ECE 6341

Spring 2016

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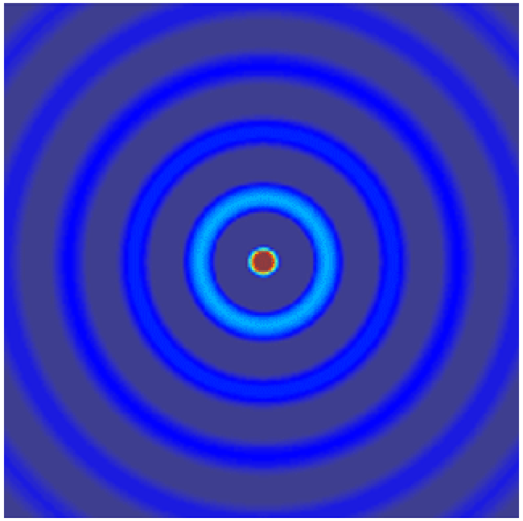
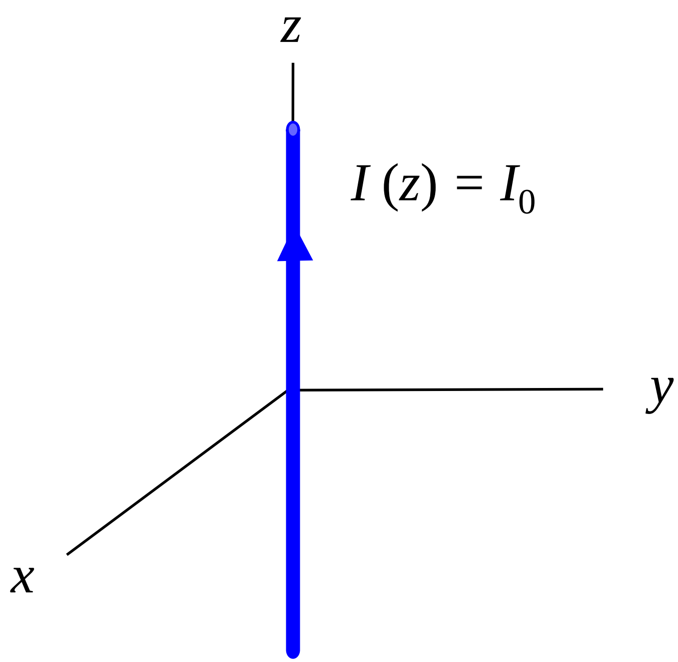
Notes 11



Current Line Source

TM$_z$: $\underline{A} = \underline{\hat{z}}\,\psi(\rho)$



Conditions:

1) Allowed angles: $\phi \in [0, 2\pi] \rightarrow \upsilon = n$

2) Symmetry: $n = 0$

3) Radiation condition: $H_n^{(2)}(k_\rho \rho)$

4) Symmetry: $k_z = 0 \quad (k_\rho = k)$

Hence

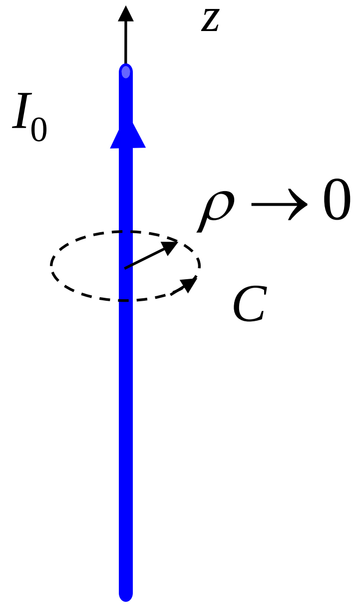
$$\psi(\rho) = a\,H_0^{(2)}(k\rho)$$

Current Line Source (cont.)

Our goal is to solve for the constant a.

$$\psi(\rho) = a\, H_0^{(2)}(k\rho)$$

Choose a small circular path:



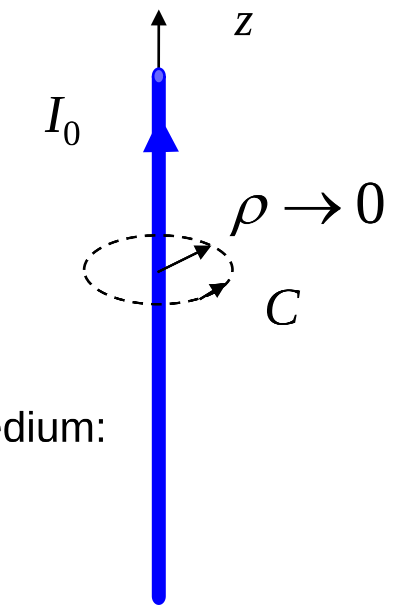
Current Line Source (cont.)

From Ampere's law and Stokes's theorem:

$$\nabla \times \underline{H} = \underline{J}^{\,i} + j\omega\varepsilon \underline{E}$$

$$\oint_C \underline{H} \cdot d\underline{r} = \int_S \left(\underline{J}^{\,i} \cdot \underline{\hat{z}} \right) dS + \int_S \left(j\omega\varepsilon \underline{E} \cdot \underline{\hat{z}} \right) dS$$

$$H_\phi (2\pi\rho) = I_0 + j\omega\varepsilon \int_S E_z \, dS$$



Conducting medium:

$\varepsilon \to \varepsilon_c$

Examine the last term (displacement current):

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{k_\rho^2}{j\omega\mu\varepsilon}\psi = \frac{k^2}{j\omega\mu\varepsilon}\psi = \frac{k^2}{j\omega\mu\varepsilon} a H_0^{(2)}(k\rho)$$

where $H_0^{(2)}(x) \sim \dfrac{-2j}{\pi}\left(\gamma + \ln\left(\dfrac{x}{2}\right)\right)$

Current Line Source (cont.)

Hence $$E_z = \mathcal{O}\left(\ln\left(\rho\right)\right)$$

so $$\int_S E_z \, dS \approx C \int_0^{2\pi} \int_0^{\rho} \ln(\rho) \rho \, d\rho \, d\phi \to 0$$

Therefore $$H_\phi (2\pi\rho) = I_0$$

Now use

$$H_\phi = -\frac{1}{\mu}\frac{\partial \psi}{\partial \rho}$$

$$\psi(\rho) = a\, H_0^{(2)}(k\rho)$$

$$H_0^{(2)}(x) \sim \frac{-2j}{\pi}\left(\gamma + \ln\left(\frac{x}{2}\right)\right)$$

$$H_0^{(2)\prime}(x) \sim \frac{-2j}{\pi}\left(\frac{2}{x}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{1}{\mu} a\, k\, H_0^{(2)\prime}(k\rho)$$

$$\sim -\frac{1}{\mu} ak \left(\frac{-2j}{\pi}\right)\left(\frac{2}{k\rho}\right)\left(\frac{1}{2}\right)$$

Current Line Source (cont.)

Hence

$$(2\pi\rho)\left[-\frac{1}{\mu}ak\left(\frac{-2j}{\pi}\right)\left(\frac{2}{k\rho}\right)\left(\frac{1}{2}\right)\right]=I_0$$

or

$$a(4j/\mu)=I_0$$

so

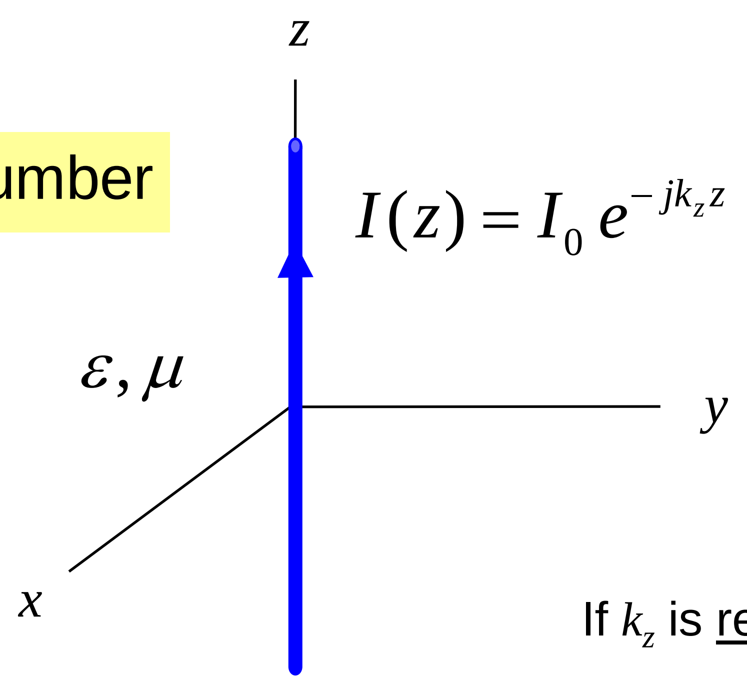
$$a=\frac{\mu I_0}{4j}$$

Thus

$$A_z=\left(\frac{\mu I_0}{4j}\right)H_0^{(2)}(k_\rho\rho)$$

Phased Line Current

k_z = vertical wavenumber



$$I(z) = I_0 \, e^{-jk_z z}$$

Assume:

$$A_z = b \, H_0^{(2)}(k_\rho \rho) \, e^{-jk_z z}$$

$$k_\rho = \left(k^2 - k_z^2\right)^{1/2}$$

If k_z is <u>real</u>, then

$$k_\rho = \begin{cases} \sqrt{k^2 - k_z^2}, & k_z \le k \\ -j\sqrt{k_z^2 - k^2}, & k_z \ge k \end{cases}$$

Phased Line Current (cont.)

The solution for "b" is the same as for "a", with $k \to k_\rho$

Hence, we have:

$$b = \frac{\mu I_0}{4j}$$

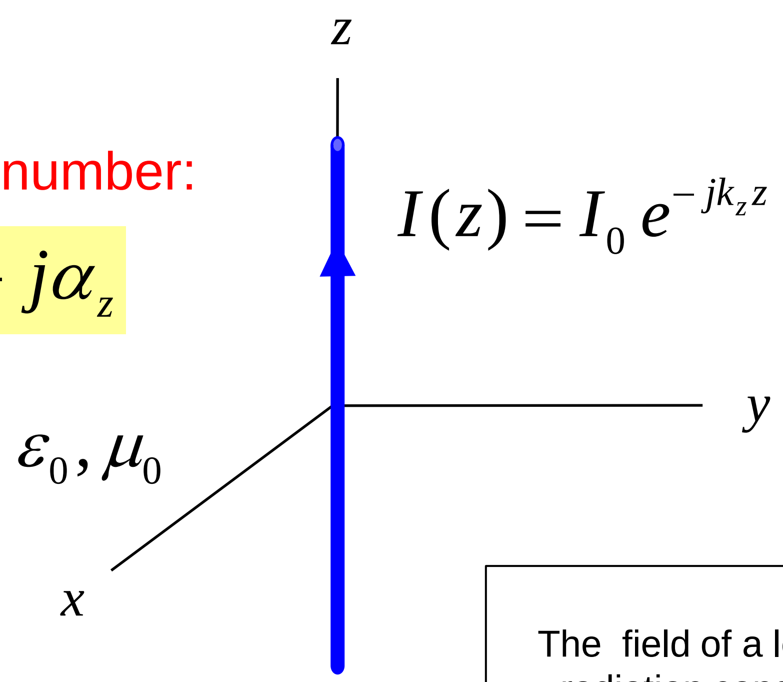
The solution for the magnetic vector potential is then

$$A_z = \left(\frac{\mu I_0}{4j}\right) H_0^{(2)}(k_\rho \rho)\, e^{-jk_z z}$$

Phased Line Current (cont.)

Complex wavenumber:

$$k_z = \beta_z - j\alpha_z$$



Note:
The field of a leaky mode does not satisfy the radiation condition at infinity (it increases to infinity in the ρ direction). We can still decide which choice of the square root gives us a "physical" field that would be measured.

$$A_z = \left(\frac{\mu I_0}{4j}\right) H_0^{(2)}(k_\rho \rho) e^{-jk_z z}$$

$$k_\rho = \left(k_0^2 - k_z^2\right)^{1/2}$$

Which sign of the square root do we choose?

Phased Line Current (cont.)

Physical choice of the square root:

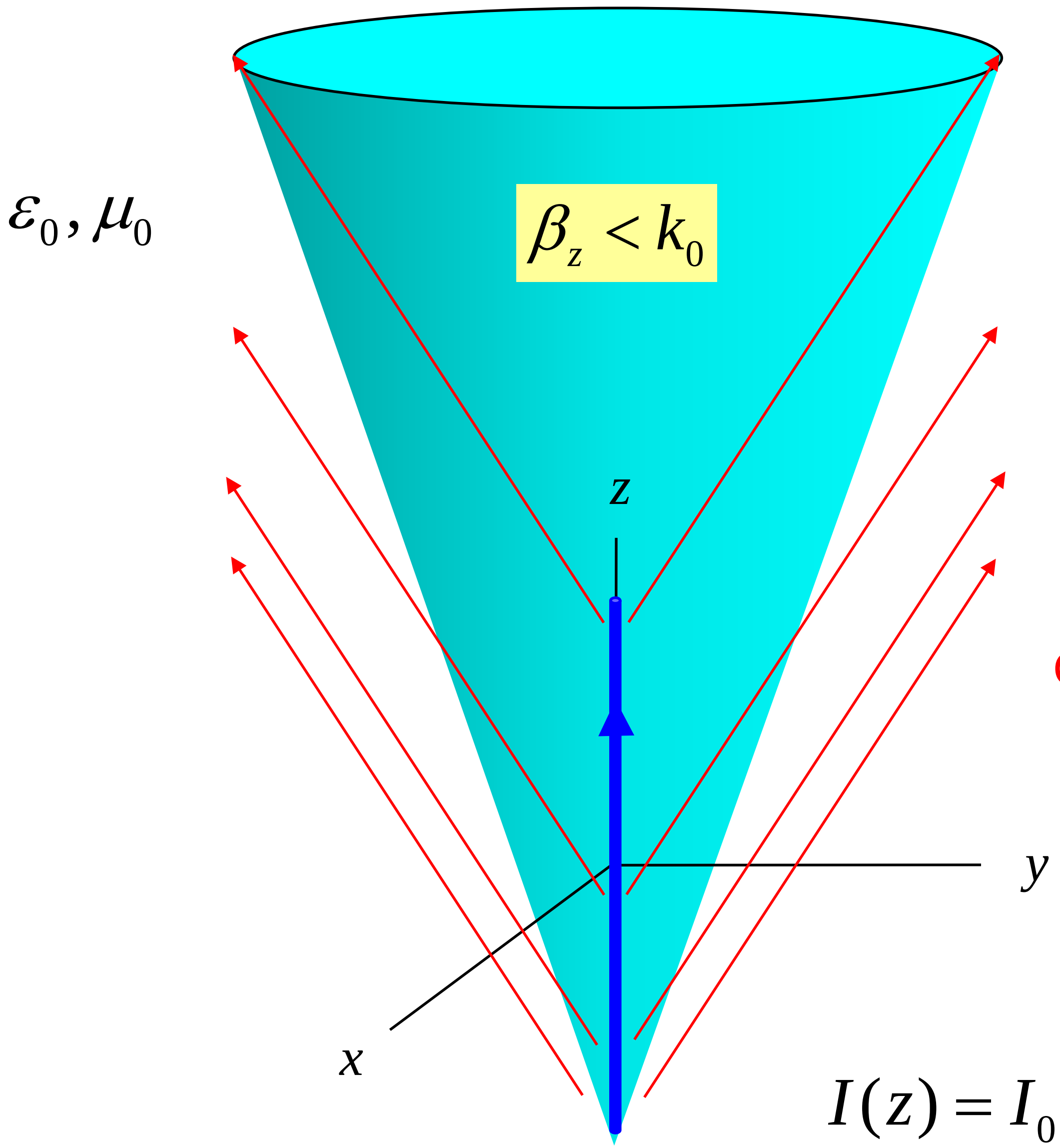
$$k_\rho = \left(k_0^2 - k_z^2\right)^{1/2} = \beta_\rho - j\alpha_\rho$$

$$\beta_z < k_0 : \quad \alpha_\rho < 0 \quad \text{(improper)}$$

$$\beta_z > k_0 : \quad \alpha_\rho > 0 \quad \text{(proper)}$$

The physical choice will agree with what one would measure.

Phased Line Current (cont.)



The fields are improper in the radial direction.

Conical radiation

$$I(z) = I_0 \, e^{-jk_z z}$$