ECE 6341

Spring 2016

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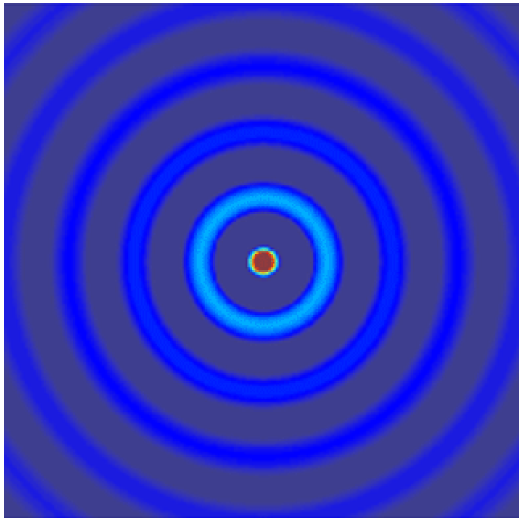
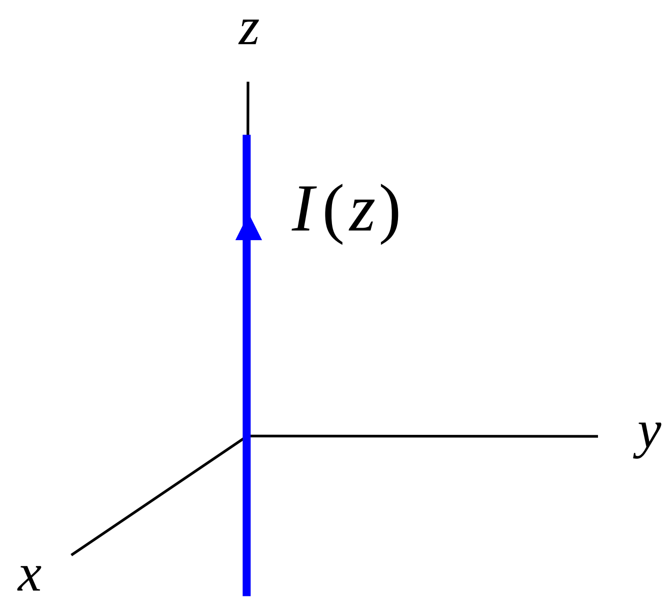
ECE Dept.

Notes 15



Arbitrary Line Current

TM$_z$: $A_z(z, \rho)$



Introduce Fourier Transform:

$$\tilde{I}(k_z) = \int_{-\infty}^{+\infty} I(z)\, e^{+jk_z z}\, dz$$

$$I(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{I}(k_z)\, e^{-jk_z z}\, dk_z$$

Arbitrary Line Current (cont.)

View this as a collection of phased line currents:

$$dI(z) = I_0 \, e^{-jk_z z}$$

$$I_0 = \frac{1}{2\pi} \tilde{I}(k_z) dk_z$$

$$I(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{I}(k_z) \, e^{-jk_z z} dk_z$$

Then

$$dA_z = \frac{\mu_0 I_0}{4j} H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z}$$

(from Notes 11)

$$= \frac{\mu_0}{4j}\left[\left(\frac{1}{2\pi}\right)\tilde{I}(k_z) \, dk_z\right] H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z}$$

Arbitrary Line Current (cont.)

Hence, from superposition, the total magnetic vector potential is

$$A_z = \frac{\mu_0}{8\pi j} \int_{-\infty}^{+\infty} \tilde{I}(k_z) H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

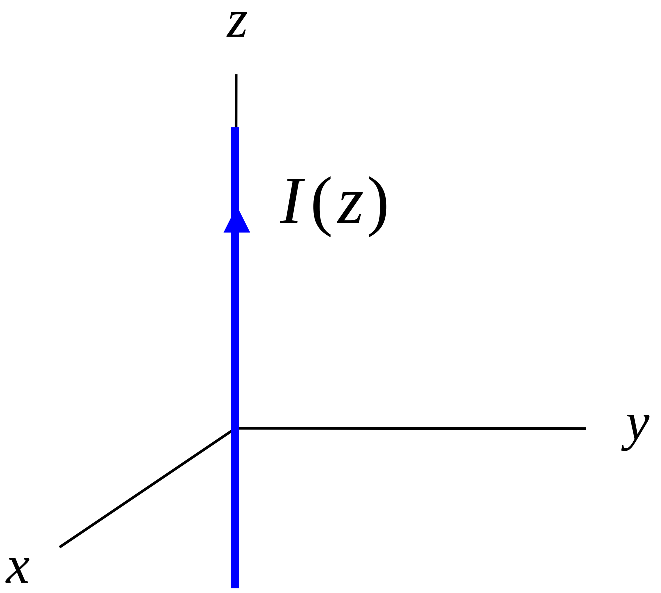
$$k_\rho = \left(k^2 - k_z^2\right)^{1/2}$$

$$k_\rho = \begin{cases} \sqrt{k^2 - k_z^2}, & |k_z| \le k \\ -j\sqrt{k_z^2 - k^2}, & |k_z| \ge k \end{cases}$$

Example

Uniform phased line current

$$I(z) = I_0 \, e^{-jk_{z0} z}$$



$$\tilde{I}(k_z) = I_0 \int_{-\infty}^{+\infty} e^{-jk_{z0}z} e^{+jk_z z} dz$$

$$= I_0 \int_{-\infty}^{+\infty} e^{j(k_z - k_{z0})z} dz$$

$$= I_0 \, 2\pi \delta(k_z - k_{z0})$$

Note:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jk_x x} \, dk_x$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{+jk_x x} \, dk_x$$

Example (cont.)

Hence

$$A_z = \frac{\mu_0}{8\pi j}\int_{-\infty}^{+\infty} \tilde{I}(k_z)\, H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

$$= \frac{\mu_0}{8\pi j}\int_{-\infty}^{+\infty} \left[I_0 2\pi\delta(k_z - k_{z0})\right] H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

$$= \frac{\mu_0}{8\pi j} I_0 2\pi H_0^{(2)}\left(k_{\rho 0} \rho\right) e^{-jk_{z0} z}$$

$$= \frac{\mu_0 I_0}{4j} H_0^{(2)}\left(k_{\rho 0} \rho\right) e^{-jk_{z0} z}$$

where $k_{\rho 0} = \left(k^2 - k_{z0}^2\right)^{1/2}$

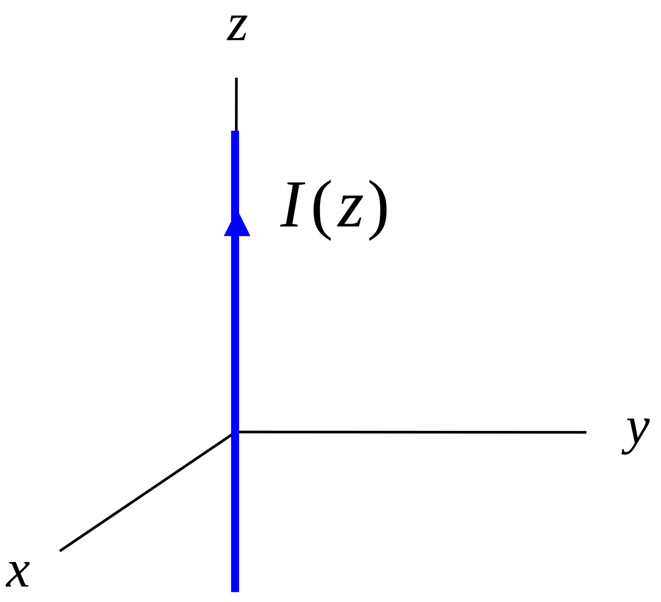
Example (cont.)

Hence

$$A_z = \left(\frac{\mu I_0}{4j} \right) H_0^{(2)}(k_{\rho 0}\rho)\, e^{-jk_{z0}z}$$

where

$$k_{\rho 0} = \left(k^2 - k_{z0}^2\right)^{1/2}$$



If k_{z0} is real, then

$$k_{\rho 0} = \begin{cases} \sqrt{k^2 - k_{z0}^2}, & |k_{z0}| \le k \\ -j\sqrt{k_{z0}^2 - k^2}, & |k_{z0}| \ge k \end{cases}$$

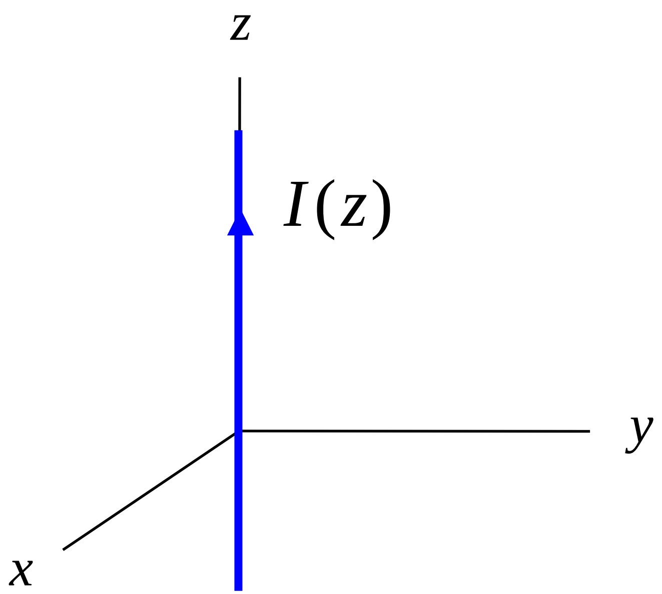
This is the correct choice of the wavenumber.

Example (cont.)

If k_{z0} is complex:

$$k_{z0} = \beta_{z0} - j\alpha_{z0}$$

$$k_{\rho 0} = \left(k^2 - k_{z0}^2\right)^{1/2}$$



$$0 < \beta_{z0} < k: \quad \operatorname{Im}\left(k_{\rho 0}\right) > 0 \quad \text{(improper)}$$

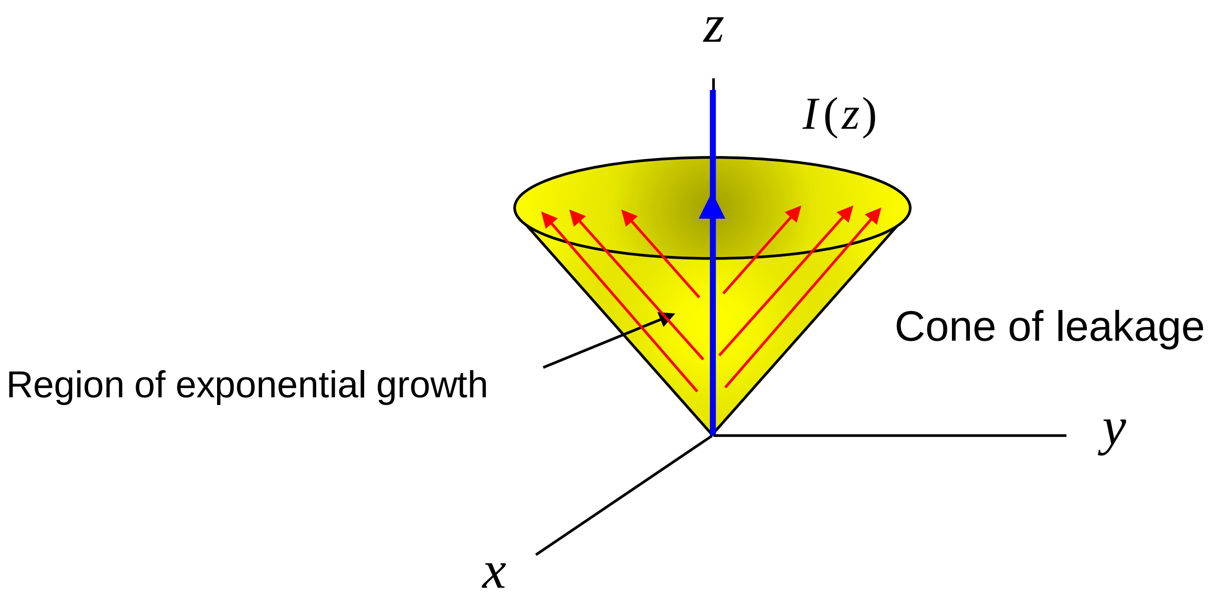
$$\beta_{z0} > k: \quad \operatorname{Im}\left(k_{\rho 0}\right) < 0 \quad \text{(proper)}$$

This is the "physical" choice of the wavenumber.

Note that the radiation condition at infinity is violated ($z \to -\infty$), so we lose uniqueness. However, we can still talk about what choice of the square root is "physical."

Example (cont.)

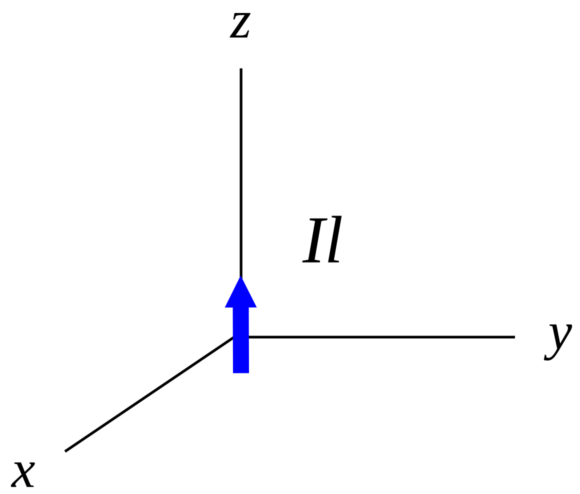
$$0 < \beta_{z0} < k: \quad \mathrm{Im}\left(k_{\rho 0}\right) > 0 \quad \text{(improper)}$$



A <u>semi-infinite</u> leaky-wave line source produces a cone of radiation in the near field.

Example

Dipole



$$I(z) = Il\,\delta(z)$$

$$\tilde{I}(k_z) = Il$$

Example (cont.)

$$A_z = \frac{\mu_0}{8\pi j} \int_{-\infty}^{+\infty} \tilde{I}(k_z) H_0^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

Hence

$$A_z = \left(I\ell\right) \frac{\mu_0}{8\pi j} \int_{-\infty}^{+\infty} H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z$$

$$k_\rho = \begin{cases} \sqrt{k^2 - k_z^2}, & |k_z| \le k \\ -j\sqrt{k_z^2 - k^2}, & |k_z| \ge k \end{cases}$$

Also,

$$A_z = \left(I\ell\right) \frac{\mu_0}{4\pi} \frac{e^{-jkr}}{r}$$

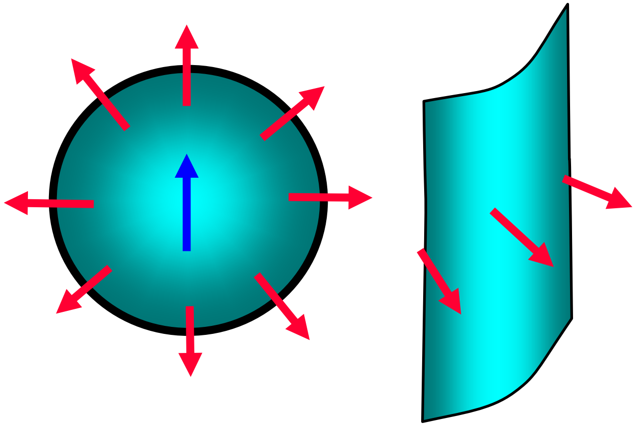
(from ECE 6340)

Example (cont.)

Hence,

$$\frac{e^{-jkr}}{r} = \frac{1}{2j}\int_{-\infty}^{+\infty} H_0^{(2)}(k_\rho \rho)\, e^{-jk_z z}\, dk_z$$

A spherical wave is thus expressed as a collection of cylindrical waves.



Example (cont.)

Use a change of variables:

$$k_z = \left(k_0^2 - k_\rho^2\right)^{1/2}$$

$$dk_z = -\frac{k_\rho}{\left(k_0^2 - k_\rho^2\right)^{1/2}} dk_\rho = -\frac{k_\rho}{k_z} dk_\rho$$

We then have

$$\frac{1}{2j}\int_{-\infty}^{+\infty} H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z = \frac{-1}{2j}\int_{C_{CCW}} \left(\frac{k_\rho}{k_z}\right) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_\rho$$

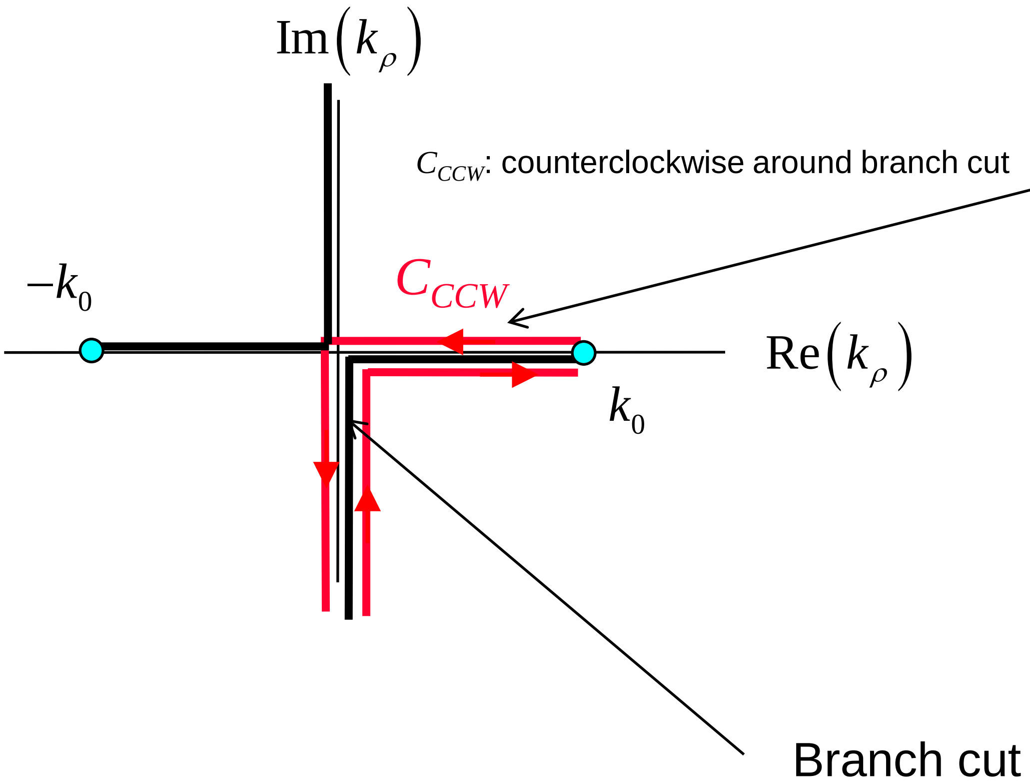
$$= \frac{1}{2j}\int_{C_{CW}} \left(\frac{k_\rho}{k_z}\right) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_\rho$$

or

$$\frac{e^{-jkr}}{r} = \frac{1}{2j}\int_{-\infty}^{+\infty} \left(\frac{k_\rho}{k_z}\right) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_\rho$$

(The path has been deformed back to the real axis, please see the next two slides)

Example (cont.)



Mapping equation for C:

$$k_z = \left(k_0^2 - k_\rho^2\right)^{1/2}$$

$$k_\rho \in C_{CCW}$$

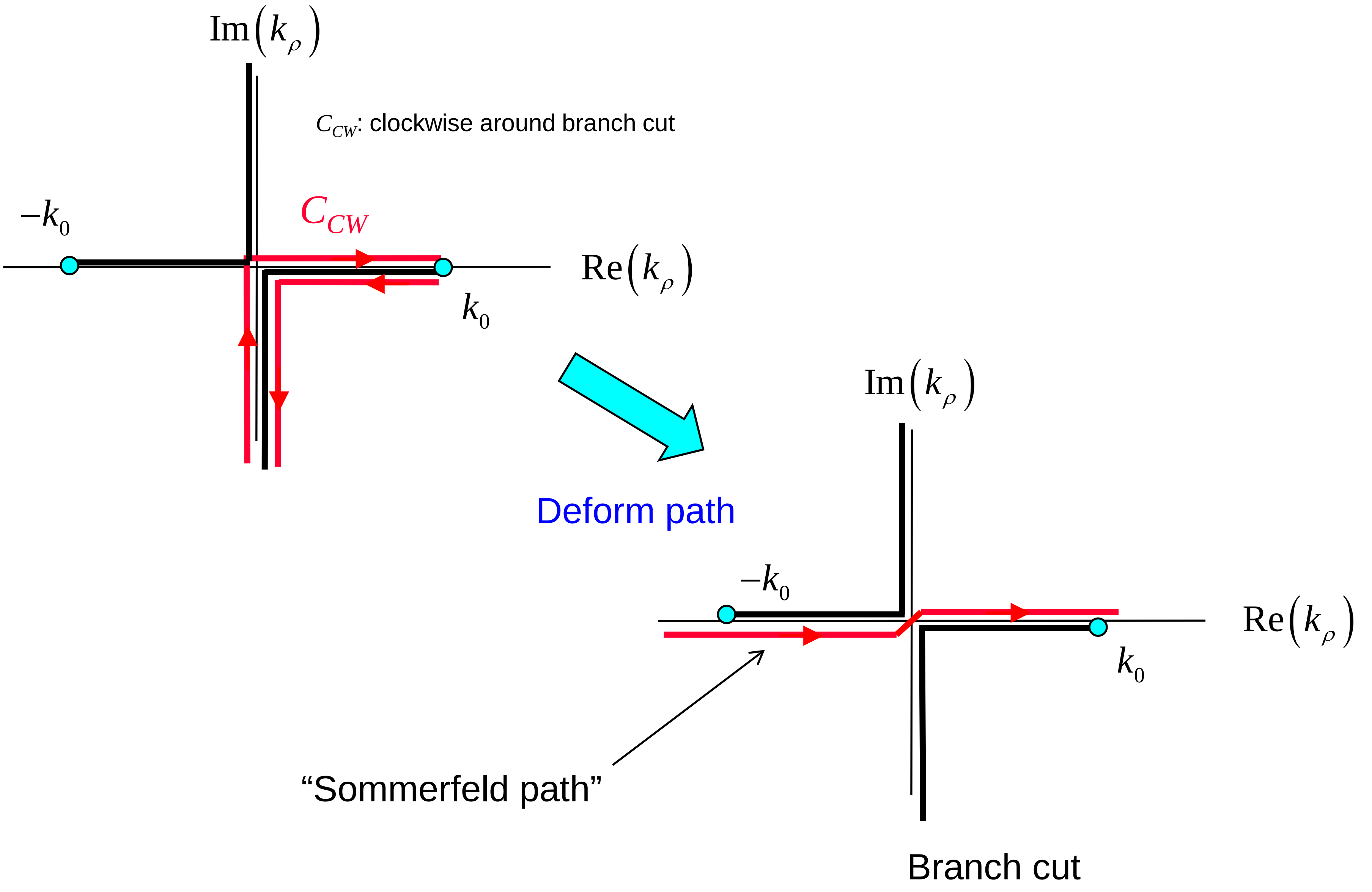
$$k_z \in (-\infty, \infty)$$

Note: The value of k_z is opposite across the branch cut.

The square root is defined so that when k_ρ is on the real axis we have:

$$k_z = -j\sqrt{k_\rho^2 - k_0^2} \quad , \quad k_\rho > k_0$$

Example (cont.)



Example (cont.)

Alternative form:

$$\frac{e^{-jkr}}{r} = -\frac{j}{2}\int_{-\infty}^{+\infty}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$$= -\frac{j}{2}\int_{-\infty}^{0}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho - \frac{j}{2}\int_{0}^{+\infty}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$k'_\rho = -k_\rho$

$$= -\frac{j}{2}\int_{\infty}^{0}\left(\frac{k'_\rho}{k_z}\right)H_0^{(2)}(-k'_\rho\rho)\,e^{-jk_z z}dk'_\rho - \frac{j}{2}\int_{0}^{+\infty}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$H_0^{(2)}(-z) = -H_0^{(1)}(+z)$

$$= +\frac{j}{2}\int_{\infty}^{0}\left(\frac{k'_\rho}{k_z}\right)H_0^{(1)}(k'_\rho\rho)\,e^{-jk_z z}dk'_\rho - \frac{j}{2}\int_{0}^{+\infty}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$$= -\frac{j}{2}\int_{0}^{\infty}\left(\frac{k'_\rho}{k_z}\right)H_0^{(1)}(k'_\rho\rho)\,e^{-jk_z z}dk'_\rho - \frac{j}{2}\int_{0}^{+\infty}\left(\frac{k_\rho}{k_z}\right)H_0^{(2)}(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$$= -j\int_{0}^{\infty}\left(\frac{k_\rho}{k_z}\right)J_0(k_\rho\rho)\,e^{-jk_z z}dk_\rho$$

$J_0(z) = \frac{1}{2}\left(H_0^{(1)}(z) + H_0^{(2)}(z)\right)$

Example (cont.)

Hence we have:

$$\frac{e^{-jkr}}{r} = \frac{1}{j} \int_0^{+\infty} \left(\frac{k_\rho}{k_z} \right) J_0(k_\rho \rho) e^{-jk_z z} dk_\rho$$

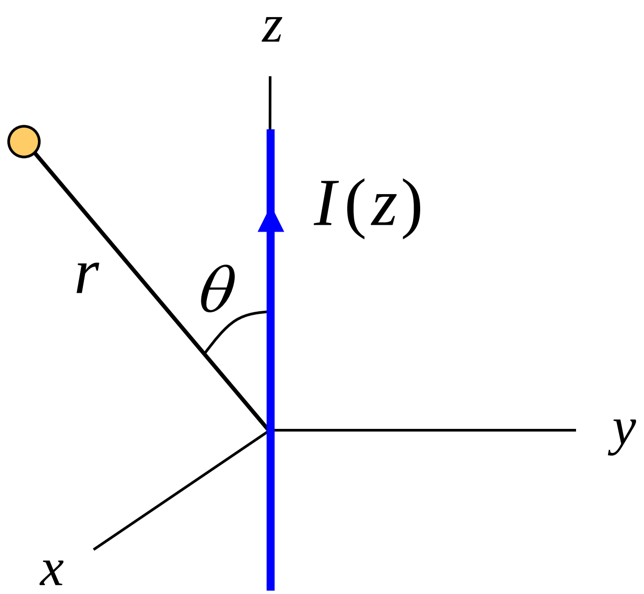
$$\frac{e^{-jkr}}{r} = \frac{1}{2j} \int_{-\infty}^{+\infty} \left(\frac{k_\rho}{k_z} \right) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_\rho$$

Sommerfeld Identities

The integrals are along a “Sommerfeld path” that stays slightly above or below the branch cuts.

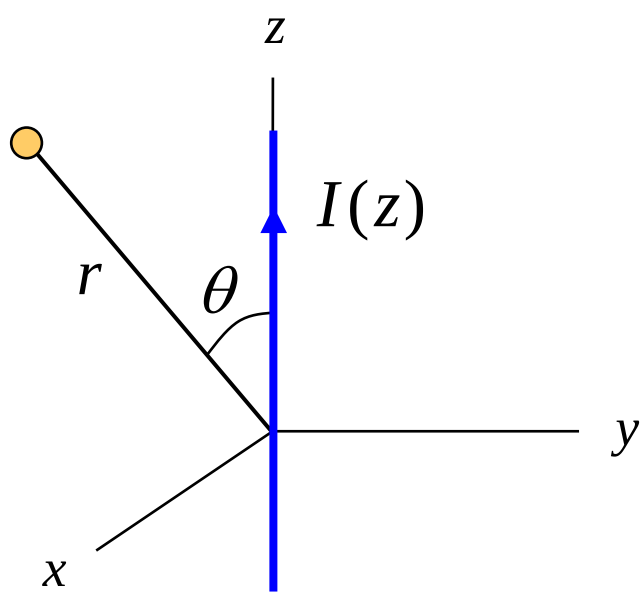
Far-Field Identity

This identity is useful for calculating the far-field of finite (3-D) sources in cylindrical coordinates.



Note: We assume that the current decays at $z = \pm\infty$ fast enough so that a 3-D far field exists.

Far-Field Identity (cont.)



Exact solution:

$$A_z = \frac{\mu_0}{8\pi j}\int_{-\infty}^{+\infty} \tilde{I}(k_z)\, H_0^{(2)}(k_\rho \rho)\, e^{-jk_z z}\, dk_z$$

Far-Field Identity (cont.)

From ECE 6340, as $r \to \infty$

$$\underline{A} \sim \frac{\mu_0}{4\pi}\left(\frac{e^{-jkr}}{r}\right)\underline{a}(\theta,\phi)$$

$$\underline{a}(\theta,\phi) = \int_V \underline{J}\left(\underline{r}'\right) e^{jk\left(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi + z'\cos\theta\right)} dx'dy'dz'$$

$$= \underline{\hat{z}}\frac{\mu_0}{4\pi}\left(\frac{e^{-jkr}}{r}\right)\int_{-\infty}^{+\infty} I(z')\, e^{jkz'\cos\theta} dz'$$

Hence

$$A_z \sim \frac{\mu_0}{4\pi}\left(\frac{e^{-jkr}}{r}\right)\tilde{I}(k\cos\theta)$$

Far-Field Identity (cont.)

Hence, comparing these two,

$$\frac{\mu_0}{8\pi j}\int_{-\infty}^{+\infty}\tilde{I}(k_z)H_0^{(2)}(k_\rho\rho)e^{-jk_z z}dk_z \sim \frac{\mu_0}{4\pi}\left(\frac{e^{-jkr}}{r}\right)\tilde{I}(k\cos\theta)$$

or

$$\int_{-\infty}^{+\infty}\tilde{I}(k_z)H_0^{(2)}(k_\rho\rho)e^{-jk_z z}dk_z \sim 2j\left(\frac{e^{-jkr}}{r}\right)\tilde{I}(k\cos\theta)$$

as $r \to \infty$

Far-Field Identity (cont.)

To generalize this identity, use

$$H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-j\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$

Hence

$$H_n^{(2)}(x) \sim j^n H_0^{(2)}(x)$$

Far-Field Identity (cont.)

Therefore, we have

$$\int_{-\infty}^{+\infty} \tilde{I}(k_z)\, H_n^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z \sim 2\, j^{n+1} \left(\frac{e^{-jkr}}{r} \right) \tilde{I}(k \cos\theta)$$

Note: This is valid for $\rho \to \infty$

Hence, this is valid for $r \to \infty, \quad \theta \neq 0, 180^\circ$

Far-Field Identity (cont.)

Since the current function is arbitrary, we can write

$$\int_{-\infty}^{+\infty} f(k_z) H_n^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z \sim 2j^{n+1} \left(\frac{e^{-jkr}}{r} \right) f(k\cos\theta)$$

for $r \to \infty, \quad \theta \neq 0, 180^\circ$