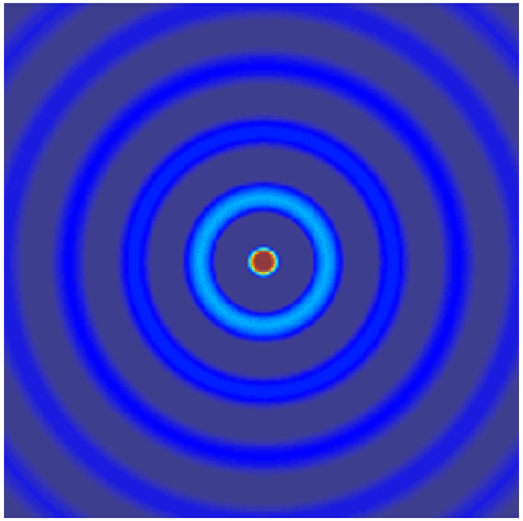
ECE 6341

Spring 2016

Prof. David R. Jackson
ECE Dept.

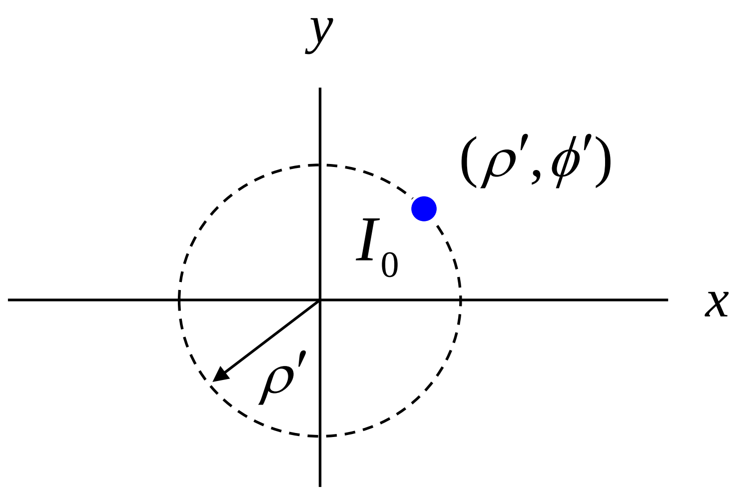
Notes 14



Addition Theorem

Allows us to "shift the origin" for line sources.

This is useful for solving problems with a line source outside of a cylinder.



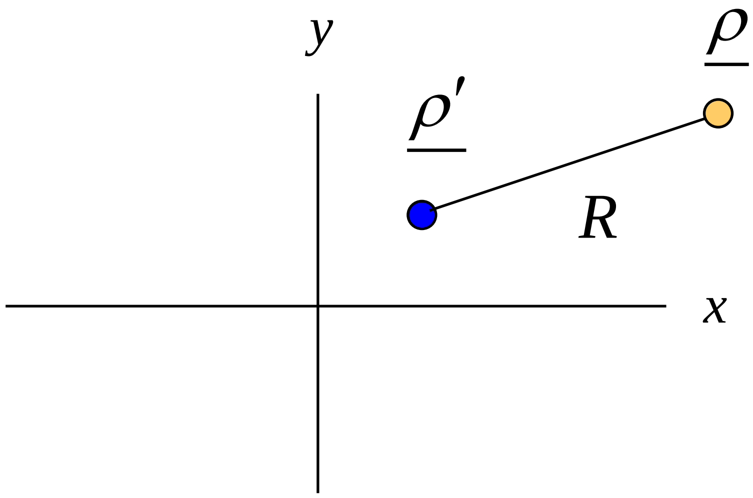
Denote $\psi = A_z$

Addition Theorem (cont.)

$$\psi = \frac{\mu I_0}{4j} H_0^{(2)}(kR) \qquad\qquad R = \left|\underline{\rho} - \underline{\rho}'\right|$$

$$\underline{\rho} = (x, y)$$

$$\underline{\rho}' = (x', y')$$

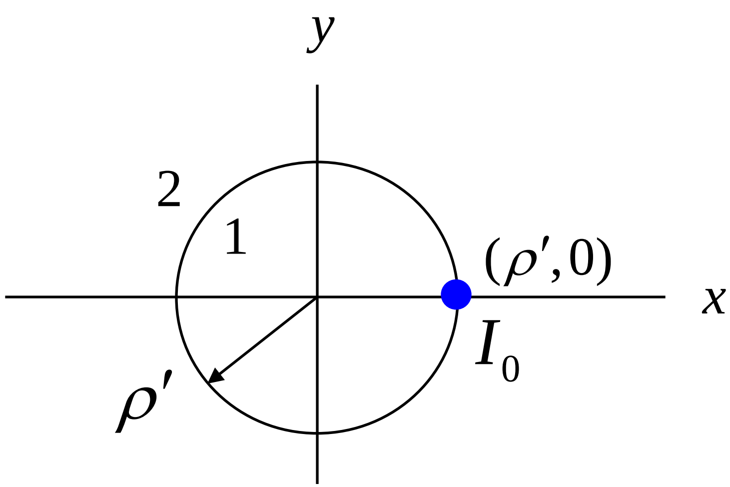


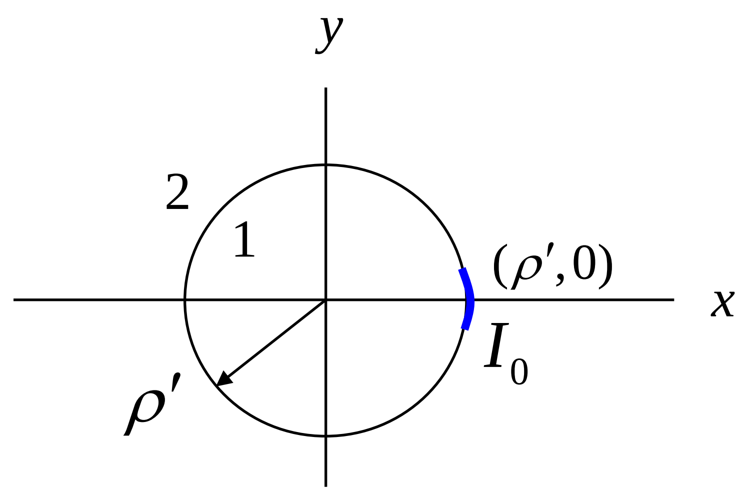
This is the exact solution, but it is not in a <u>useful</u> form.

Addition Theorem (cont.)

Assume $\phi' = 0$ for simplicity

Tube of current model





Surface-current:

$$J_{sz}(\phi') = A_1 \, \delta(\phi')$$

Addition Theorem (cont.)

Determine A_1: $$\int_{-\pi}^{+\pi} J_{sz}(\phi')\rho' d\phi' = I_0$$

$$J_{sz}(\phi') = A_1\,\delta(\phi')$$

so $$A_1\rho' = I_0$$

Hence $$J_{sz}(\phi') = \frac{I_0}{\rho'}\delta(\phi')$$

Using the “tube of current” concept,

$$J_{sz}(\phi') = \sum_{n=-\infty}^{+\infty} c_n e^{jn\phi}$$

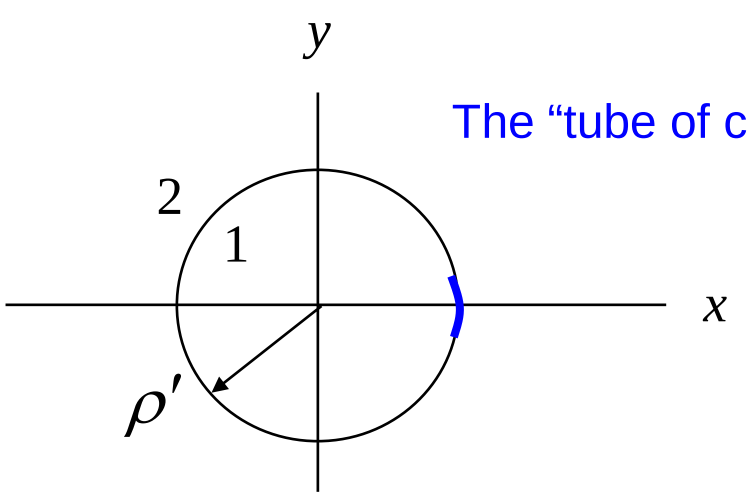
Addition Theorem (cont.)

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} J_{sz}(\phi')\, e^{-jn\phi'}\, d\phi'$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{I_0}{\rho'}\delta(\phi')\right) e^{-jn\phi'}\, d\phi'$$

$$= \frac{1}{2\pi}\frac{I_0}{\rho'}$$

$$c_n = \frac{1}{2\pi}\frac{I_0}{\rho'}$$



The "tube of current" representation

Addition Theorem (cont.)

$\rho < \rho'$ $$\psi_1 = \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi} J_n(k_\rho)$$

$\rho > \rho'$ $$\psi_2 = \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi} H_n^{(2)}(k\rho)$$

From previous "tube of current" derivation: $(a \rightarrow \rho')$

$$a_n = -j\mu\left(\frac{\pi\rho'}{2}\right) c_n H_n^{(2)}(k\rho')$$

$$b_n = -j\mu\left(\frac{\pi\rho'}{2}\right) c_n J_n(k\rho')$$

$$c_n = \frac{1}{2\pi}\frac{I_0}{\rho'}$$

Addition Theorem (cont.)

Therefore, we have

$$a_n = -j\mu\left(\frac{I_0}{4}\right)H_n^{(2)}\left(k\rho'\right)$$

$$b_n = -j\mu\left(\frac{I_0}{4}\right)J_n\left(k\rho'\right)$$

Hence

$$\psi_1 = -j\frac{\mu I_0}{4}\sum_{n=-\infty}^{+\infty} H_n^{(2)}\left(k_\rho \rho'\right)e^{jn\phi}J_n\left(k_\rho \rho\right)$$

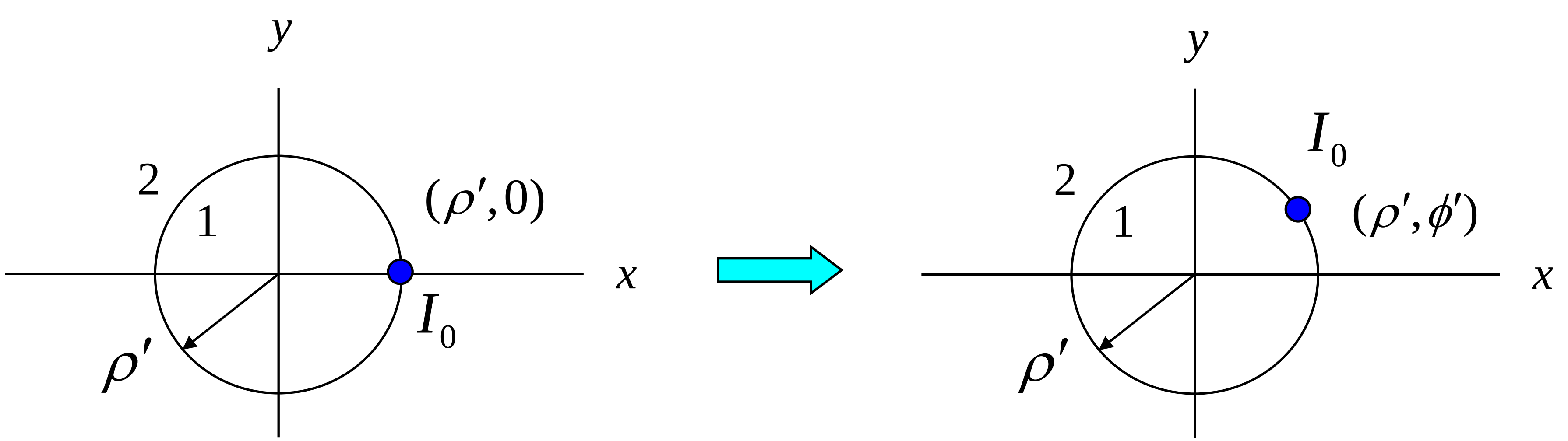
$$\psi_2 = -j\frac{\mu I_0}{4}\sum_{n=-\infty}^{+\infty} J_n\left(k_\rho \rho'\right)e^{jn\phi}H_n^{(2)}(k_\rho \rho)$$

Addition Theorem (cont.)

For $\phi' \neq 0$ replace $\phi \to \phi - \phi'$

$$\psi_1 = -j\frac{\mu I_0}{4}\sum_{n=-\infty}^{+\infty} H_n^{(2)}\left(k_\rho \rho'\right) e^{jn(\phi-\phi')} J_n\left(k_\rho \rho\right)$$

$$\psi_2 = -j\frac{\mu I_0}{4}\sum_{n=-\infty}^{+\infty} J_n\left(k_\rho \rho'\right) e^{jn(\phi-\phi')} H_n^{(2)}\left(k_\rho \rho\right)$$



Addition Theorem (cont.)

Comparing with the original expression for the potential of the line current, we have

$$H_0^{(2)}(k|\underline{\rho}-\underline{\rho}'|)=\begin{cases}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(k\rho')J_n(k\rho)e^{jn(\phi-\phi')} & \rho<\rho' \\ \sum_{n=-\infty}^{+\infty} J_n(k\rho')H_n^{(2)}(k\rho)e^{jn(\phi-\phi')} & \rho>\rho'\end{cases}$$

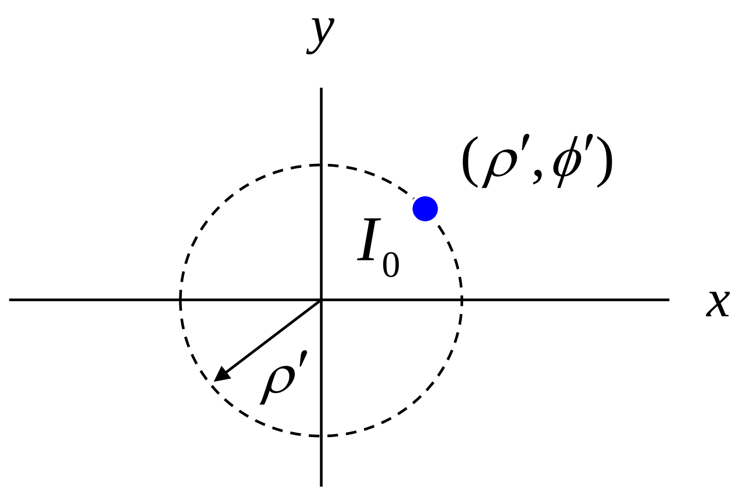
This is the "Graf addition theorem."

It also holds if k is replaced with k_ρ.

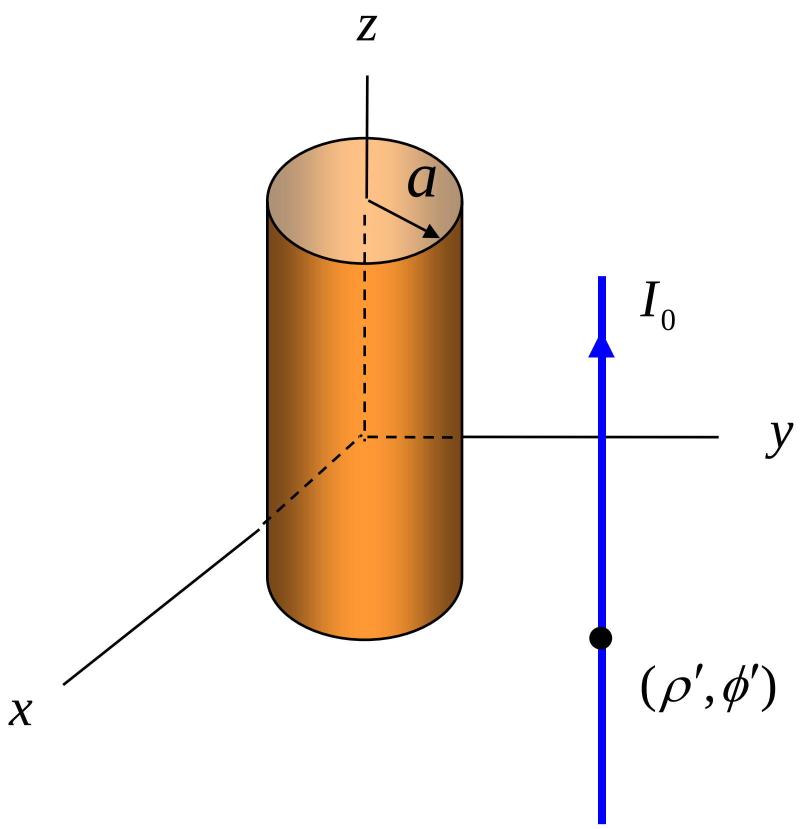
Johann Heinrich Graf
(1852-1918)

Addition Theorem (cont.)

$$H_0^{(2)}(k|\underline{\rho}-\underline{\rho}'|)=\begin{cases}\sum\limits_{n=-\infty}^{+\infty} H_n^{(2)}(k\rho')\,J_n(k\rho)\,e^{jn(\phi-\phi')} & \rho<\rho' \\ \sum\limits_{n=-\infty}^{+\infty} J_n(k\rho')\,H_n^{(2)}(k\rho)\,e^{jn(\phi-\phi')} & \rho>\rho'\end{cases}$$



Scattering From a Line Current



TM$_z$: $\psi = A_z$

Scattering From a Line Current (cont.)

$\rho < \rho'$

$$\psi^i = \frac{\mu I_0}{4j} \sum_{n=-\infty}^{+\infty} H_n^{(2)}(k\rho') J_n(k\rho) e^{jn(\phi-\phi')}$$

(This is only valid for $\rho < \rho'$)

Assume

$$\psi^s = \frac{\mu I_0}{4j} \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(k\rho) e^{jn(\phi-\phi')}$$

(This is only valid for $\rho \geq a$)

Scattering From a Line Current (cont.)

Boundary Conditions ($\rho = a$):

$$\psi^{s}(a,\phi,z) = -\psi^{i}(a,\phi,z)$$

Hence

$$a_n H_n^{(2)}(ka) = -H_n^{(2)}(k\rho')\, J_n(ka)$$

or

$$a_n = -\frac{H_n^{(2)}(k\rho')}{H_n^{(2)}(ka)}\, J_n(ka)$$

Scattering From a Line Current (cont.)

Final result:

$$\psi^{s} = \frac{\mu I_0}{4j} \sum_{n=-\infty}^{+\infty} \left[-\frac{H_n^{(2)}(k\rho')}{H_n^{(2)}(ka)} J_n(ka) \right] H_n^{(2)}(k\rho) e^{jn(\phi-\phi')}$$