ECE 6341

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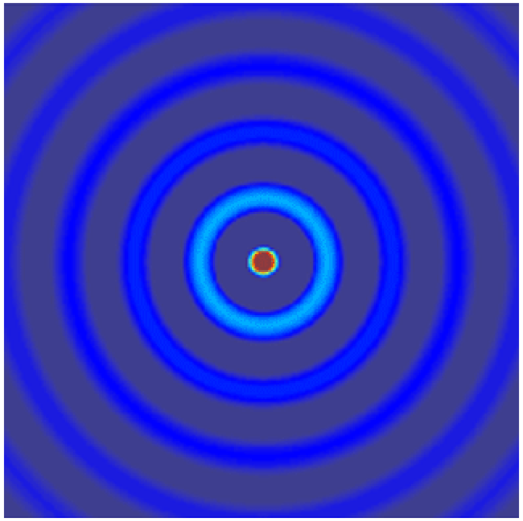
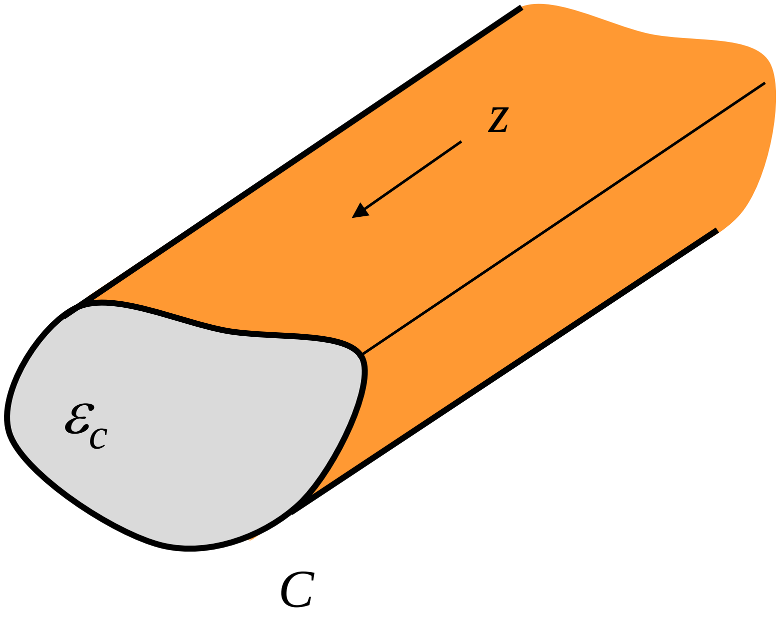
Notes 3



Guided Wave

We consider a waveguiding structure that has a constant cross section.



$$\psi = E(z)\, f(x, y)$$

Guided wave:

The field can change amplitude and phase as z changes, but the shape of the wave remains the same in the (x,y) plane.

Guided Wave (cont.)

The z variation of a guided wave

Assume some component of the guided wave is described by:

$$\psi = E(z)\, f(x, y)$$

where

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

so

$$E(z)\left(\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right) + \frac{\partial^2 E(z)}{\partial z^2} f(x,y) + k^2 E(z)\, f(x,y) = 0$$

Guided Wave (cont.)

$$E(z)\left(\frac{\partial^2 f(x,y)}{\partial x^2}+\frac{\partial^2 f(x,y)}{\partial y^2}\right)+\frac{\partial^2 E(z)}{\partial z^2}f(x,y)+k^2E(z)f(x,y)=0$$

$$\Downarrow$$

$$\frac{\partial^2 E(z)}{\partial z^2}=-k^2E(z)-E(z)\frac{1}{f(x,y)}\left(\frac{\partial^2 f(x,y)}{\partial x^2}+\frac{\partial^2 f(x,y)}{\partial y^2}\right)$$

$$\Downarrow$$

$$\frac{\partial^2 E(z)}{\partial z^2}=-E(z)\left[k^2+\frac{1}{f(x,y)}\left(\frac{\partial^2 f(x,y)}{\partial x^2}+\frac{\partial^2 f(x,y)}{\partial y^2}\right)\right]$$

Guided Wave (cont.)

$$\frac{\partial^2 E(z)}{\partial z^2} = -E(z)\left[k^2 + \frac{1}{f(x,y)}\left(\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right)\right]$$

This has the following form:

$$\frac{\partial^2 E(z)}{\partial z^2} = -E(z)F(x,y)$$

Define: $k_z^2 \equiv F(x,y)$

We then have $$\frac{\partial^2 E(z)}{\partial z^2} = -k_z^2 E(z)$$

so that $$E(z) = Ae^{-jk_z z}$$

Guided Wave (cont.)

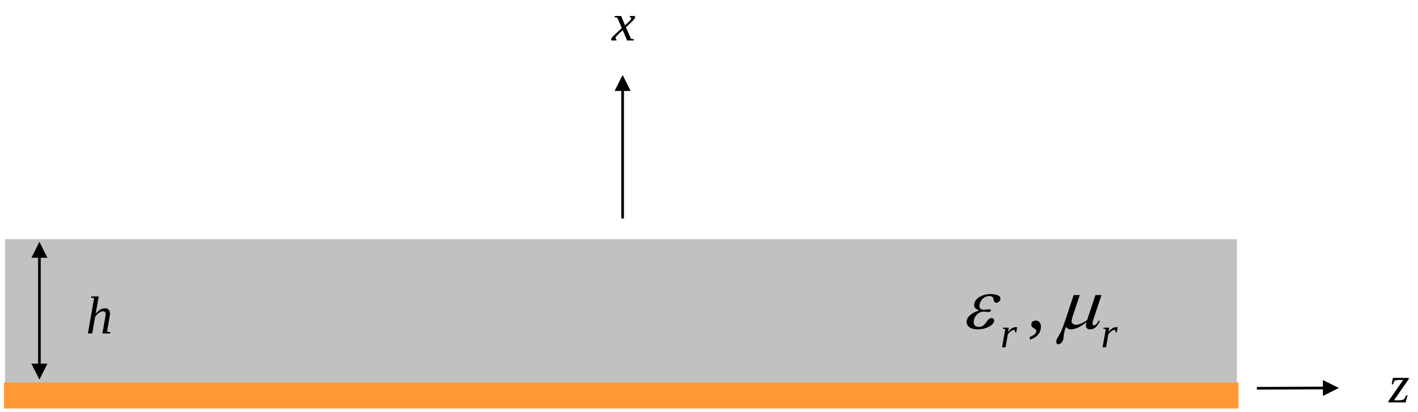
The guided wave field therefore has the following form:

$$\psi = f\left(x, y\right) e^{-jk_z z}$$

The assumption of a guided wave implies that

$$k_z^2 \equiv F\left(x, y\right) = \text{constant}$$

Grounded Dielectric Slab



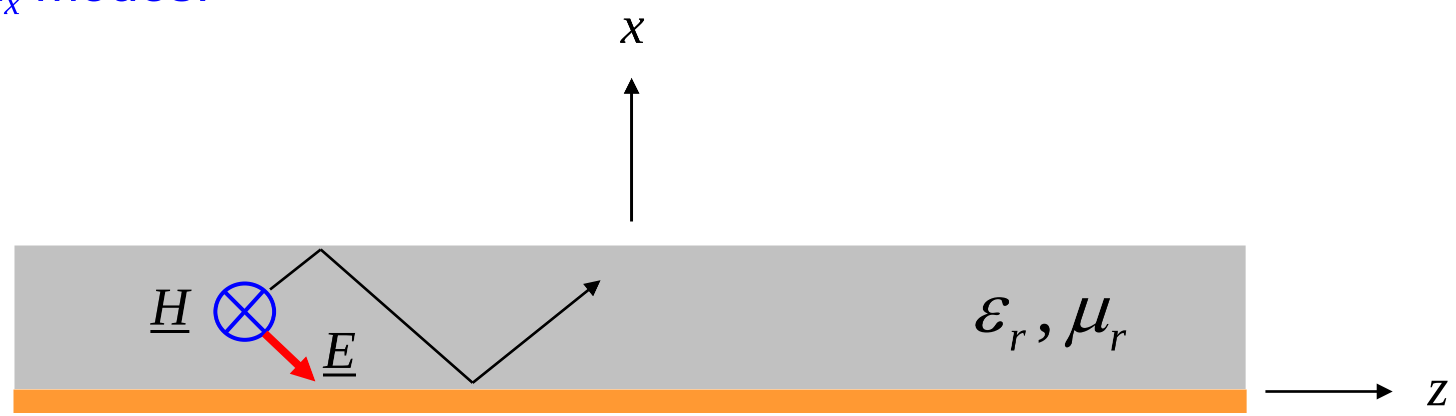
Goal: Determine the modes of propagation and their wavenumbers.

Assumption: There is no variation of the fields in the *y* direction, and propagation is along the *z* direction.

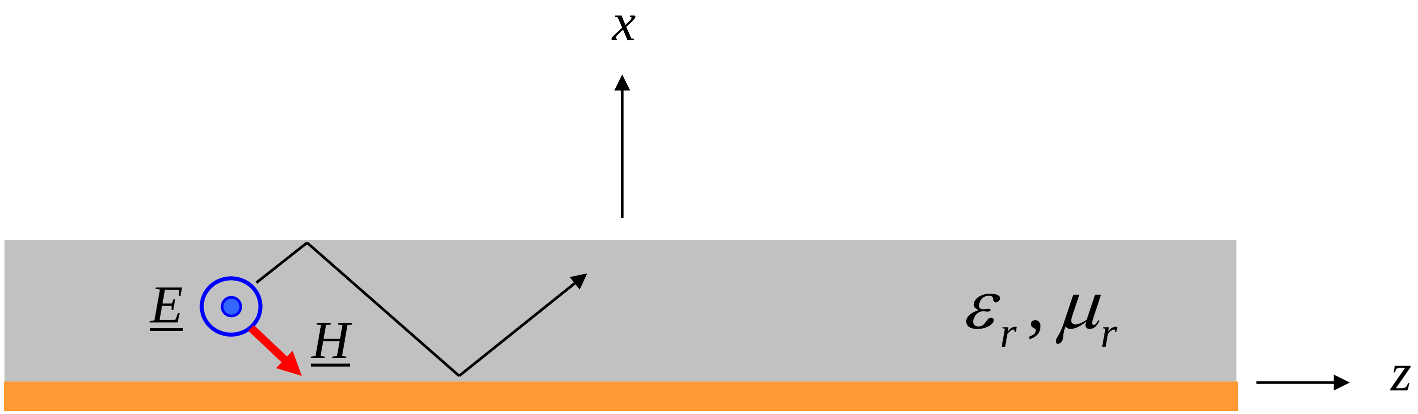
Dielectric Slab

TM$_x$ & TE$_x$ modes:

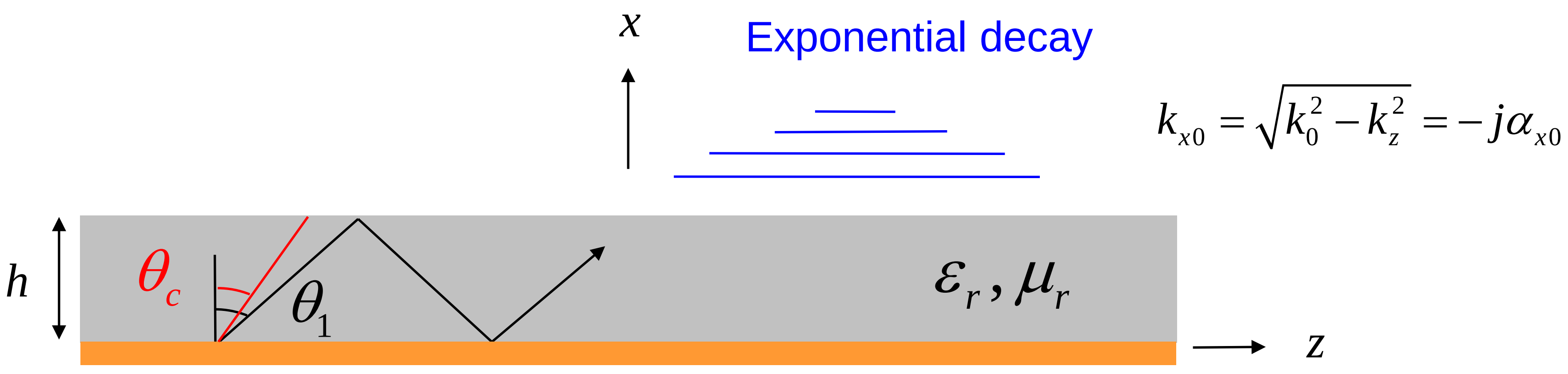
TM$_x$



TE$_x$



Surface Wave



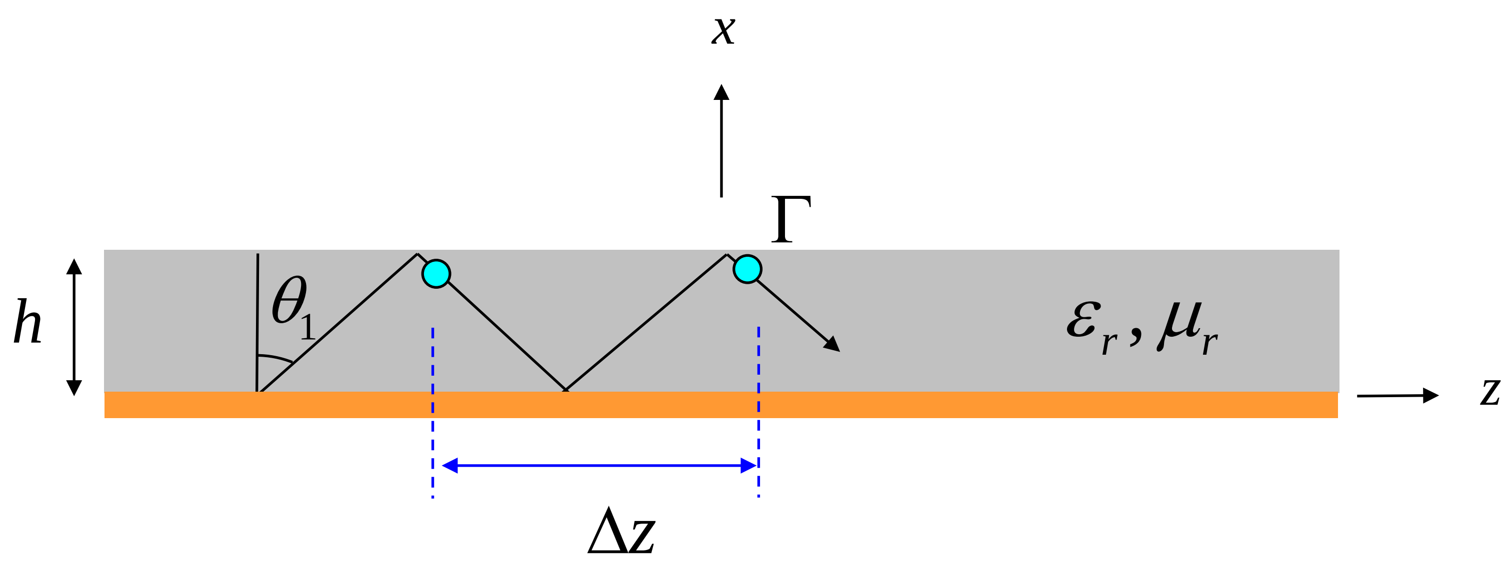
$$\theta_1 > \theta_c$$

The internal angle is greater than the critical angle, so there is exponential decay in the air region.

$$k_z = k_1 \sin\theta_1 > k_0$$

The surface wave is a "slow wave".

Surface Wave



The wave must also satisfy a “consistency condition”:

$$2k_1 h \sec\theta_1 + \pi + \angle\Gamma\left(\theta_1\right) = k_z \Delta z + 2\pi n$$

or

$$2k_1 h \sec\theta_1 + \pi + \angle\Gamma\left(\theta_1\right) = k_1 \sin\theta_1 \left(2h\tan\theta_1\right) + 2\pi n$$

This forces the angle θ_1 to be a discrete value, depending on *n*.

TM$_x$ Solution

Assume TM$_x$ $\quad \underline{A} = \underline{\hat{x}}\, A_x(x, z)$

B.C. :
(see TM$_x$-TE$_x$ tables in Appendix)

$$\left. \frac{\partial A_x}{\partial x} = 0 \right|_{x=0} \qquad (E_y = E_z = 0)$$

Assume $\quad A_x = e^{-jk_z z} f\left(x\right)$

$$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} + k^2 A_x = 0 \quad \Longrightarrow \quad \frac{\partial^2 A_x}{\partial x^2} + \left(k^2 - k_z^2\right) A_x = 0$$

TM$_x$ Solution (cont.)

Hence $$\frac{\partial^2 f(x)}{\partial x^2} + \left(k^2 - k_z^2\right) f(x) = 0$$

Denote $$\begin{cases} k_{x0} = \left(k_0^2 - k_z^2\right)^{1/2} \\ k_{x1} = \left(k_1^2 - k_z^2\right)^{1/2} \end{cases}$$

$x \geq h$ $$\frac{\partial^2 f(x)}{\partial x^2} + k_{x0}^2 f(x) = 0$$

$x \leq h$ $$\frac{\partial^2 f(x)}{\partial x^2} + k_{x1}^2 f(x) = 0$$

TM$_x$ Solution (cont.)

Applying boundary conditions at the ground plane,

$$x \le h \qquad A_{x1} = e^{-jk_z z}\cos(k_{x1}x)$$

$$x \ge h \qquad A_{x0} = Ae^{-jk_z z}e^{-jk_{x0}x}$$

Note: Since the surface wave is a slow wave, we have:

$$k_{x0} = \left(k_0^2 - k_z^2\right)^{1/2} \quad \Rightarrow \quad k_{x0} = -j\alpha_{x0}$$

$$\alpha_{x0} = \sqrt{k_z^{\,2} - k_0^{\,2}} \ge 0$$

Boundary Conditions

BC 1)

$$H_{y0} = H_{y1} \quad @\, x = h$$

$$\frac{1}{\mu_0}\frac{\partial A_{x0}}{\partial z} = \frac{1}{\mu_1}\frac{\partial A_{x1}}{\partial z}$$

$$\mu_r A_{x0} = A_{x1}$$

BC 2)

$$E_{z0} = E_{z1} \quad @\, x = h$$

$$\frac{1}{\mu_0 \varepsilon_0}\frac{\partial A_{x0}}{\partial x} = \frac{1}{\mu_1 \varepsilon_1}\frac{\partial A_{x1}}{\partial x}$$

$$\mu_r \varepsilon_r \frac{\partial A_{x0}}{\partial x} = \frac{\partial A_{x1}}{\partial x}$$

TM$_x$ Solution (cont.)

Hence we have:

$$x \le h \qquad A_{x1} = e^{-jk_z z}\cos(k_{x1}x)$$

$$x \ge h \qquad A_{x0} = Ae^{-jk_z z}e^{-\alpha_{x0}x}$$

$$@\, x = h$$

$$\mu_r A_{x0} = A_{x1}$$ BC #1

$$\mu_r \varepsilon_r \frac{\partial A_{x0}}{\partial x} = \frac{\partial A_{x1}}{\partial x}$$ BC #2

$$\mu_r\, Ae^{-\alpha_{x0}h} = \cos(k_{x1}h)$$

$$\mu_r\, \varepsilon_r \left(-\alpha_{x0}Ae^{-\alpha_{x0}h}\right) = (-k_{x1})\,\sin(k_{x1}h)$$

Boundary Conditions (cont.)

The two equations again:

$$\mu_r \, A e^{-\alpha_{x0} h} = \cos(k_{x1} h)$$

$$\mu_r \, \varepsilon_r \left(-\alpha_{x0} A e^{-\alpha_{x0} h} \right) = (-k_{x1}) \, \sin(k_{x1} h)$$

Divide second by first:

$$-\alpha_{x0} \varepsilon_r = (-k_{x1}) \tan(k_{x1} h)$$

or

$$\alpha_{x0} \, \varepsilon_r = k_{x1} \tan(k_{x1} h)$$

Final Result: TM_x

This may be written as:

$$\varepsilon_r \sqrt{k_z^2 - k_0^2} = \left(k_1^2 - k_z^2\right)^{1/2} \tan\left[h\left(k_1^2 - k_z^2\right)^{1/2}\right]$$

This is a transcendental equation for the unknown wavenumber k_z.

Note: The choice of square root for k_{x1} is not important, but it is for k_{x0}:

$$k_{x0} = -j\sqrt{k_z^2 - k_0^2}$$

$$\sqrt{k_z^2 - k_0^2} > 0$$

Appendix

TM$_x$

$$\psi = A_x$$

$$E_x = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + k^2\right)\psi \qquad H_x = 0$$

$$E_y = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial x\partial y} \qquad H_y = \frac{1}{\mu}\frac{\partial\psi}{\partial z}$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial x\partial z} \qquad H_z = -\frac{1}{\mu}\frac{\partial\psi}{\partial y}$$

Note: There is a factor μ difference with the Harrington text.

Appendix (cont.)

TE$_x$

$$\psi = F_x$$

$$H_x = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + k^2\right)\psi$$

$$E_x = 0$$

$$H_y = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial x \partial y}$$

$$E_y = -\frac{1}{\varepsilon}\frac{\partial\psi}{\partial z}$$

$$H_z = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial x \partial z}$$

$$E_z = \frac{1}{\varepsilon}\frac{\partial\psi}{\partial y}$$

Note: There is a factor μ difference with the Harrington text.