ECE 6341

Spring 2016

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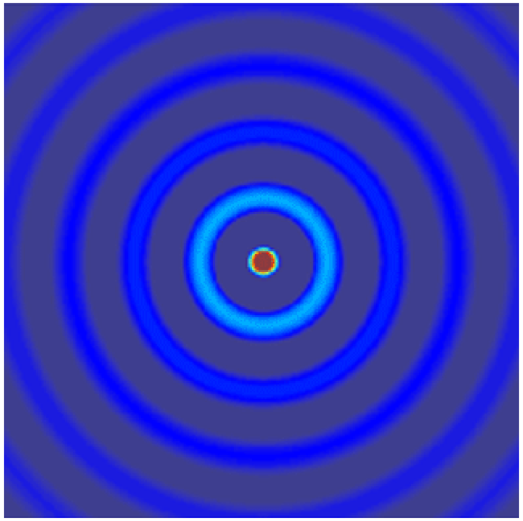
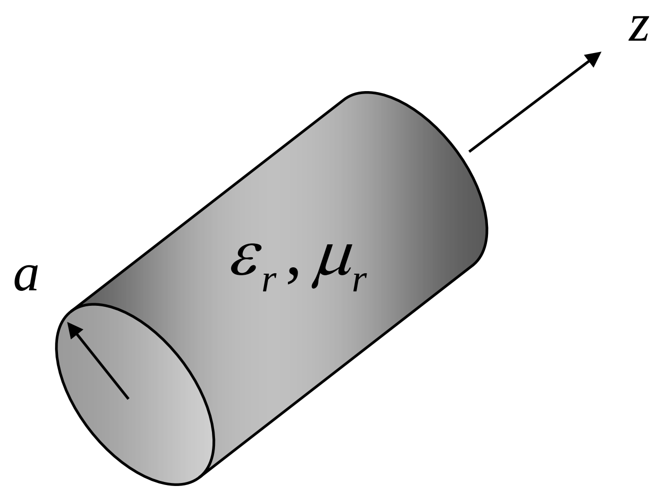
ECE Dept.

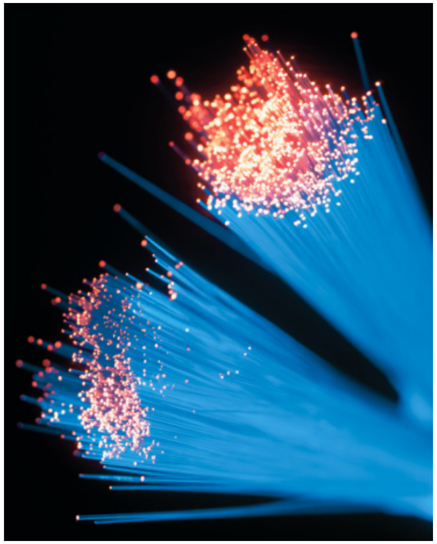
Notes 10



Dielectric Rod



This serves as a model for a fiber-optic guide.



Fiber Optic Guides

Two types of fiber-optic guides:

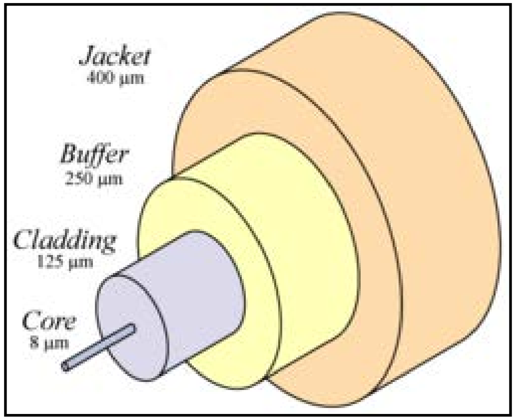
1) Single-mode fiber

This fiber carries a single mode (HE_{11}). This requires the fiber diameter to be on the order of a wavelength. It has less loss, dispersion, and signal distortion. It is often used for long-distances (e.g., greater than 1 km). The diameter is typically 8 μm. It requires more expensive electro-optic equipment.

2) Multi-mode fiber

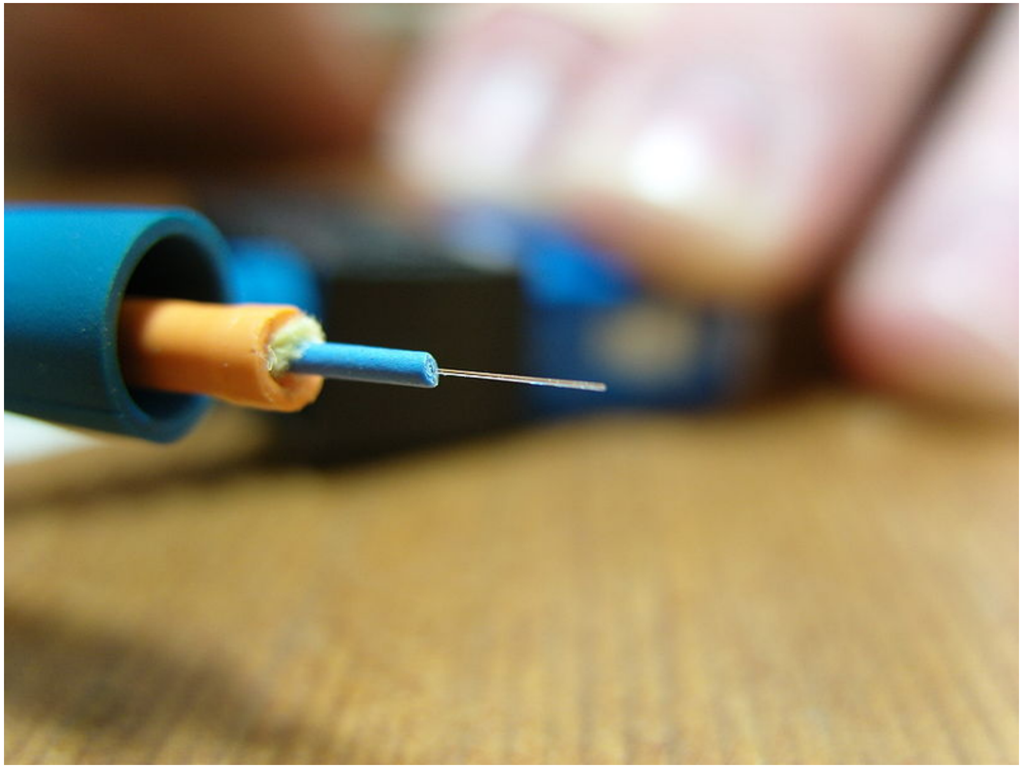
This fiber has a diameter that is large relative to a wavelength (e.g., 10 wavelengths). It operates on the principle of total internal reflection (critical-angle effect). It can handle more power than the single-mode fiber, but has more dispersion. The diameter is typically 50 μm. It is often used for LANs, etc.

Single-mode Fiber



It usually has a yellow jacket.

Multi-mode Fiber



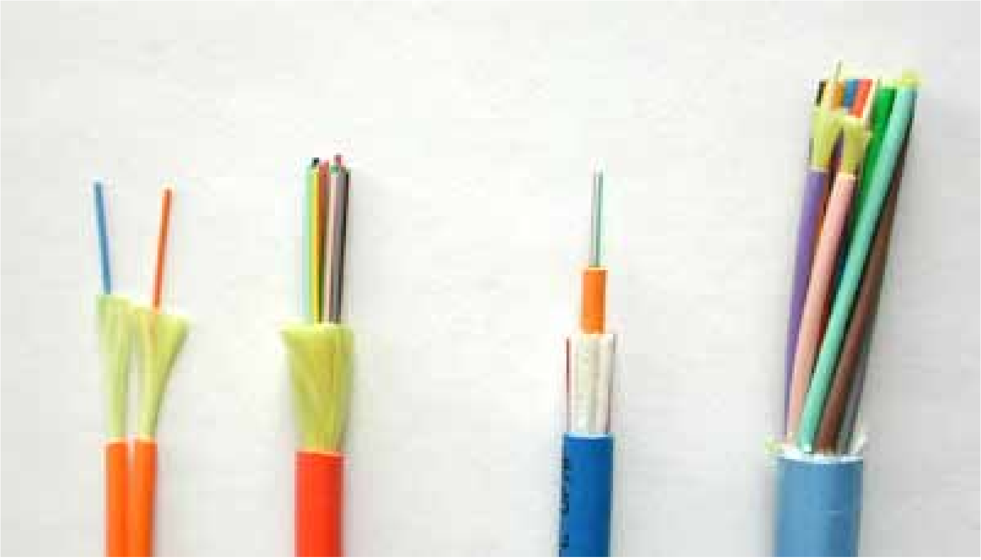
A 1.25 Gbit/s multi-mode fiber

http://en.wikipedia.org/wiki/Multi-mode_fiber

Fiber-optic Cable

Fibers (single-mode or multi-mode) may be bundled together into a “fiber optic cable” that has one or more fibers.

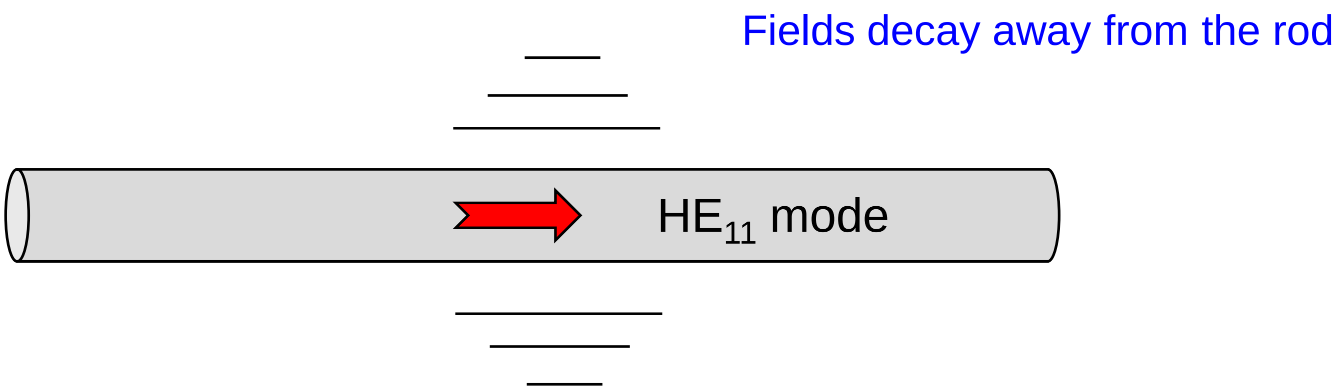
Simplex: single fiber in a cable
Duplex: two fibers in a cable



Cable Types: (L to R): Zipcord, Distribution, Loose Tube, Breakout

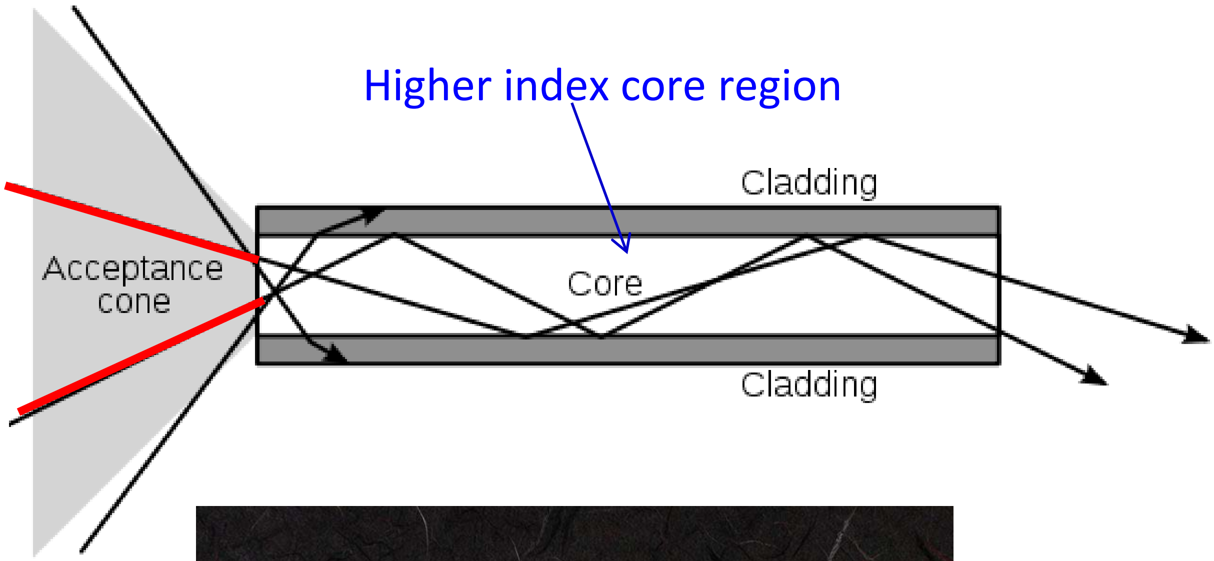
http://www.thefoa.org/tech/ref/basic/cable.html

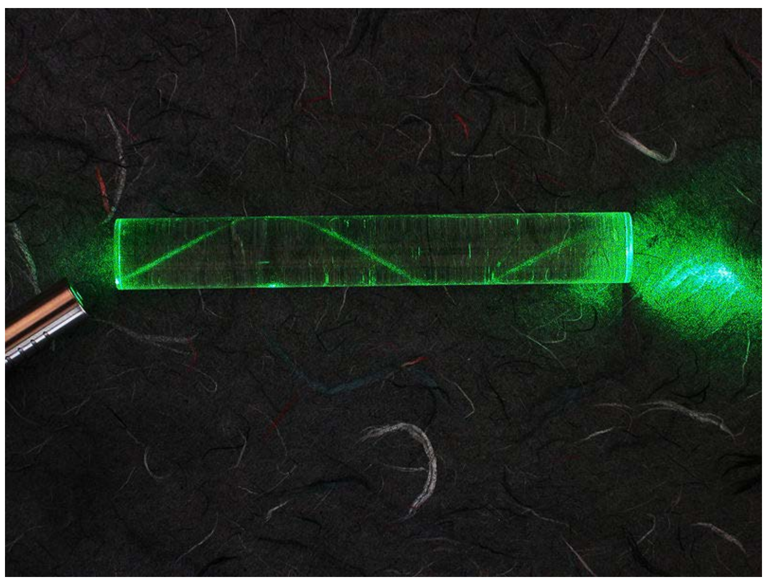
Single-mode Fiber: Operation



A single mode (HE_{11} mode) propagates on the dielectric rod. The mode is similar to the TM_0 mode on a grounded slab. It has a zero cutoff frequency.

Multi-mode Fiber: Operation

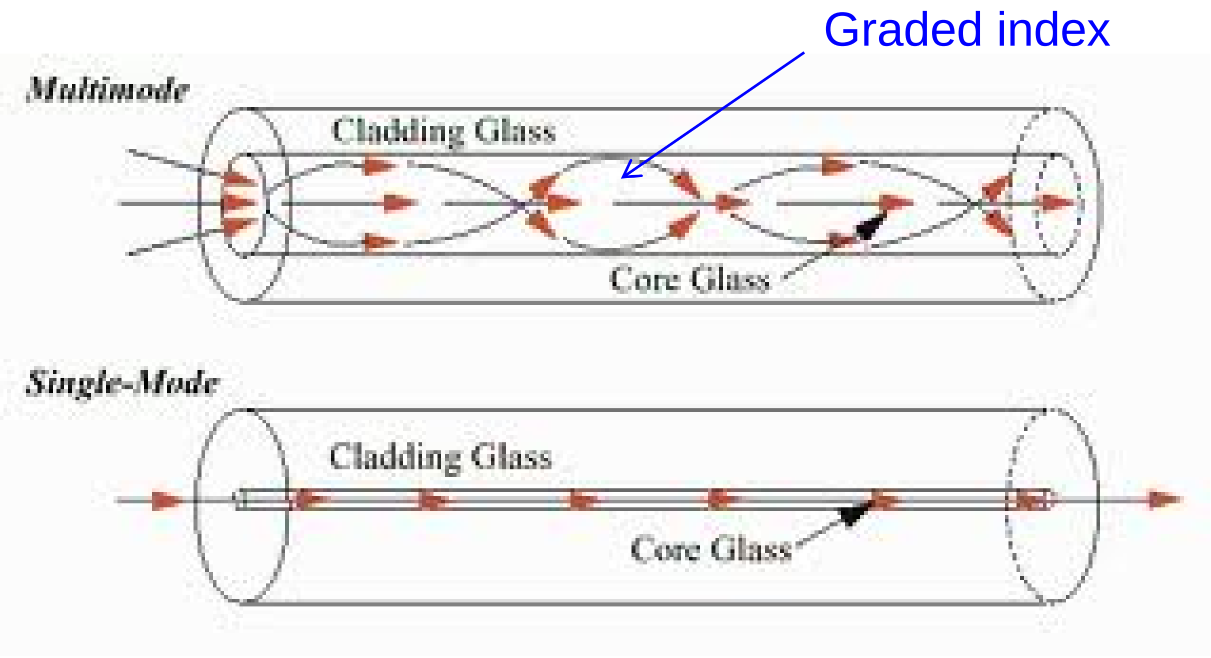




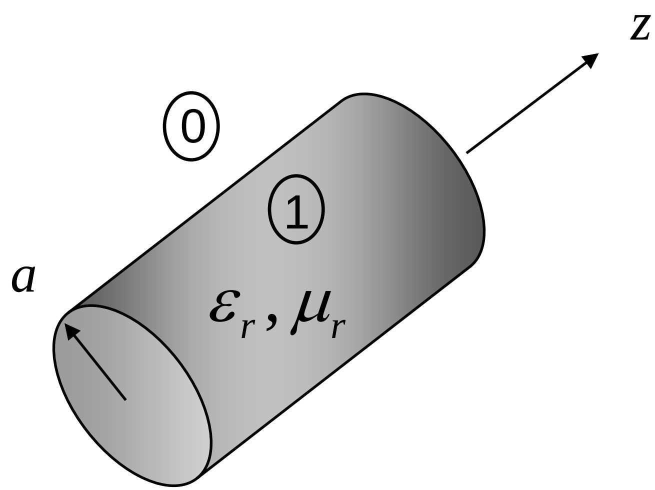
A laser bouncing down an acrylic rod, illustrating the total internal reflection of light in a multi-mode optical fiber

http://en.wikipedia.org/wiki/Optical_fiber

Single-mode vs. Multi-mode Fibers



Dielectric Rod



Modes are hybrid unless $\dfrac{\partial}{\partial \phi} = 0 \quad (n = 0)$

Note:

We can have TE_{0p}, TM_{0p} modes

For example, assume TM_z: $\psi = A_z$

$$H_\rho = \frac{1}{\mu \rho} \frac{\partial \psi}{\partial \phi}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon\rho} \frac{\partial \psi}{\partial \phi \partial z} = \frac{-jk_z}{j\omega\mu\varepsilon\rho} \frac{\partial \psi}{\partial \phi}$$

Dielectric Rod (cont.)

At $\rho = a$:

$$\mu_1 H_{\rho 1} = \mu_0 H_{\rho 0}$$

$$E_{\phi 1} = E_{\phi 0}$$

so

$$\frac{\partial \psi_1}{\partial \phi} = \frac{\partial \psi_0}{\partial \phi}$$

$$\frac{1}{\mu_1 \varepsilon_1} \frac{\partial \psi_1}{\partial \phi} = \frac{1}{\mu_0 \varepsilon_0} \frac{\partial \psi_0}{\partial \phi}$$

Hence, for $n > 0$ we have $\mu_1 \varepsilon_1 = \mu_0 \varepsilon_0$ (Not true in general!)

Dielectric Rod (cont.)

Representation of potentials inside the rod:

$$A_{z1} = A\, J_n\left(k_{\rho 1}\rho\right)\sin\left(n\phi\right)e^{-jk_z z}$$

$$F_{z1} = B\, J_n\left(k_{\rho 1}\rho\right)\cos\left(n\phi\right)e^{-jk_z z}$$

$\rho < a$:

where $k_{\rho 1}^2 = k_1^2 - k_z^2$ (k_z is unknown)

Dielectric Rod (cont.)

To see choice of sin/cos, examine the field components (for example E_ρ):

$$E_\rho = -\frac{1}{\varepsilon\rho}\frac{\partial F_z}{\partial\phi} + \frac{-k_z}{\omega\mu\varepsilon}\frac{\partial A_z}{\partial\rho}$$

The field E_ρ is assumed (arbitrarily) to vary as $\sin(n\phi)$ for our choice of potentials.

$$\Rightarrow \quad F_z \propto \cos(n\phi) \qquad A_z \propto \sin(n\phi)$$

We then have for the field components:

$$E_\rho \propto \sin(n\phi) \qquad E_\phi \propto \cos(n\phi) \qquad E_z \propto \sin(n\phi)$$

$$H_\rho \propto \cos(n\phi) \qquad H_\phi \propto \sin(n\phi) \qquad H_z \propto \cos(n\phi)$$

Dielectric Rod (cont.)

Representation of potentials outside the rod: $\rho > a$:

Use $$H_n^{(2)}(k_{\rho 0}\rho) = H_n^{(2)}(-j\alpha_{\rho 0}\rho)$$

where

$$k_{\rho 0} = \left(k_0^2 - k_z^2\right)^{1/2} = -j\alpha_{\rho 0}$$

$$\alpha_{\rho 0} = \sqrt{k_z^2 - k_0^2}$$

Note: $\alpha_{\rho 0}$ is interpreted as a positive real number in order to have decay radially in the air region, for a bound (non-leaky) mode.

Dielectric Rod (cont.)

Useful identity:

$$H_n^{(2)}(-jx) = (-1)^{n+1} H_n^{(1)}(+jx)$$

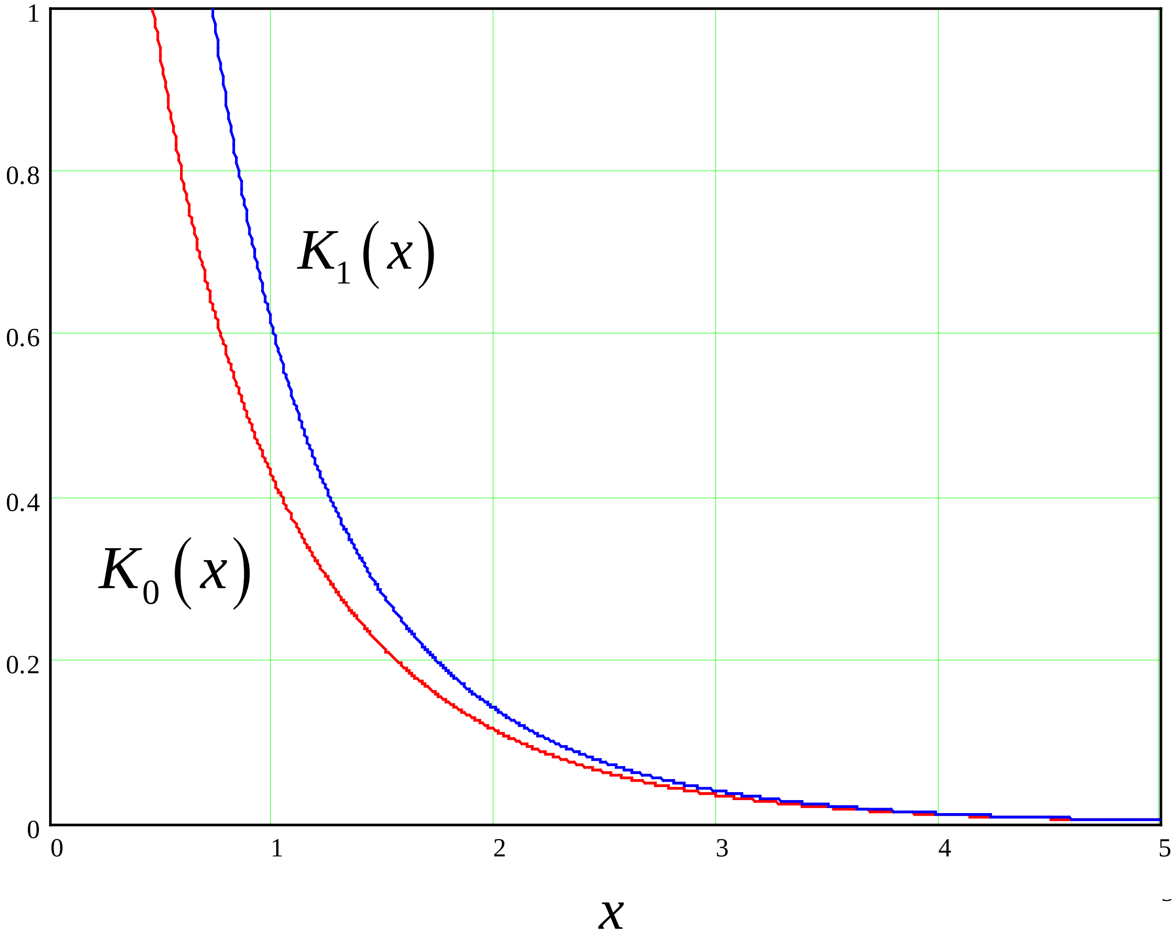
Another useful identity:

$$K_n(x) \equiv \frac{\pi}{2} j^{n+1} H_n^{(1)}(jx)$$

$K_n(x)$ = Modified Bessel function of the second kind.

Dielectric Rod (cont.)

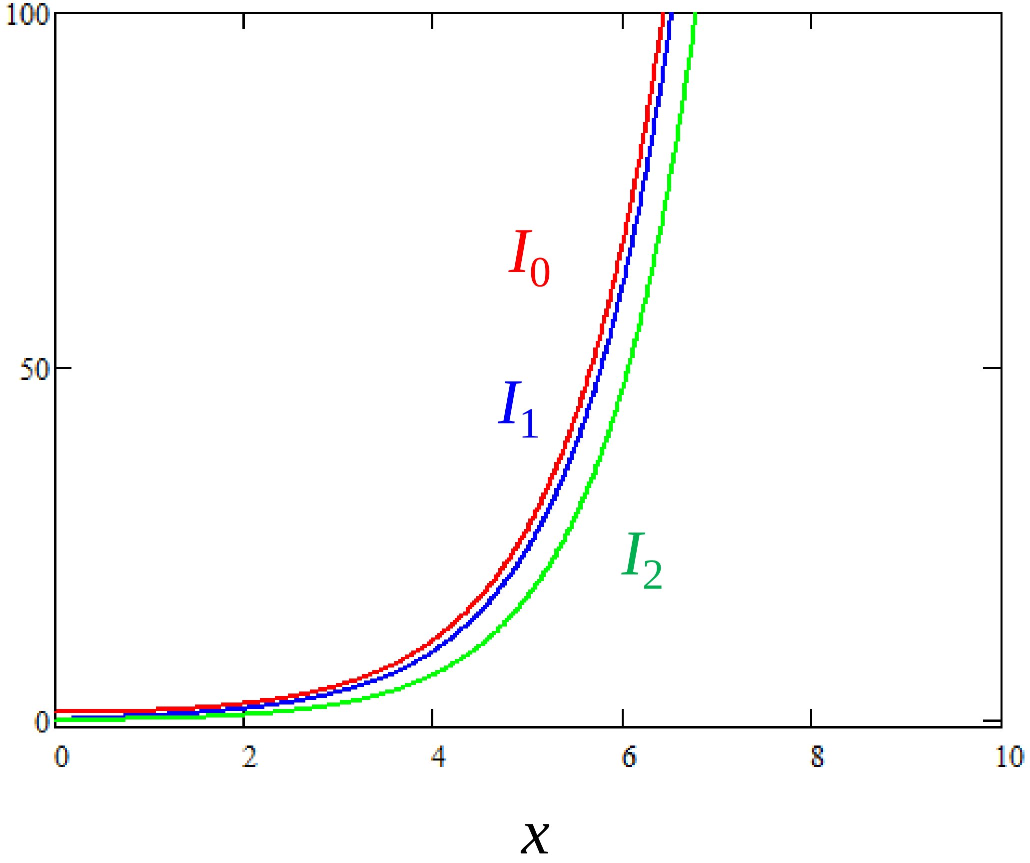
Modified Bessel function of the second kind



Dielectric Rod (cont.)

The modified Bessel function of the *first kind* grows exponentially, so we don't want this one.

$$I_n(x) \equiv (-i)^n J_n(ix)$$



Dielectric Rod (cont.)

Hence, we choose

$$A_{z0} = C K_n(\alpha_{\rho 0}\rho)\sin(n\phi)\, e^{-jk_z z}$$

$$F_{z0} = D K_n(\alpha_{\rho 0}\rho)\cos(n\phi)\, e^{-jk_z z}$$

Dielectric Rod (cont.)

Summary of potentials:

$$A_{z1} = A\,J_n\left(k_{\rho 1}\rho\right)\sin\left(n\phi\right)e^{-jk_z z}$$

$$F_{z1} = B\,J_n\left(k_{\rho 1}\rho\right)\cos\left(n\phi\right)e^{-jk_z z}$$

$$A_{z0} = CK_n(\alpha_{\rho 0}\rho)\sin(n\phi)\,e^{-jk_z z}$$

$$F_{z0} = DK_n(\alpha_{\rho 0}\rho)\cos(n\phi)\,e^{-jk_z z}$$

Dielectric Rod (cont.)

Match E_z, H_z, E_ϕ, H_ϕ at $\rho = a$:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: $E_{z1} = E_{z0}$ $\quad E_z = \dfrac{1}{j\omega\mu\varepsilon}\left(\dfrac{\partial^2}{\partial z^2} + k^2\right) A_z$

$$\Rightarrow \quad \frac{1}{j\omega\mu_1\varepsilon_1}\left(k_1^2 - k_z^2\right) A J_n\left(k_{\rho 1} a\right) = \frac{1}{j\omega\mu_0\varepsilon_0}\left(k_0^2 - k_z^2\right) C K_n\left(\alpha_{\rho 0} a\right)$$

so

$$M_{11} = \frac{1}{j\omega\mu_1\varepsilon_1}\left(k_1^2 - k_z^2\right) J_n\left(k_{\rho 1} a\right) \qquad M_{13} = \frac{-1}{j\omega\mu_0\varepsilon_0}\left(k_0^2 - k_z^2\right) K_n\left(\alpha_{\rho 0} a\right) \qquad M_{12} = M_{14} = 0$$

Dielectric Rod (cont.)

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To have a non-trivial solution, we require that

$$\det\left[M(k_z, \omega) \right] = 0$$

k_z = unknown (for a given frequency ω)

Dielectric Rod (cont.)

Note:
This is an open structure, so cutoff means the boundary between proper and improper behavior ($k_z = k_0$).

Cutoff frequency:

Set $$k_z = k_0 = \omega_c \sqrt{\mu_0 \varepsilon_0}$$

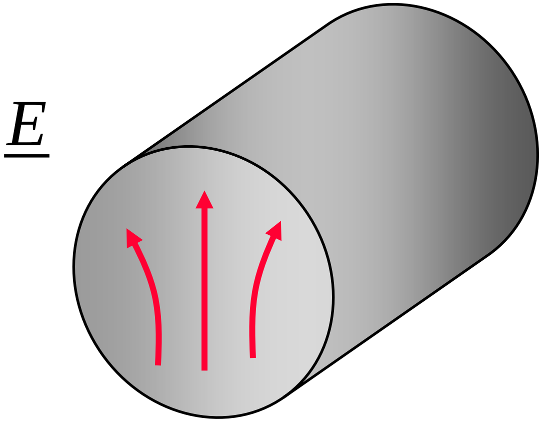
Then

$$\det\left[M\left(\omega_c \sqrt{\mu_0 \varepsilon_0}, \omega_c\right) \right] = 0$$

The unknown is now ω_c.

Dielectric Rod (cont.)

<u>Dominant mode</u> (lowest cutoff frequency): HE_{11} $(f_c = 0)$



Note: The notation HE means that the mode is hybrid, and has both E_z and H_z, although H_z is stronger. (For an EH mode, E_z would be stronger.)

The <u>field shape</u> is somewhat similar to the TE_{11} circular waveguide mode.

The <u>physical properties</u> of the fields are similar to those of the TM_0 surface wave on a slab (For example, at low frequency the field is loosely bound to the rod.)

Dielectric Rod (cont.)

When will the next mode be at cutoff?

This determines the upper frequency limit for the single-mode fiber.

The next mode (i.e., with the next lowest cutoff frequency) is the TM_{01} mode.

(Recall: For $n = 0$, the modes are TM_z and TE_z.)

Cutoff: $$k_z = k_0$$

$$\Rightarrow \quad k_\rho = k_{\rho 0} = 0 \text{ (in air region)}$$

$$\psi = A_z$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon\rho}\frac{\partial^2\psi}{\partial\phi\partial z}$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi = \frac{1}{j\omega\mu\varepsilon}k_\rho^2\,\psi$$

The tangential field in the air region at the boundary becomes zero at the cutoff frequency: the rod acts like a circular waveguide with a PEC wall.

Dielectric Rod (cont.)

TM_{01} mode at cutoff: $$k_{\rho 1} a = x_{01} = 2.405$$

$$\Rightarrow \quad \sqrt{k_1^2 - k_0^2}\, a = 2.405$$

or

$$k_0 a \sqrt{n_1^2 - 1} = 2.405$$

Hence, to have <u>only</u> the HE_{11} mode, we have the following frequency restriction:

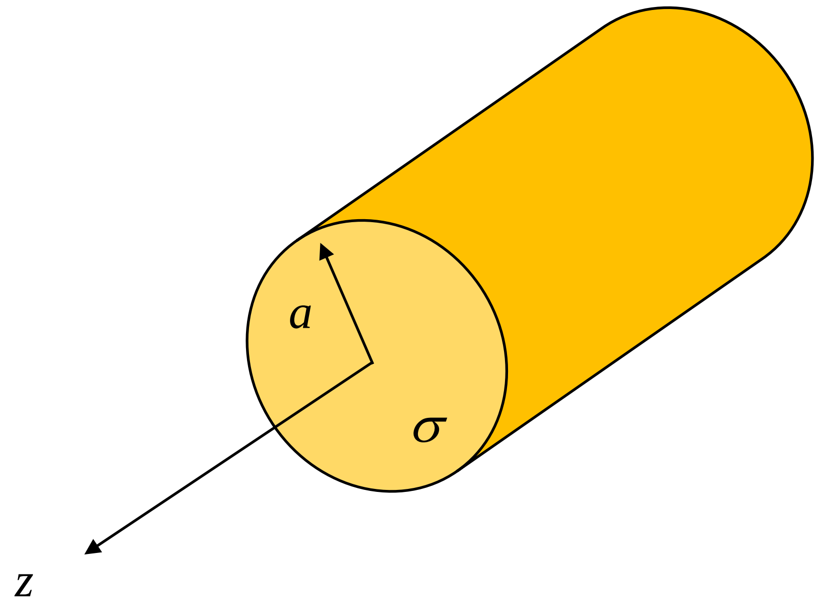
$$k_0 a < \frac{2.405}{\sqrt{n_1^2 - 1}}$$

n_1 = index of refraction of rod

Impedance of Wire

A round wire made of conducting material is examined.

Goal: Determine the impedance per unit length (in the z direction).



The wire has a conductivity of σ.

We neglect the z variation of the fields inside the wire.
(The wire is short compared with a wavelength.)

Impedance of Wire (cont.)

Inside the wire:

$$E_z = AJ_0(k\rho)$$

(The field must be finite on the z axis.)

$$k = \omega\sqrt{\mu\varepsilon_c}$$

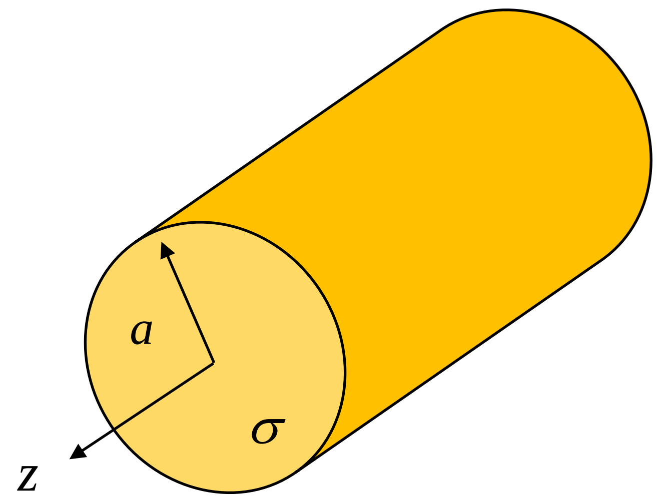
$$= \omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$\approx \omega\sqrt{\mu\varepsilon\left(-j\frac{\sigma}{\omega\varepsilon}\right)}$$

$$= \sqrt{-j\omega\mu\sigma}$$

$$= \sqrt{\omega\mu\sigma}\left(e^{-j\pi/4}\right)$$

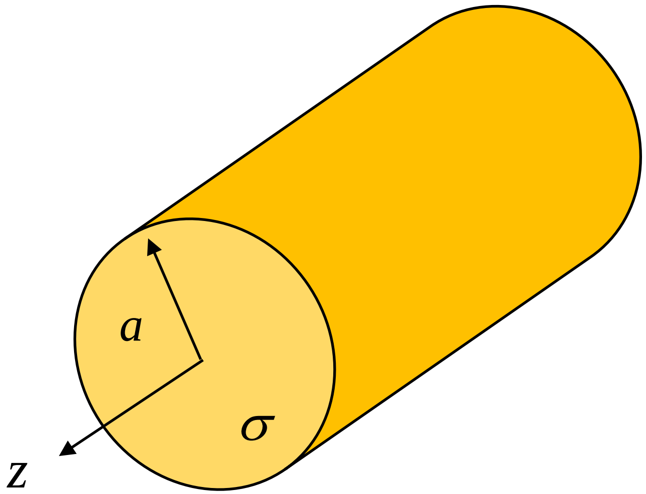
$$= \sqrt{2}\sqrt{\frac{\omega\mu\sigma}{2}}\left(e^{-j\pi/4}\right)$$



Impedance of Wire (cont.)

Hence, we have

$$E_z = AJ_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{-j\pi/4}\right)$$



where

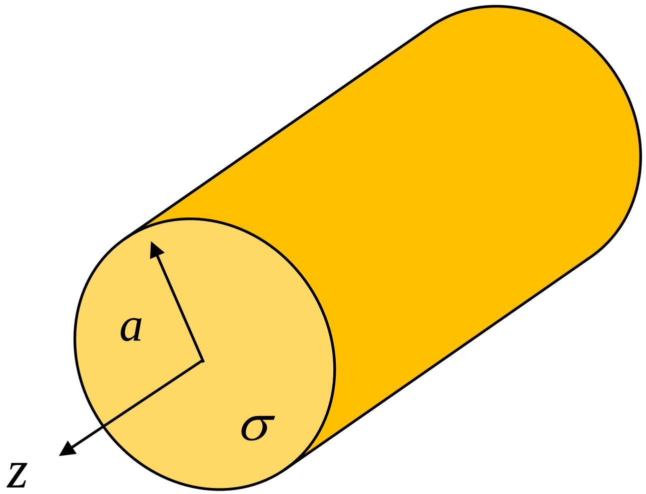
$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$ (skin depth)

We can also write the field as

$$E_z = AJ_0\left(-\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right) = AJ_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right)$$

Impedance of Wire (cont.)

$$E_z = AJ_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right)$$



Definition of Kelvin functions:

$$\mathrm{Ber}_\upsilon(x) \equiv \mathrm{Re}\left(J_\upsilon\left(xe^{j3\pi/4}\right)\right)$$

$$\mathrm{Bei}_\upsilon(x) \equiv \mathrm{Im}\left(J_\upsilon\left(xe^{j3\pi/4}\right)\right)$$

Therefore, we can write

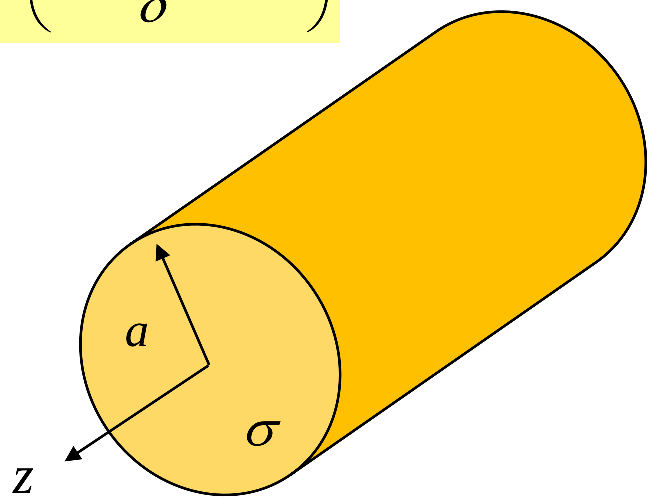
$$E_z = A\left(\mathrm{Ber}_0\left(\sqrt{2}\,\frac{\rho}{\delta}\right) + j\mathrm{Bei}_0\left(\sqrt{2}\,\frac{\rho}{\delta}\right)\right)$$

Impedance of Wire (cont.)

The current flowing in the wire is

$$
\begin{aligned}
I &= \int_S J_z \, dS \\
&= \int_0^{2\pi} \int_0^a J_z \, \rho \, d\rho \, d\phi \\
&= 2\pi \int_0^a J_z \, \rho \, d\rho \\
&= 2\pi\sigma \int_0^a E_z \, \rho \, d\rho
\end{aligned}
$$

$$E_z = A J_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right)$$

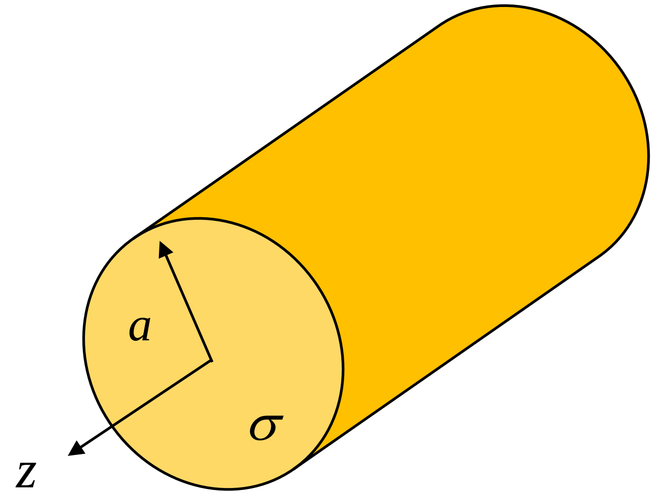


Hence $$I = 2\pi\sigma A \int_0^a J_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right)\rho \, d\rho$$

Impedance of Wire (cont.)

The internal impedance per unit length is defined as:

$$Z_l \equiv \frac{E_z(a)}{I}$$



Hence,

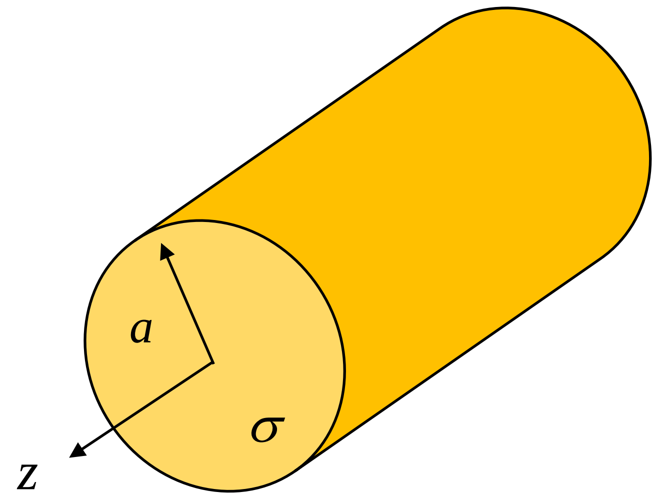
$$Z_l = \frac{A J_0\left(\sqrt{2}\,\frac{a}{\delta}\,e^{j3\pi/4}\right)}{2\pi\sigma A \int_0^a J_0\left(\sqrt{2}\,\frac{\rho}{\delta}\,e^{j3\pi/4}\right)\rho\, d\rho}$$

Note: The internal impedance accounts for the internal stored energy and power dissipation.

Impedance of Wire (cont.)

We also have the following helpful integration identity:

$$\int J_0(x)\,x\,dx = xJ_1(x)$$



Hence

$$\int_0^a J_0\left(\sqrt{2}\,\frac{\rho}{\delta}e^{j3\pi/4}\right)\rho\, d\rho = \left(\frac{\delta}{\sqrt{2}}e^{-j3\pi/4}\right)^2 \int_0^L J_0(x)\,x\,dx$$

$$= xJ_1(x)\Big|_0^L$$

$$= LJ_1(L)$$

Use:

$$x = \sqrt{2}\,\frac{\rho}{\delta}e^{j3\pi/4}$$

$$dx = d\rho\left(\frac{\sqrt{2}}{\delta}\right)e^{j3\pi/4}$$

where $L \equiv \sqrt{2}\,\dfrac{a}{\delta}e^{j3\pi/4}$

Impedance of Wire (cont.)

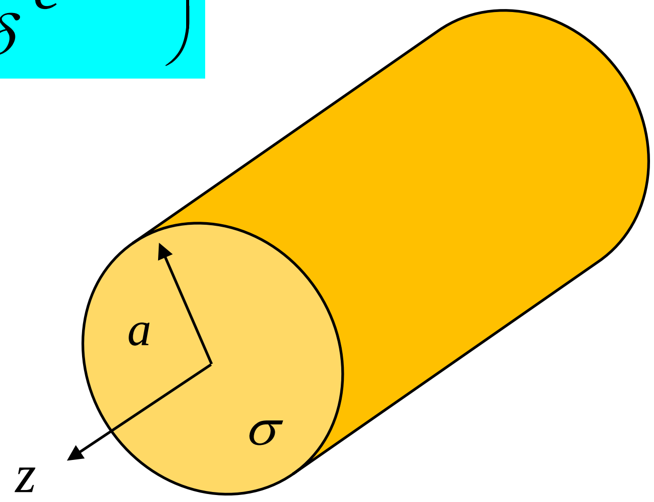
Hence, we have

$$Z_l = \frac{J_0\left(\sqrt{2}\,\dfrac{a}{\delta}e^{j3\pi/4}\right)}{2\pi\sigma\left(\dfrac{\delta}{\sqrt{2}}e^{-j3\pi/4}\right)^2\left(\sqrt{2}\,\dfrac{a}{\delta}e^{j3\pi/4}\right)J_1\left(\sqrt{2}\,\dfrac{a}{\delta}e^{j3\pi/4}\right)}$$

where

$$J_0\left(\frac{a}{\delta}e^{j3\pi/4}\right) = \mathrm{Ber}_0\left(\frac{a}{\delta}\right) + j\mathrm{Bei}_0\left(\frac{a}{\delta}\right)$$

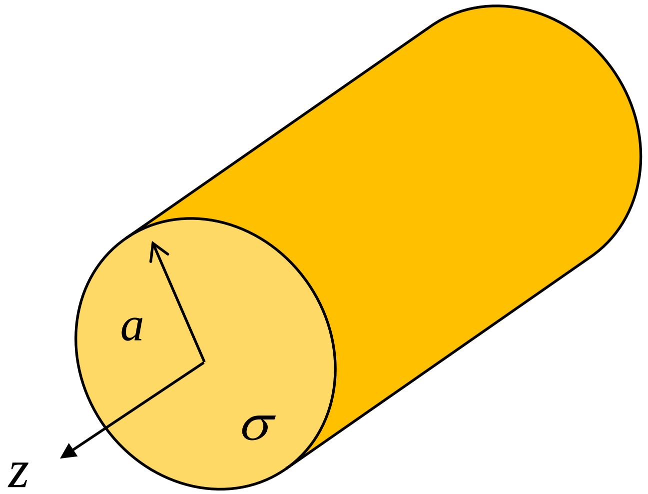
$$J_1\left(\frac{a}{\delta}e^{j3\pi/4}\right) = \mathrm{Ber}_1\left(\frac{a}{\delta}\right) + j\mathrm{Bei}_1\left(\frac{a}{\delta}\right)$$



Impedance of Wire (cont.)

At low frequency ($a << \delta$):

$$Z_l \approx \frac{1}{\sigma\left(\pi a^2\right)} + j\omega\left(\frac{\mu_0 \mu_r}{8\pi}\right)$$



At high frequency ($a >> \delta$):

$$Z_l \approx \frac{Z_s}{2\pi a}$$

where

$$Z_s = R_s\left(1 + j\right)$$

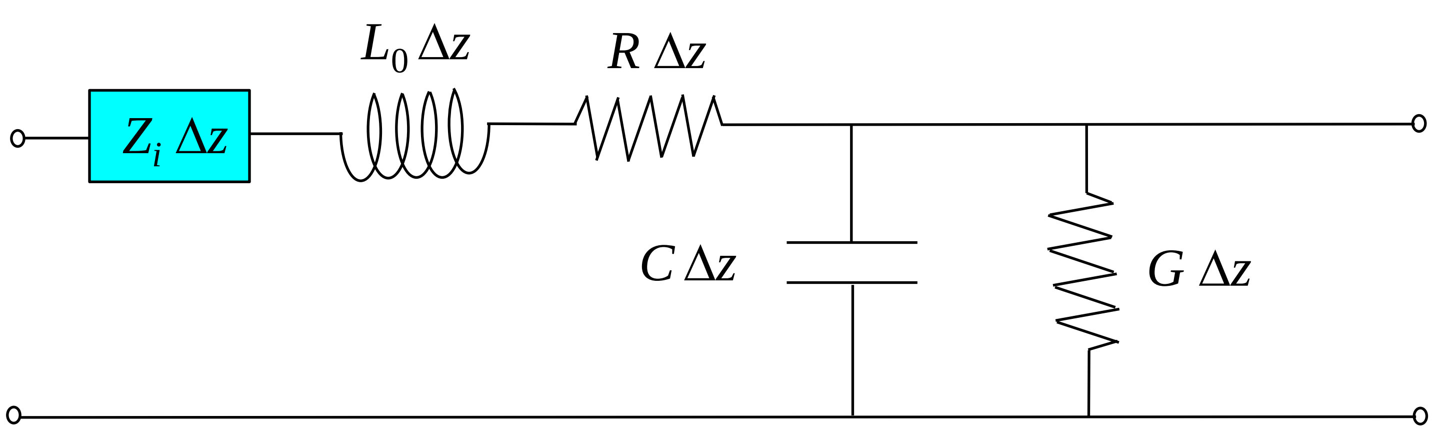
$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$$ (surface resistance of metal)

Impedance of Wire (cont.)

Transmission Line Model

An extra impedance per unit length Z_i is added to the TL model in order to account for the internal impedance of the conductors.

Note: The form of Z_i will depend on the shape of the conductors: we have only analyzed the case of a round wire.



The external inductance per unit length L_0 is calculated assuming perfectly conducting wires.

Impedance of Wire (cont.)

Transmission Line Model

For a twin lead (two wires running parallel) we have:

$$Z_i \approx 2Z_l$$

We assume that the wires are far enough apart so that the current is uniform inside each one.