials as a *spectrum of plane waves*.

# Example (cont.)

Derivative property:

$$\frac{\partial A_z}{\partial x} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-jk_x)\, \tilde{A}_z(k_x, k_y, 0)\, e^{-j(k_x x + k_y y + k_z z)} dk_x dk_y$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-jk_x)\, \tilde{A}_z(k_x, k_y, z)\, e^{-j(k_x x + k_y y)} dk_x dk_y$$

Also $$\frac{\partial A_z}{\partial x} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{F}\left[\frac{\partial A_z}{\partial x}\right] e^{-j(k_x x + k_y y)} dk_x dk_y$$

Hence $$\mathscr{F}\left[\frac{\partial A_z}{\partial x}\right] = (-jk_x)\, \tilde{A}_z(k_x, k_y, z)$$

Similarly for the $y$ and $z$ derivatives.

# Example (cont.)

Apply B.C.'s at $z = 0$:

$$E_x = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x \partial z} - \frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}$$

$$E_y = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y \partial z} + \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}$$

Hence:

$$\tilde{E}_x = \frac{1}{j\omega\mu\varepsilon}(-jk_x)(-jk_z)\tilde{A}_z - \frac{1}{\varepsilon}(-jk_y)\tilde{F}_z$$

$$\tilde{E}_y = \frac{1}{j\omega\mu\varepsilon}(-jk_y)(-jk_z)\tilde{A}_z + \frac{1}{\varepsilon}(-jk_x)\tilde{F}_z$$

# Example (cont.)

Hence, we have (A zero subscript denotes $z = 0$.)

$$\tilde{E}_{x0}\left(k_x,k_y\right)=\left(\frac{-k_x k_z}{j\omega\mu\varepsilon}\right)\tilde{A}_{z0}(k_x,k_y)+\left(\frac{jk_y}{\varepsilon}\right)\tilde{F}_{z0}(k_x,k_y)$$

$$\tilde{E}_{y0}\left(k_x,k_y\right)=\left(\frac{-k_y k_z}{j\omega\mu\varepsilon}\right)\tilde{A}_{z0}(k_x,k_y)+\left(\frac{-jk_x}{\varepsilon}\right)\tilde{F}_{z0}(k_x,k_y)$$

The left-hand sides are known.

We have two equations in two unknowns, $(\tilde{A}_{z0},\tilde{F}_{z0})$ .

Hence, we can solve for $(\tilde{A}_{z0},\tilde{F}_{z0})$ .

# Example (cont.)

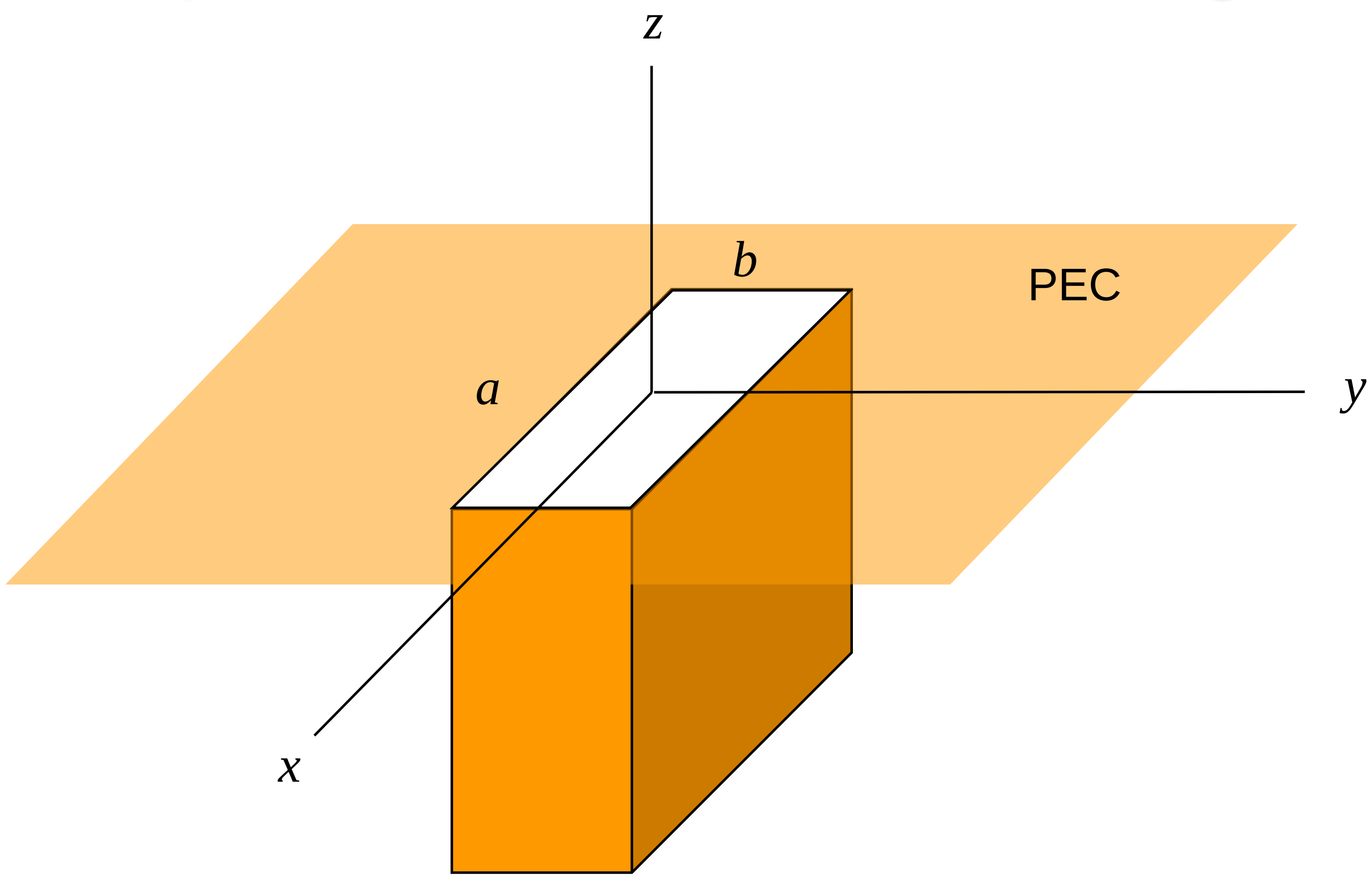
In the space domain,

$$A_z(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{A}_{z0}(k_x,k_y) e^{-jk_z z} e^{-j(k_x x + k_y y)} dk_x dk_y$$

and similarly for $F_z(x,y,z)$

We can then find the fields from the $\text{TE}_z$ –$\text{TM}_z$ equations.

# Example: Radiation from Waveguide




$$\underline{E}_{t0}(x, y) = \underline{\hat{y}}\, E_0 \cos\left(\frac{\pi x}{a}\right)$$

Find the complex power radiated by the aperture.

# Example (cont.)

$$\tilde{E}_{x0} = \left( \frac{-k_x k_z}{j\omega\mu_0\varepsilon_0} \right) \tilde{A}_{z0} + \left( \frac{jk_y}{\varepsilon_0} \right) \tilde{F}_{z0} = 0$$

$$\tilde{E}_{y0} = \left( \frac{-k_y k_z}{j\omega\mu_0\varepsilon} \right) \tilde{A}_{z0} + \left( \frac{-jk_x}{\varepsilon_0} \right) \tilde{F}_{z0}$$

Hence, from the first equation

$$\boxed{\tilde{F}_{z0} = \tilde{A}_{z0} \left( \frac{-k_x k_z}{\omega\mu_0 k_y} \right)}$$

From the second equation,

$$\tilde{E}_{y0} = \left[ \left( \frac{-k_y k_z}{j\omega\mu_0\varepsilon_0} \right) + \left( \frac{-jk_x}{\varepsilon_0} \right) \left( \frac{-k_x k_z}{\omega\mu_0 k_y} \right) \right] \tilde{A}_{z0}$$

# Example (cont.)

Hence

$$\tilde{E}_{y0} = \tilde{A}_{z0}\left[\left(\frac{-1}{j\omega\mu_0\varepsilon_0}\right)\left(k_y k_z + \frac{k_x^2 k_z}{k_y}\right)\right]$$

$$= \tilde{A}_{z0}\left[\left(\frac{-k_z}{j\omega\mu_0\varepsilon_0}\right)\left(k_y + \frac{k_x^2}{k_y}\right)\right]$$

$$= \tilde{A}_{z0}\left[\left(\frac{-k_z}{j\omega\mu_0\varepsilon_0 k_y}\right)\left(k_y^2 + k_x^2\right)\right]$$

$$= \tilde{A}_{z0}\left[\left(\frac{-k_z}{j\omega\mu_0\varepsilon_0 k_y}\right)\left(k_0^2 - k_z^2\right)\right]$$

Therefore

$$\tilde{A}_{z0} = \tilde{E}_{y0}\left[j\omega\mu_0\varepsilon_0\left(-\frac{k_y}{k_z}\right)\left(\frac{1}{k_0^2 - k_z^2}\right)\right]$$

# Example (cont.)

For the Fourier transform of the aperture field, we have:

$$\tilde{E}_{y0}(k_x,k_y) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} E_0 \cos\left(\frac{\pi x}{a}\right) e^{j(k_x x + k_y y)} dy\, dx$$

$$= E_0 \left[ b \frac{\sin\left(\dfrac{k_y b}{2}\right)}{\left(\dfrac{k_y b}{2}\right)} \right] \left[ a \frac{\dfrac{\pi}{2}\cos\left(k_x \dfrac{a}{2}\right)}{\left(\dfrac{\pi}{2}\right)^2 - \left(k_x \dfrac{a}{2}\right)^2} \right]$$

# Example (cont.)

Complex power radiated by the aperture:

$$P_c = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right) \cdot \underline{\hat{z}}\, dxdy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} -\frac{1}{2} E_{y0} H_{x0}^*\, dxdy$$

# Example (cont.)

Parseval's theorem:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, g^{*}(x,y)\, dx dy = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{f}(k_x,k_y)\,\tilde{g}^{*}(k_x,k_y)\, dk_x dk_y$$

Hence

$$P_c = -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}_{y0}\tilde{H}_{x0}^{*}\, dk_x dk_y$$

Also, recall:

$$\tilde{A}_{z0} = \tilde{E}_{y0}\left[ j\omega\mu_0\varepsilon_0 \left(-\frac{k_y}{k_z}\right)\left(\frac{1}{k_0^2 - k_z^{\,2}}\right)\right]$$

# Example (cont.)

Now find $\tilde{H}_{x0}$ :

$$H_x = \frac{1}{\mu_0}\frac{\partial A_z}{\partial y} + \frac{1}{j\omega\mu_0\varepsilon_0}\frac{\partial^2 F_z}{\partial x \partial z}$$

$$\tilde{H}_x = \frac{1}{\mu_0}(-jk_y)\tilde{A}_z + \frac{1}{j\omega\mu_0\varepsilon_0}(-jk_x)(-jk_z)\tilde{F}_z$$

Hence,

$$\tilde{H}_{x0} = \left[\frac{1}{\mu_0}(-jk_y) + \frac{1}{j\omega\mu_0\varepsilon_0}(-jk_x)(-jk_z)\left(\frac{-k_x k_z}{\omega\mu_0 k_y}\right)\right]\tilde{A}_{z0}$$

or

$$\tilde{H}_{x0} = \left[\frac{1}{\mu_0}(-jk_y) + \frac{1}{j\omega\mu_0\varepsilon_0}(-jk_x)(-jk_z)\left(\frac{-k_x k_z}{\omega\mu_0 k_y}\right)\right]\left[j\omega\mu_0\varepsilon_0\left(-\frac{k_y}{k_z}\right)\left(\frac{1}{k_0^2 - k_z{}^2}\right)\right]\tilde{E}_{y0}$$

# Example (cont.)

Simplifying,

$$\tilde{H}_{x0} = -\tilde{E}_{y0}\left(\frac{\omega\varepsilon_0}{k_z}\right)\left(\frac{1}{k_0^2 - k_z^2}\right)\left[k_y^2 + \frac{k_x^2 k_z^2}{k_0^2}\right]$$

$$P_c = -\frac{1}{2}\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{E}_{y0}\tilde{H}_{x0}^{*}\,dk_x dk_y$$

Hence,

$$P_c = \frac{1}{2}\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}\left|\tilde{E}_{y0}(k_x,k_y)\right|^2\left(\frac{\omega\varepsilon_0}{k_z^*}\right)\left(\frac{1}{k_0^2 - k_z^{*2}}\right)\left[k_y^2 + \frac{k_x^2 k_z^{*2}}{k_0^2}\right]dk_x dk_y$$

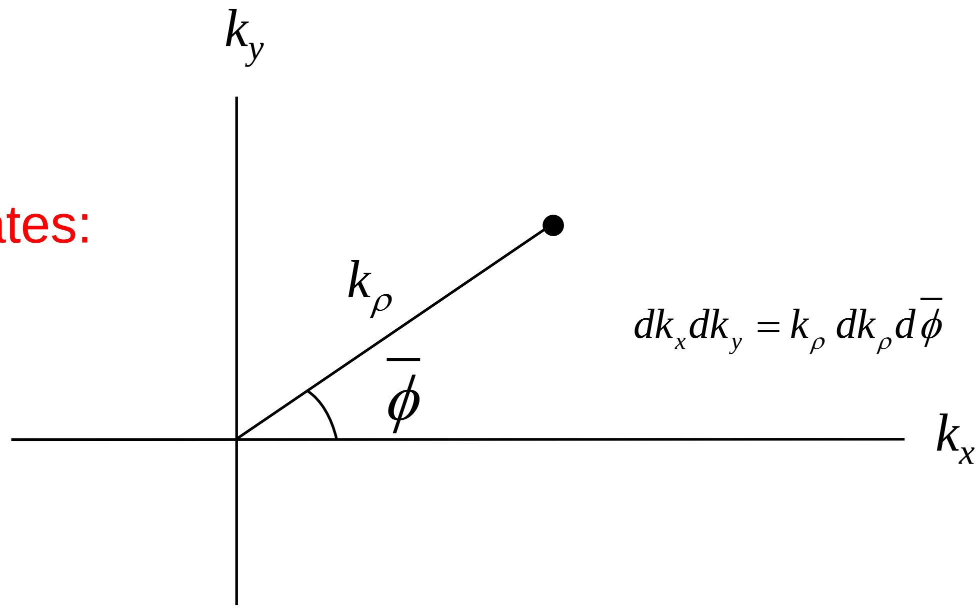
Notes:

$$\left(k_z^2\right)^* = \left(k_z^*\right)^2$$

$$k_0^* = k_0 \quad (k_0 \text{ is real})$$

# Example (cont.)

Polar coordinates:




$$dk_x dk_y = k_\rho \, dk_\rho d\bar{\phi}$$

$$k_x = k_\rho \cos\bar{\phi}$$

$$k_y = k_\rho \sin\bar{\phi}$$

$$k_\rho = \sqrt{k_x^{\,2} + k_y^{\,2}}$$

New notation: $\tilde{E}_{y0}(k_x, k_y) \rightarrow \tilde{E}_{y0}(k_\rho, \bar{\phi})$

# Example (cont.)

$$k_z^{\,2} = k^2 - k_x^2 - k_y^2 = \text{real} \qquad \left( \Rightarrow k_z^{*2} = k_z^2 \right)$$

$$k^2 - k_z^{*2} = k^2 - k_z^2 = k_x^2 + k_y^2 = k_\rho^2$$

Also,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\,)\, dk_x dk_y = \int_0^{2\pi} \int_0^{\infty} (\,)\, k_\rho dk_\rho \, d\overline{\phi}$$

Hence

$$P_c = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{\infty} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 \left( \frac{\omega \varepsilon}{k_z^*} \right) \left( \frac{k_\rho^2}{k_\rho^2} \right) \left[ \sin^2 \overline{\phi} + \left( \frac{k_z}{k} \right)^2 \cos^2 \overline{\phi} \right] k_\rho \, dk_\rho d\overline{\phi}$$

# Example (cont.)

$$P_c = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{\infty} \left|\tilde{E}_{y0}(k_\rho,\overline{\phi})\right|^2 F\left(k_\rho,\overline{\phi}\right) dk_\rho d\overline{\phi}$$

where

$$F\left(k_\rho,\overline{\phi}\right) = \left(\frac{\omega\varepsilon}{k_z^{\,*}}\right) k_\rho \left[\sin^2\overline{\phi} + \left(\frac{k_z}{k}\right)^2 \cos^2\overline{\phi}\right]$$

$$k_z = \left(k^2 - k_\rho^2\right)^{1/2}$$

$k_z = \text{real}, \quad k_\rho < k$

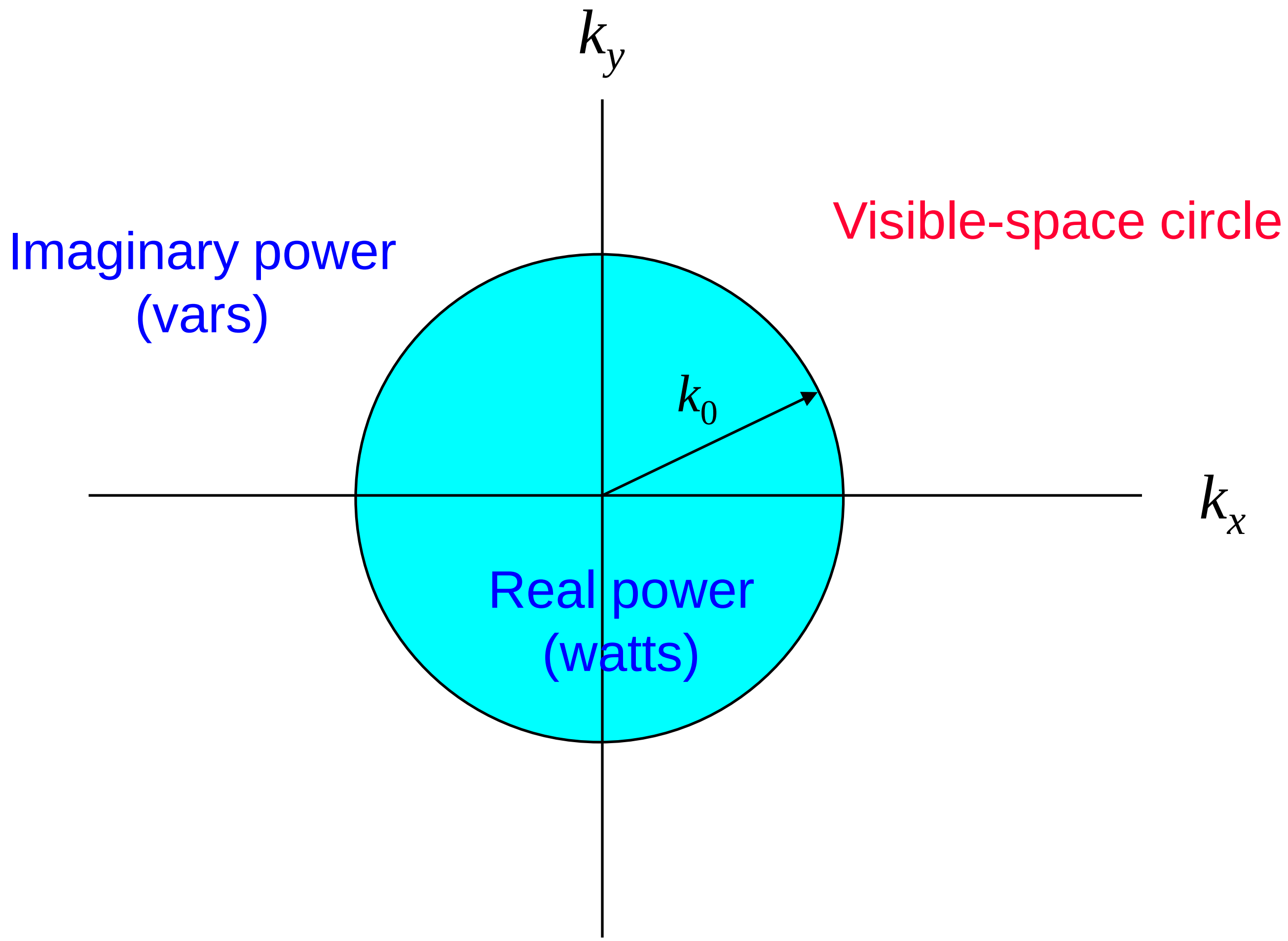
$k_z = \text{imaginary}, \quad k_\rho > k$

$\Rightarrow$

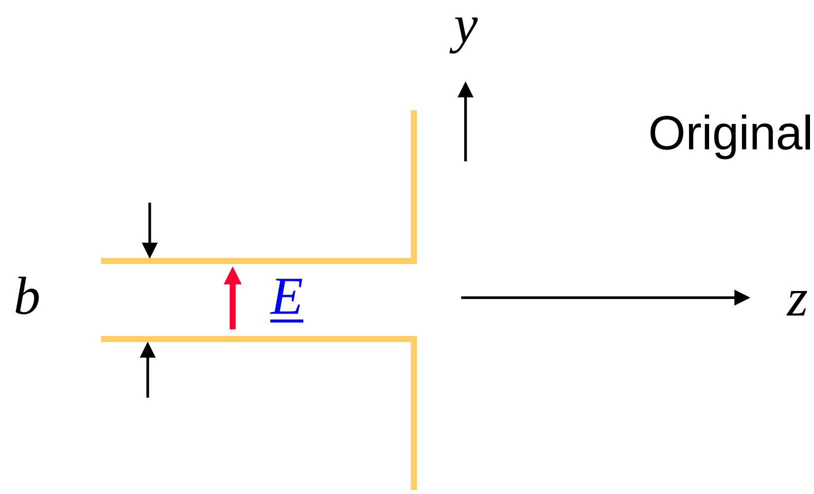
$F\left(k_\rho,\overline{\phi}\right) = \text{real}, \quad k_\rho < k$

$F\left(k_\rho,\overline{\phi}\right) = \text{imaginary}, \quad k_\rho > k$

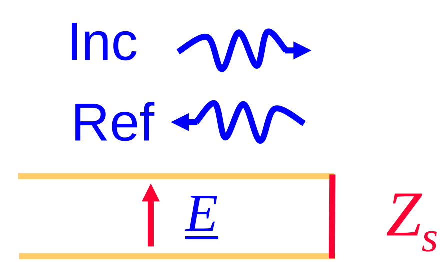
# Example (cont.)



# Equivalent Circuit




Original problem




Equivalent problem

(Impedance surface)

$$\frac{E_{y0}}{-H_{x0}} = Z_s$$

$$P_c = \int_S \frac{1}{2} (\underline{E} \times \underline{H}^*) \cdot \underline{\hat{z}} \, dS$$

# Equivalent Circuit (cont.)

so

$$P_c = -\frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} E_{y0}H_{x0}^{*}\,dx\,dy$$

$$= +\frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{Z_s^{*}}\left|E_{y0}\right|^2 dx\,dy$$

$$= \frac{\left|E_0\right|^2}{2Z_s^{*}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x}{a}\right) dx\,dy$$

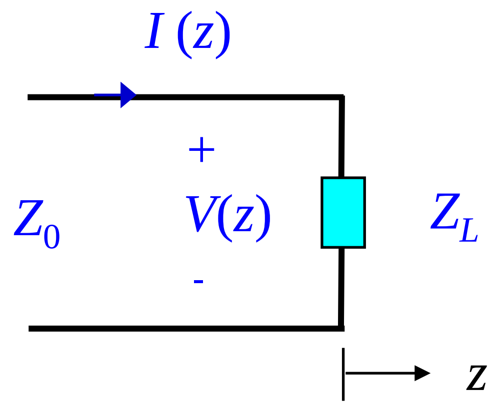
$$= \frac{\left|E_0\right|^2}{2Z_s^{*}}(b)\left(\frac{a}{2}\right)$$

Hence

$$Z_s = \frac{ab}{4}\left(\frac{\left|E_0\right|^2}{P_c^{*}}\right)$$

# TEN Model

TEN equivalent circuit




TEN modeling equations:

$$E_y(x,y,z) = V(z)\cos\left(\frac{\pi x}{a}\right)$$

$$-H_x(x,y,z) = I(z)\cos\left(\frac{\pi x}{a}\right)$$

# TEN Model (cont.)

$$Z_0 = \frac{V^+(z)}{I^+(z)} = -\frac{E_y{}^+(x,y,z)}{H_x{}^+(x,y,z)}$$

$$= Z_0{}^{TE} = \frac{\omega\mu_0}{k_z^{10}}$$

where $k_z^{10} = \sqrt{k_0{}^2 - \left(\frac{\pi}{a}\right)^2}$

Hence

$$Z_0^{TE} = \frac{\omega\mu_0}{\sqrt{k_0^2 - \left(\frac{\pi}{a}\right)^2}}$$

# TEN Model

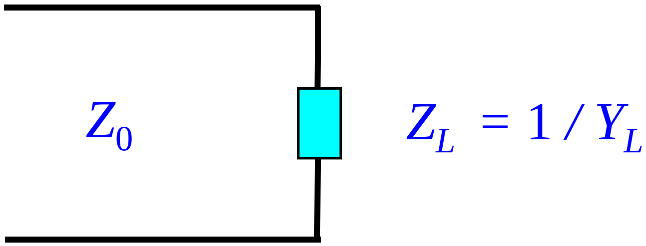
Therefore

$$Z_0 = Z_0^{TE} = \frac{\eta_0}{\sqrt{1-\left(\dfrac{\pi}{k_0 a}\right)^2}}$$

Also,

$$\boxed{Z_L = \frac{V(0)}{I(0)} = \frac{E_{y0}}{-H_{x0}} = Z_s}$$

# TEN Model (cont.)




$$Z_0 = Z_0^{TE} = \frac{\eta_0}{\sqrt{1-\left(\frac{\pi}{k_0 a}\right)^2}}$$

$$Y_L = \frac{1}{Z_L} = \frac{1}{Z_s} = \frac{4}{ab}\left(\frac{P_c^*}{|E_0|^2}\right)$$

# TEN Model (cont.)

$$Y_L = \left( \frac{4}{ab\left|E_0\right|^2} \right) \frac{1}{2} \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_0^{\infty} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 f\left(k_\rho, \overline{\phi}\right) dk_\rho \, d\overline{\phi}$$

where

$$f\left(k_\rho, \overline{\phi}\right) = F^*\left(k_\rho, \overline{\phi}\right) = \left( \frac{\omega \varepsilon_0}{k_z} \right) k_\rho \left[ \sin^2 \overline{\phi} + \left( \frac{k_z}{k_0} \right)^2 \cos^2 \overline{\phi} \right]$$

$$Y_L = G_L + jB_L$$

Visible-space region

Invisible-space region

# TEN Model (cont.)

Hence

$$G_L = \left( \frac{4}{ab\left|E_0\right|^2} \right) \frac{1}{2} \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_0^{k} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 f\left(k_\rho, \overline{\phi}\right) dk_\rho \, d\overline{\phi}$$

$$jB_L = \left( \frac{4}{ab\left|E_0\right|^2} \right) \frac{1}{2} \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_k^{\infty} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 f\left(k_\rho, \overline{\phi}\right) dk_\rho \, d\overline{\phi}$$

where

$$f\left(k_\rho, \overline{\phi}\right) = \left( \frac{\omega \varepsilon_0}{k_z} \right) k_\rho \left[ \sin^2 \overline{\phi} + \left( \frac{k_z}{k_0} \right)^2 \cos^2 \overline{\phi} \right]$$

# TEN Model (cont.)

Explicitly writing out the $k_z$ term, the result can be written as

$$G_L = \left( \frac{4}{ab\left|E_0\right|^2} \right) \frac{1}{2} \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_0^{k_0} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 F_1\left(k_\rho, \overline{\phi}\right) \left( \frac{1}{\sqrt{k_0^2 - k_\rho^2}} \right) dk_\rho \, d\overline{\phi}$$

$$B_L = \left( \frac{4}{ab\left|E_0\right|^2} \right) \frac{1}{2} \left( \frac{1}{2\pi} \right)^2 \int_0^{2\pi} \int_{k_0}^{\infty} \left| \tilde{E}_{y0}(k_\rho, \overline{\phi}) \right|^2 F_1\left(k_\rho, \overline{\phi}\right) \left( \frac{1}{\sqrt{k_\rho^2 - k_0^2}} \right) dk_\rho \, d\overline{\phi}$$

Note: There is a square-root branch-point singularity at $k_\rho = k_0$.

where

$$F_1\left(k_\rho, \overline{\phi}\right) = k_z \; f\left(k_\rho, \overline{\phi}\right) = \left(\omega\varepsilon_0\right) k_\rho \left[ \sin^2 \overline{\phi} + \left( \frac{k_z}{k_0} \right)^2 \cos^2 \overline{\phi} \right]$$