ECE 6341

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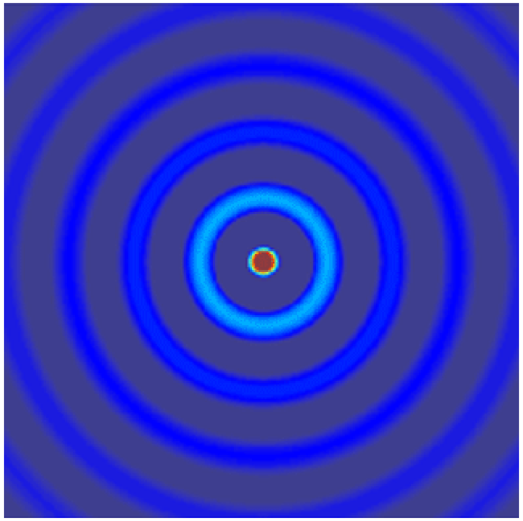
Notes 13



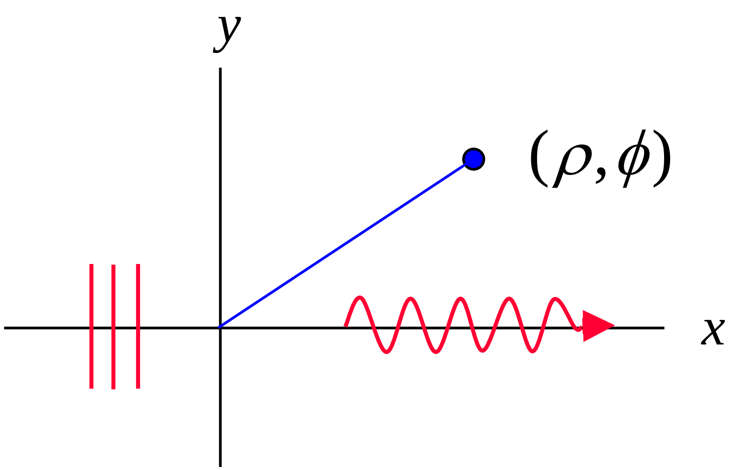
Plane Wave Expansion

The goal is to represent a plane wave in cylindrical coordinates as a series of cylindrical waves (to help us do scattering problems).

$$\underline{E} = \underline{E}_0 \, \psi(x)$$

$$\psi = e^{-jkx}$$

$$x = \rho \cos\phi$$



Generating function: (Schaum's Outline Eq. (24.16))

$$e^{\frac{\alpha}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(\alpha)\, t^n$$

Plane Wave Expansion (cont.)

Let
$$\begin{cases} \alpha = k\rho \\ t = -je^{j\phi} \end{cases}$$

$$t - \frac{1}{t} = -je^{j\phi} - \frac{1}{-je^{j\phi}} = -je^{j\phi} - je^{-j\phi} = -2j\cos\phi$$

Hence, the generating function identity gives us

$$e^{\frac{\alpha}{2}\left(t - \frac{1}{t}\right)} = e^{\left(\frac{k\rho}{2}\right)(-2j\cos\phi)} = e^{-j(k\rho)\cos\phi} = e^{-jkx} = \sum_{n=-\infty}^{+\infty} J_n(k\rho)(-j)^n e^{jn\phi}$$

Plane Wave Expansion (cont.)

or

$$e^{-jkx} = \sum_{n=-\infty}^{+\infty} \frac{1}{j^n} J_n(k\rho)\, e^{jn\phi}$$

"Jacobi-Anger Expansion"

Generalization:

$$e^{-jk_x x} e^{-jk_z z} = e^{-jk_z z} \sum_{n=-\infty}^{+\infty} \frac{1}{j^n} J_n(k_\rho \rho)\, e^{jn\phi}$$

where

$$k_\rho = k_x = \left(k^2 - k_z^2\right)^{1/2}$$

Alternative Derivation

Let

$$e^{-jkx} = \sum_{n=-\infty}^{+\infty} a_n J_n(k\rho) e^{jn\phi}$$

Note: The plane-wave field on the LHS is finite on the z axis.

Multiple by $e^{-jm\phi}$ and integrate over $\phi \in [0, 2\pi]$

Note that

$$\int_0^{2\pi} e^{-jm\phi} e^{jn\phi} d\phi = \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$$

Hence

$$\int_0^{2\pi} e^{-jkx} e^{-jm\phi} d\phi = 2\pi a_m J_m(k\rho)$$

Alternative Derivation

$$\int_0^{2\pi} e^{-jkx} e^{-jm\phi} d\phi = 2\pi a_m J_m(k\rho)$$

Hence
$$a_m = \frac{1}{2\pi J_m(k\rho)} \int_0^{2\pi} e^{-jkx} e^{-jm\phi} d\phi$$

or
$$a_m = \frac{1}{2\pi J_m(k\rho)} \int_0^{2\pi} e^{-jk\rho\cos\phi} e^{-jm\phi} d\phi$$

Note: It is not obvious, but a_m should be a constant (not a function of ρ).

Alternative Derivation (cont.)

Identity (adapted from Schaum's Mathematical Handbook Eq. (24.99)):

$$\int_0^{2\pi} e^{-j(x\cos\phi + m\phi)}\, d\phi = \frac{2\pi}{j^m} J_m(x)$$

Hence

$$a_m = \frac{1}{2\pi J_m(k\rho)}\left(\frac{2\pi}{j^m} J_m(k\rho)\right)$$

or

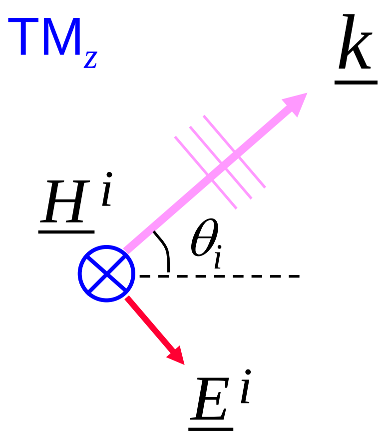
$$a_m = \frac{1}{j^m}$$

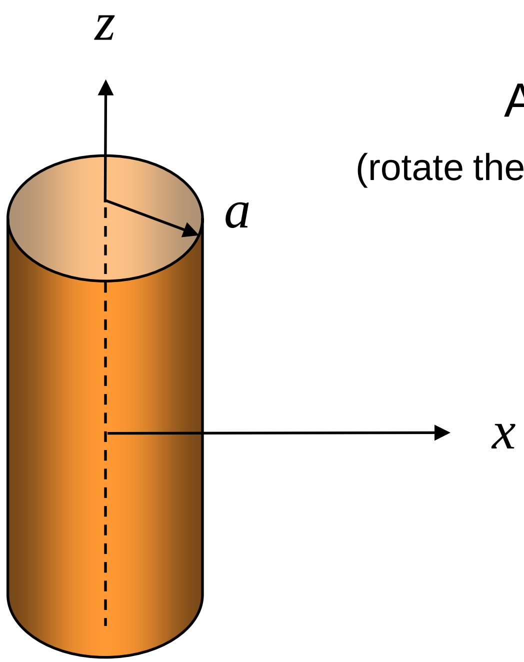
We then have

$$e^{-jkx} = \sum_{n=-\infty}^{+\infty} \frac{1}{j^m} J_n(k\rho)\, e^{jn\phi}$$

Scattering by Cylinder

A TM_z plane wave is incident on a PEC cylinder.





Assume $k_y = 0$

(rotate the coordinates if necessary)

$$\underline{H}^i = \underline{\hat{y}}\, H_{y0}\, e^{-j(k_x x + k_z z)}$$

$$k_x = k \cos\theta_i$$

$$k_z = k \sin\theta_i$$

Scattering by Cylinder (cont.)

Let

$$A_z^i = A_1 e^{-j(k_x x + k_z z)}$$

To find A_1:

$$H_y^i = -\frac{1}{\mu}\frac{\partial A_z^i}{\partial x}$$

$$= -\frac{1}{\mu}\left(-jk_x\right) A_1 e^{-j(k_x x + k_z z)}$$

Scattering by Cylinder (cont.)

Hence

$$H_{y0} = \frac{j}{\mu} k_x A_1$$

or

$$A_1 = -j\mu \left(\frac{1}{k_x} \right) H_{y0}$$

The incident potential is

$$A_z^i = A_1 e^{-j(k_x x + k_z z)}$$

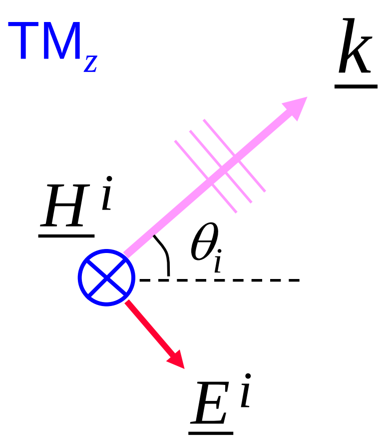
Scattering by Cylinder (cont.)

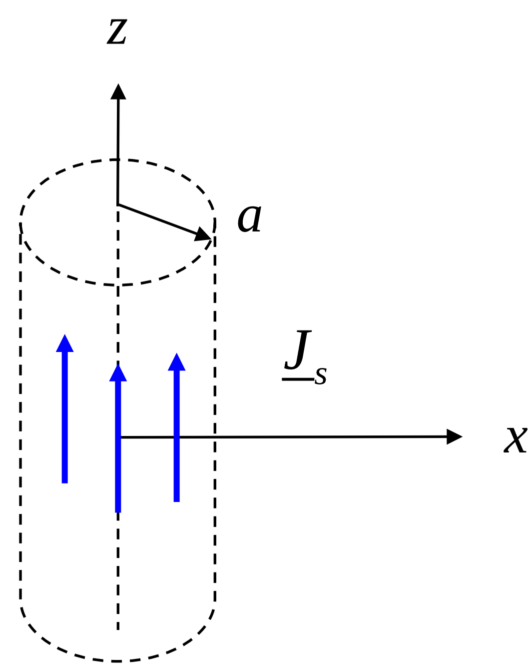
For $\rho \geq a$ denote

$$A_z = A_z^i + A_z^s$$

- The <u>incident</u> potential is that which exists assuming that the cylinder is not there.
- The <u>scattered</u> potential is that produced by the currents on the cylinder, which radiate.

Scattering by Cylinder (cont.)





$$\underline{H}^i = \underline{\hat{y}}\, H_{y0}\, e^{-j(k_x x + k_z z)}$$

According to the equivalence principle, we can remove the metal cylinder and keep the surface currents.

Scattering by Cylinder (cont.)

To solve for A_z^s , first put A_z^i into cylindrical form (Jacobi-Anger identity):

$$A_z^i = A_1 e^{-jk_z z} \sum_{n=-\infty}^{+\infty} \frac{1}{j^n} J_n(k_\rho \rho) e^{jn\phi}$$

where $k_\rho = \sqrt{k^2 - k_z^2} = \sqrt{k^2 - k^2 \sin^2\theta_i} = k\cos\theta_i = k_x$

Assume the following form for the scattered field:

$$A_z^s = A_1 e^{-jk_z z} \sum_{n=-\infty}^{+\infty} a_n \left(\frac{1}{j^n}\right) H_n^{(2)}(k_\rho \rho) e^{jn\phi}$$

Scattering by Cylinder (cont.)

At $\rho = a$ $\begin{cases} E_z = 0 \\ E_\phi = 0 \end{cases}$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right)$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z}$$

Both will be satisfied if

$$A_z(a,\phi,z) = 0$$

Scattering by Cylinder (cont.)

Hence

$$A_z^s(a,\phi,z) = -A_z^i(a,\phi,z)$$

This yields

$$J_n(k_\rho a) = -a_n\, H_n^{(2)}(k_\rho a)$$

or

$$a_n = -\frac{J_n(k_\rho a)}{H_n^{(2)}(k_\rho a)}$$

We then have

$$A_z^s = e^{-jk_z z} A_1 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{j^n}\right) \left(\frac{-J_n(k_\rho a)}{H_n^{(2)}(k_\rho a)}\right) H_n^{(2)}(k_\rho \rho)\, e^{jn\phi}$$

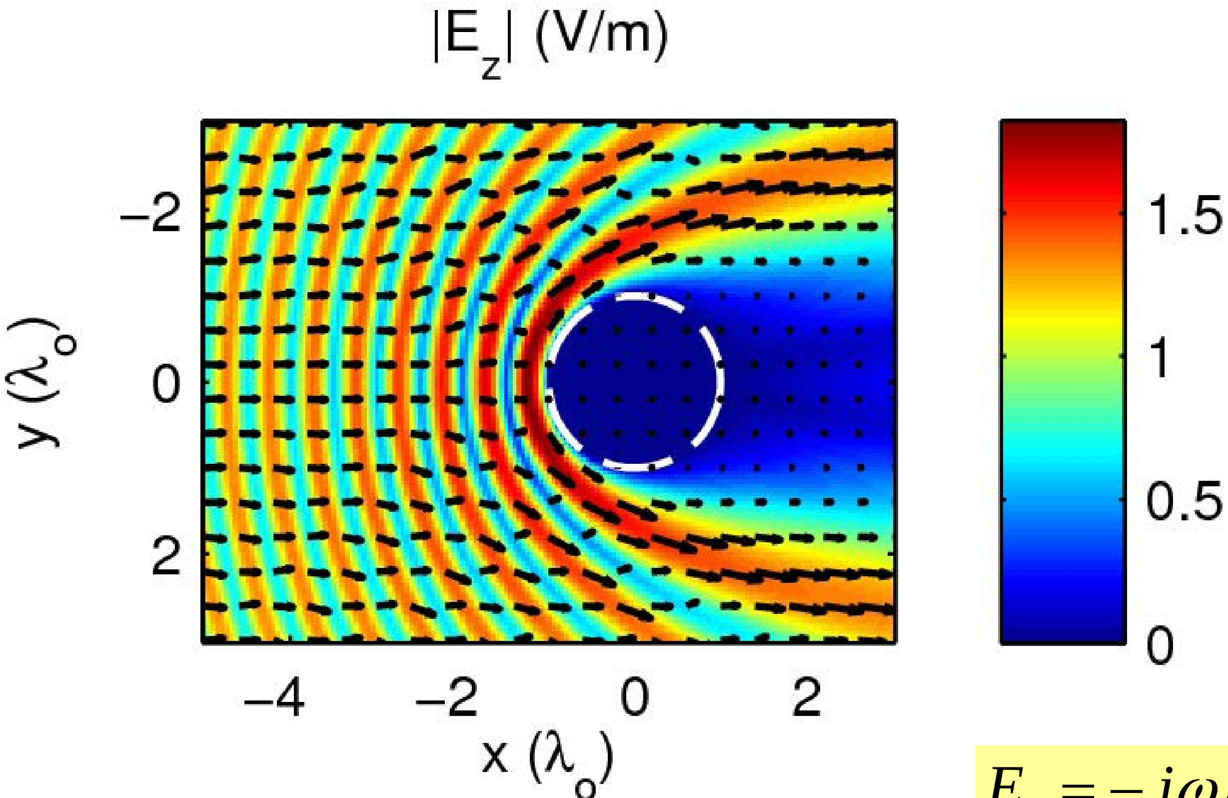
Scattering by Cylinder (cont.)

Note:

We were successful in solving the scattering problem using only a TM_z scattered field. This is because the cylinder was perfectly conducting. For a dielectric cylinder, the scattered field must have BOTH A_z and F_z (unless the incident plane wave has $k_z = 0$).

High-Frequency Scattering by Cylinder (cont.)

The <u>total</u> field near a conducting cylinder is shown (normal incidence).



$$E_z = -j\omega A_z$$

http://www.mathworks.com/matlabcentral/fileexchange/30162-cylinder-scattering