ECE 6341

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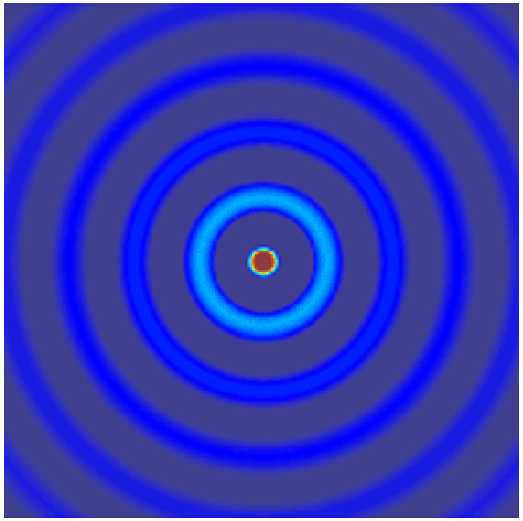
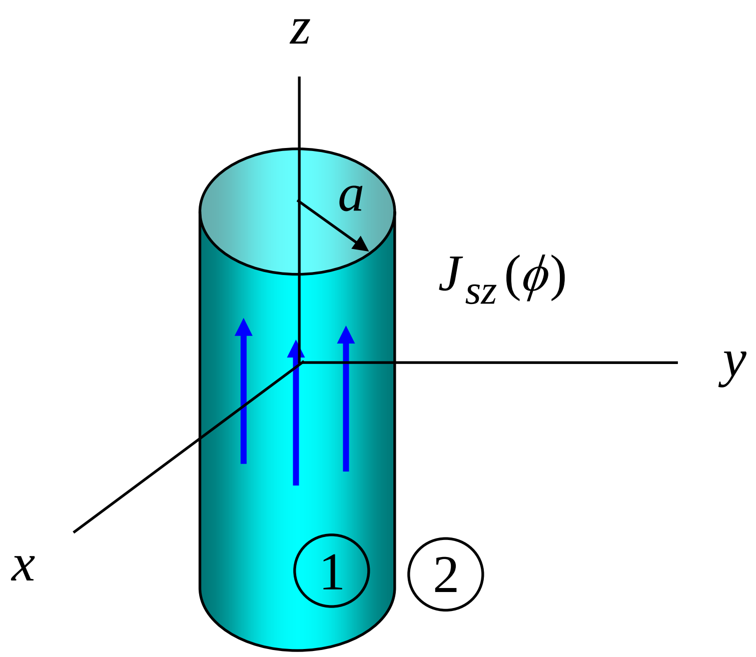
Notes 12

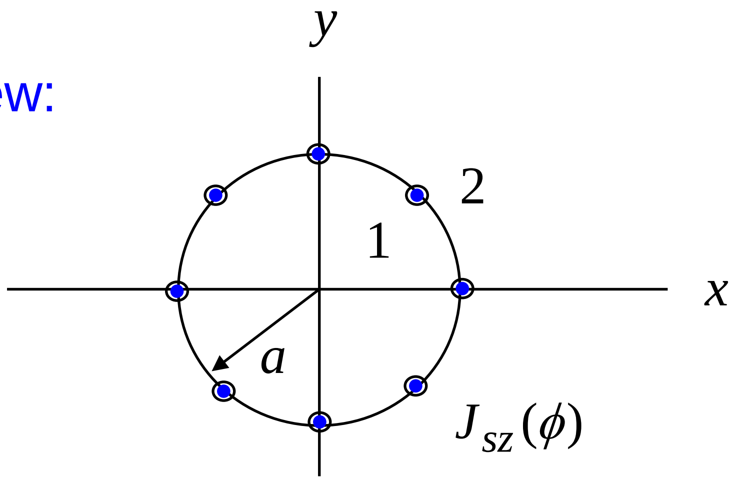


Tube of Surface Current

An infinite hollow tube of surface current, flowing in the z direction:



Top view:



From superposition, we know we can solve the problem using

$$\underline{A} = \underline{\hat{z}}\,\psi \quad (\text{TM}_z)$$

Tube of Current (cont.)

Represent $J_{sz}(\phi)$ as a (complex exponential) Fourier series:

$$J_{sz}(\phi) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\phi}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} J_{sz}(\phi) \, e^{-jn\phi} d\phi$$

The coefficients c_n are assumed to be known.

Tube of Current (cont.)

$\rho < a$

$$\psi_1 = \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi} J_n(k\rho)$$

$\rho > a$

$$\psi_2 = \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi} H_n^{(2)}(k\rho)$$

Apply B.C.'s (at $\rho = a$):

$$E_{z1} = E_{z2}$$

$$H_{\phi 2} - H_{\phi 1} = J_{sz}(\phi)$$

Tube of Current (cont.)

The first B.C. gives

$$\psi_1 = \psi_2$$

The second B.C. gives

TM_z Table : $H_\phi = -\frac{1}{\mu}\left(\frac{\partial \psi}{\partial \rho}\right)$

$$-\frac{1}{\mu}\left(\frac{\partial \psi_2}{\partial \rho} - \frac{\partial \psi_1}{\partial \rho}\right) = J_{sz}(\phi)$$

Hence we have, from the first BC,

$$a_n J_n(ka) = b_n H_n^{(2)}(ka)$$

Tube of Current (cont.)

From the second BC we have

$$-\frac{1}{\mu}k\left[b_n H_n^{(2)\,\prime}(ka) - a_n J_n'(ka)\right] = c_n$$

Substituting for a_n from the first equation, we have

$$b_n\left[H_n^{(2)\,\prime}(ka) - \left(\frac{H_n^{(n)}(ka)}{J_n(ka)}\right)J_n'(ka)\right] = -\frac{\mu}{k}c_n$$

$$\Downarrow$$

$$b_n\left[J_n(ka)\,H_n^{(2)\,\prime}(ka) - J_n'(ka)\,H_n^{(2)}(ka)\right] = -\frac{\mu}{k}c_n J_n(ka)$$

Next, look at the term

$$[\;\;] = J_n(ka)\,H_n^{(2)\,\prime}(ka) - J_n'(ka)\,H_n^{(2)}(ka)$$

Tube of Current (cont.)

This may be written as (using $x = ka$)

$$\left[\;\right] = J_n(x)\left[J_n'(x) - jY_n'(x)\right] - J_n'(x)\left[J_n(x) - jY_n(x)\right]$$

$$= -j\left[J_n(x)Y_n'(x) - J_n'(x)Y_n(x)\right]$$

$$= -j\left[\frac{2}{\pi x}\right]$$

(This follows from the Wronskian identity below.)

$$J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = \frac{2}{\pi x}$$

Hence,

$$b_n\left[\frac{-2j}{\pi ka}\right] = -\frac{\mu}{k}c_n J_n(ka)$$

Tube of Current (cont.)

The coefficient is then

$$b_n = -j\mu\pi\left(\frac{a}{2}\right)c_n J_n(ka)$$

Using

$$a_n J_n(ka) = b_n H_n^{(2)}(ka)$$

we have

$$a_n = -j\mu\pi\left(\frac{a}{2}\right)c_n H_n^{(2)}(ka)$$

Tube of Current (cont.)

Generalization: a phased tube of current

$$J_{sz}(\phi) = e^{-jk_z z} \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\phi}$$

$\rho < a$

$$\psi_1 = e^{-jk_z z} \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi} J_n(k_\rho \rho)$$

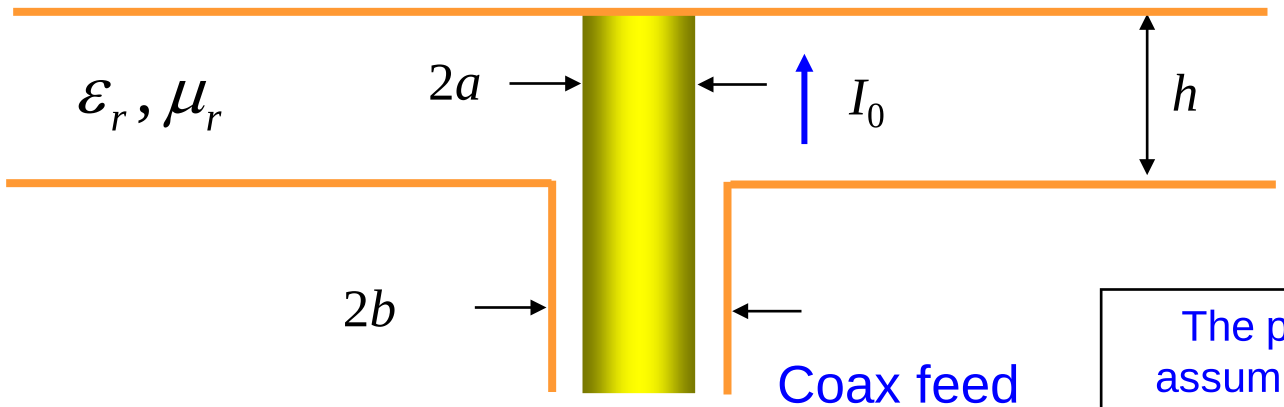
$\rho > a$

$$\psi_2 = e^{-jk_z z} \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi} H_n^{(2)}(k_\rho \rho)$$

$$a_n = -j\mu\pi \left(\frac{a}{2}\right) c_n H_n^{(2)}(k_\rho a)$$

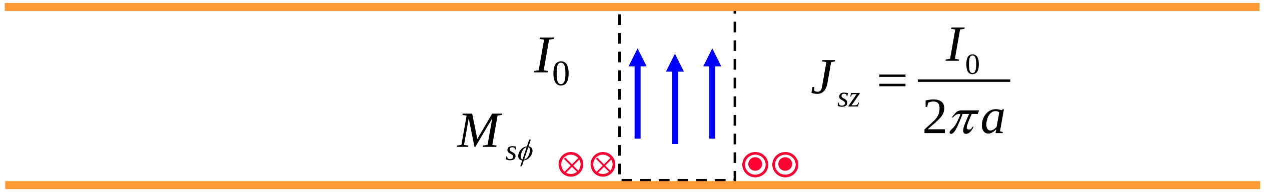
$$b_n = -j\mu\pi \left(\frac{a}{2}\right) c_n J_n(k_\rho a)$$

Inductive Post



The probe current is assumed to be uniform (no z or ϕ variation).

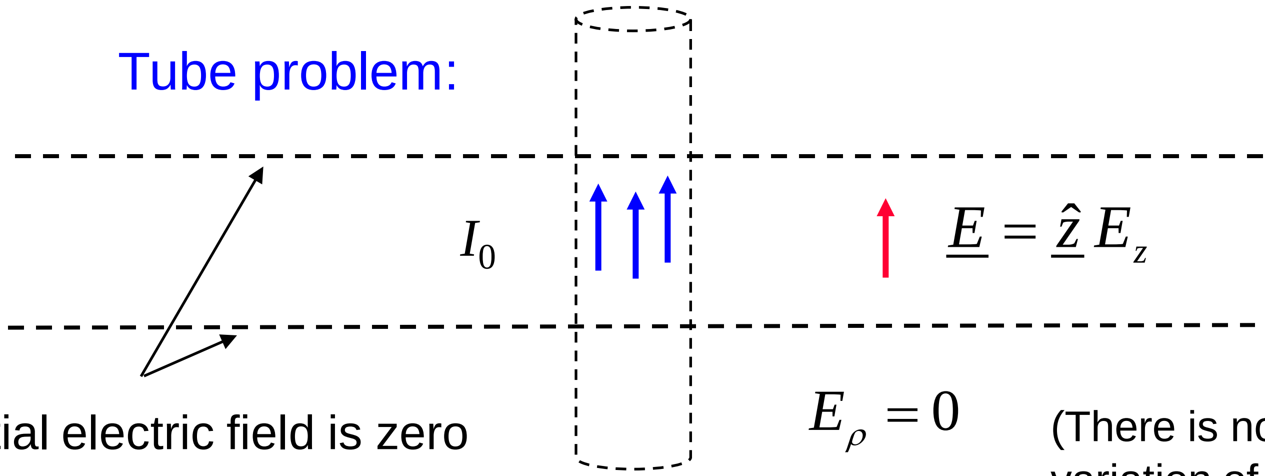
Equivalence principle: remove coax aperture and probe metal



The magnetic-current frill is ignored in calculating the fields.

Inductive Post (cont.)

Tube problem:



Tangential electric field is zero

$E_\rho = 0$

$E_\phi = 0$

(There is no z or ϕ variation of the fields.)

$$E_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial\rho\partial z}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon\rho}\frac{\partial^2\psi}{\partial\phi\partial z}$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

The probe problem is thus equivalent to an infinite tube of current in free space.

For $\rho \geq a$: $$A_z = \psi_2 = \sum_{n=-\infty}^{\infty} b_n e^{jn\phi} H_n^{(2)}(k\rho)$$

where $$b_n = -j\mu\pi\left(\frac{a}{2}\right) c_n J_n(ka)$$

Inductive Post (cont.)

and

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} J_{sz}(\phi) e^{-jn\phi} d\phi$$

$$= \frac{1}{2\pi} \left(\frac{I_0}{2\pi a} \right) \int_{-\pi}^{+\pi} e^{-jn\phi} d\phi$$

$$= \begin{cases} \dfrac{I_0}{2\pi a}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Hence

$$A_z = b_0 H_0^{(2)}(k\rho)$$

$$b_0 = -j\mu\pi \left(\frac{a}{2} \right) c_0 J_0(ka)$$

$$c_0 = \frac{I_0}{2\pi a}$$

Inductive Post (cont.)

We then have

$$A_z = \left[-j\mu\pi \left(\frac{a}{2} \right) \left(\frac{I_0}{2\pi a} \right) J_0(ka) \right] H_0^{(2)}(k\rho)$$

$$E_z = \frac{1}{j\omega\mu\varepsilon} \left(k^2 + \cancel{\frac{\partial^2}{\partial z^2}} \right) A_z = -j\omega A_z$$

so

$$E_z = -\omega\mu \left(\frac{I_0}{4} \right) J_0(ka) H_0^{(2)}(k\rho)$$

Inductive Post (cont.)

Complex source power radiated by tube of current (height h):

$$P_s = -\frac{1}{2}\int_s \underline{E}\cdot \underline{J}_s^* dS$$

$$= -\frac{1}{2}(2\pi a)h\, J_{sz}^*\, E_z\Big|_{\rho=a}$$

or

$$P_s = -\frac{1}{2}(2\pi a)h\left(\frac{I_0^*}{2\pi a}\right)E_z\Big|_{\rho=a}$$

$$= -\frac{1}{2}(2\pi a)h\left(\frac{I_0^*}{2\pi a}\right)\left[-\omega\mu\left(\frac{I_0}{4}\right)J_0(ka)H_0^{(2)}(ka)\right]$$

Inductive Post (cont.)

or

$$P_s = \left(\frac{h}{8}\right)(\omega\mu)\left|I_0\right|^2 J_0(ka)H_0^{(2)}(ka)$$

or

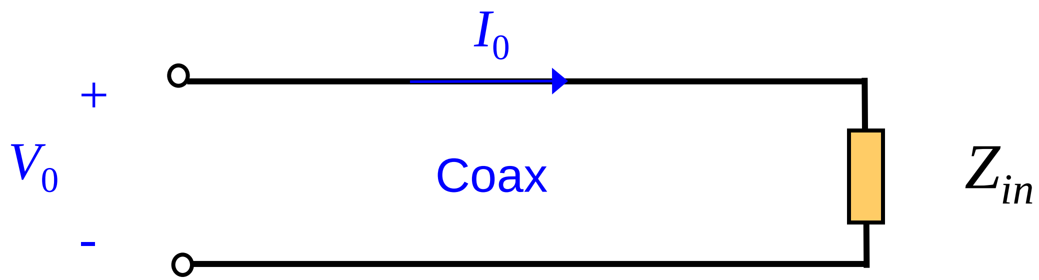
$$P_s = \frac{(kh)}{8}(\eta)\left|I_0\right|^2 J_0(ka)H_0^{(2)}(ka)$$

where

$$k = k_0\sqrt{\varepsilon_r\mu_r} \qquad \eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Inductive Post (cont.)

Equivalent circuit seen by coax:



$$P_s = \frac{1}{2} V_0 I_0^* = \frac{1}{2} Z_{in} \left| I_0 \right|^2$$

so

$$Z_{in} = \frac{2P_s}{\left| I_0 \right|^2}$$

Inductive Post (cont.)

Therefore

$$Z_{in} = \eta \frac{(kh)}{4} J_0(ka) H_0^{(2)}(ka)$$

For $ka << 1$

$$Z_{in} \approx \eta \frac{(kh)}{4} \left[1 - j\frac{2}{\pi} \left(\gamma + \ln\left(\frac{ka}{2} \right) \right) \right]$$

$$\gamma = 0.5772156$$

We thus have the "probe reactance": $X_p = \text{Im}\left(Z_{in}\right)$

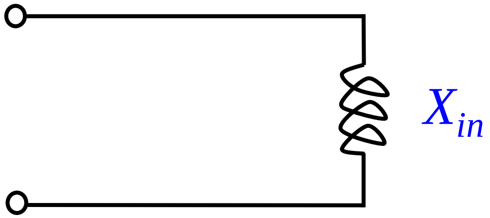
$$X_p \approx \eta \frac{(kh)}{4} \left(\frac{2}{\pi} \right) \left(-\gamma - \ln\left(\frac{ka}{2} \right) \right)$$

Inductive Post (cont.)

or

$$X_p \approx \frac{\eta}{2\pi}(kh)\left(-\gamma + \ln\left(\frac{2}{ka}\right)\right)$$

$$\gamma = 0.5772156$$



Probe inductance:

$$L_p = \frac{X_p}{\omega} = \left(\frac{1}{\omega}\right)\frac{\eta}{2\pi}(kh)\left(-\gamma + \ln\left(\frac{2}{ka}\right)\right)$$

Inductive Post (cont.)

Summary

Probe reactance:

$$X_p \approx \mu_r \frac{\eta_0}{2\pi}(k_0 h)\left(-\gamma + \ln\left(\frac{2}{k_0 a\sqrt{\varepsilon_r \mu_r}}\right)\right)$$

Probe inductance:

$$L_p \approx \mu_0 \mu_r \frac{h}{2\pi}\left(-\gamma + \ln\left(\frac{2}{k_0 a\sqrt{\varepsilon_r \mu_r}}\right)\right)$$

Inductive Post (cont.)

Comments:

- The probe inductance increases as the length of the probe increases.
- The probe inductance increases as the radius of the probe decreases.
- The probe inductance is usually positive since there is a net stored magnetic energy in the vicinity of the probe.
- This problem provides a good approximation for the probe reactance of a microstrip antenna (as long as $kh << 1$).

Example

Practical patch antenna case:

$$\varepsilon_r = 2.94$$

$$h = 0.1524 \ [\text{cm}] \qquad (60 \, \text{mils})$$

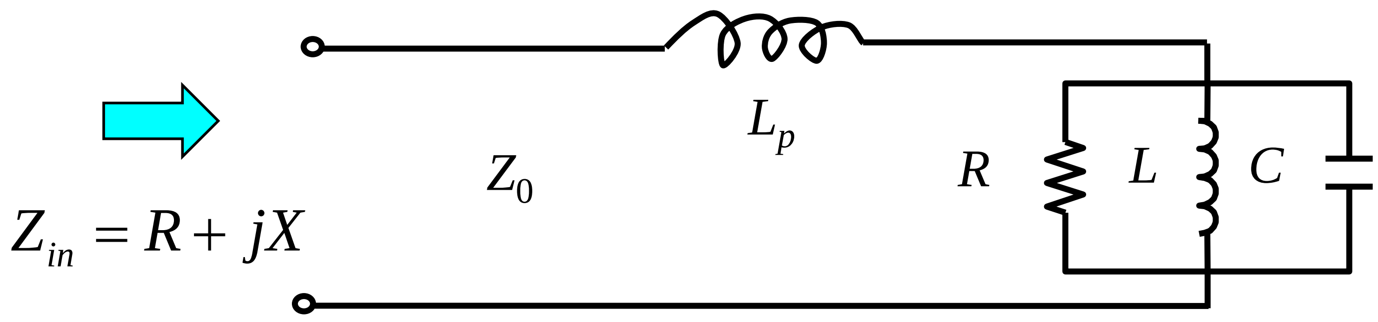
$$a = 0.0635 \ [\text{cm}] \qquad (50 [\Omega] \ \text{SMA Connector})$$

$$f = 2.0 \ [\text{GHz}] \qquad (\lambda_0 = 14.9896 \ [\text{cm}])$$

$$X_p = 12.3 \ [\Omega]$$

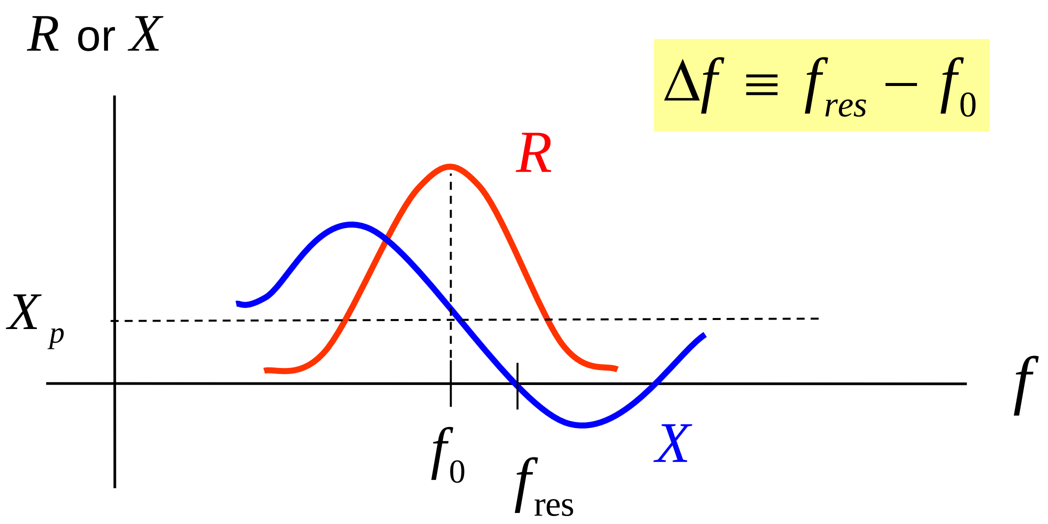
$$L_p = 0.979 \ [\text{nH}]$$

Microstrip Antenna Model



f_0 = resonance frequency of RLC circuit

$$Z_{in} = jX_p + Z_{RLC}$$



$$\Delta f \equiv f_{res} - f_0$$

The frequency where R is maximum is different from the frequency where the reactance is zero, due to the probe reactance.

Lossy Post

If the post has loss (finite conductivity), then we need to account for the skin effect, which accounts for the magnetic energy stored <u>inside</u> the post. This gives us an <u>internal</u> reactance.

$$X_p^{ext} \approx \mu_r \frac{\eta_0}{2\pi}(k_0 h)\left(-\gamma + \ln\left(\frac{2}{k_0 a\sqrt{\varepsilon_r \mu_r}}\right)\right)$$

$$X_p^{int} = X_s\left(\frac{h}{2\pi a}\right)$$

$$X_s = R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Example (Revisited)

Practical patch antenna case:

$$\varepsilon_r = 2.94$$

$$h = 0.1524 \;\; [\text{cm}] \qquad (60 \,\text{mils})$$

$$a = 0.0635 \;\; [\text{cm}] \qquad (50[\Omega] \;\; \text{SMA Connector})$$

$$f = 2.0 \;\; [\text{GHz}] \qquad (\lambda_0 = 14.9896 \; [\text{cm}])$$

Assume: $\sigma = 3.0 \times 10^7 \;\; [\text{S/m}]$

$$X_p^{ext} = 12.3 \;\; [\Omega]$$

$$X_p^{int} = 0.0062 \;\; [\Omega]$$

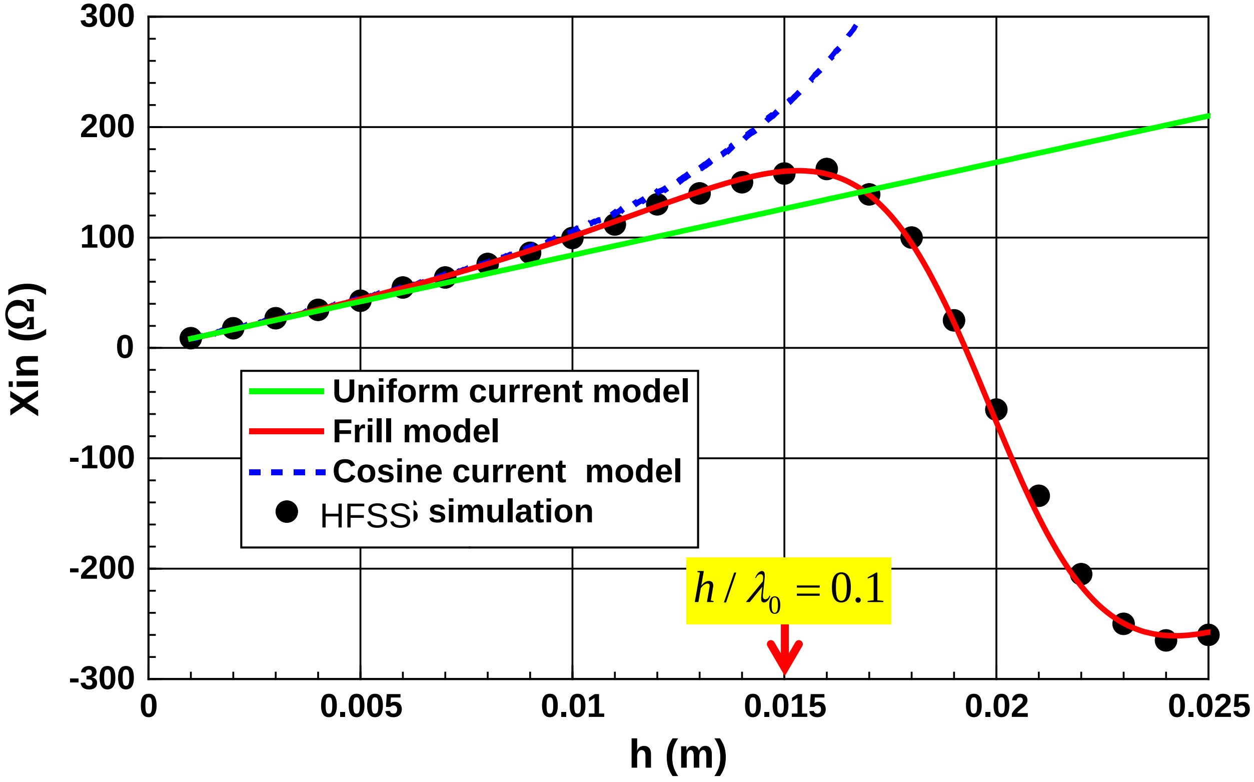
Accuracy of Probe Formula

$$\varepsilon_r = 2.2$$

$$f = 2.0 \ \ [\text{GHz}]$$

$$a = 0.635 \ [\text{mm}] \ (\text{SMA connector})$$

$$b = 2.188 \ [\text{mm}] \ (50 \ [\Omega] \ \text{coax})$$



H. Xu, D. R. Jackson, and J. T. Williams, "Comparison of Models for the Probe Inductance for a Parallel Plate Waveguide and a Microstrip Patch," *IEEE Trans. Antennas and Propagation*, vol. 53, pp. 3229-3235, Oct. 2005.