ECE 6341

Spring 2016

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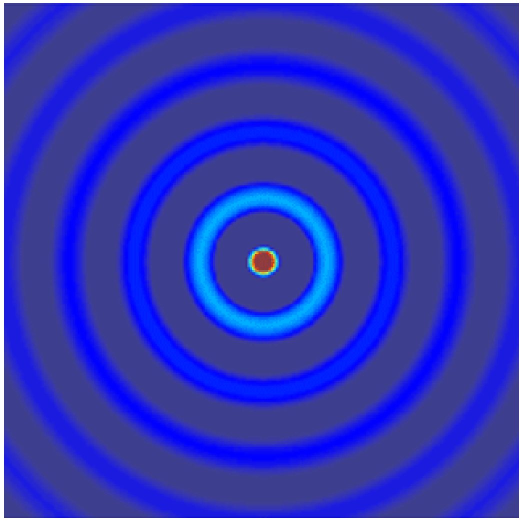
ECE Dept.

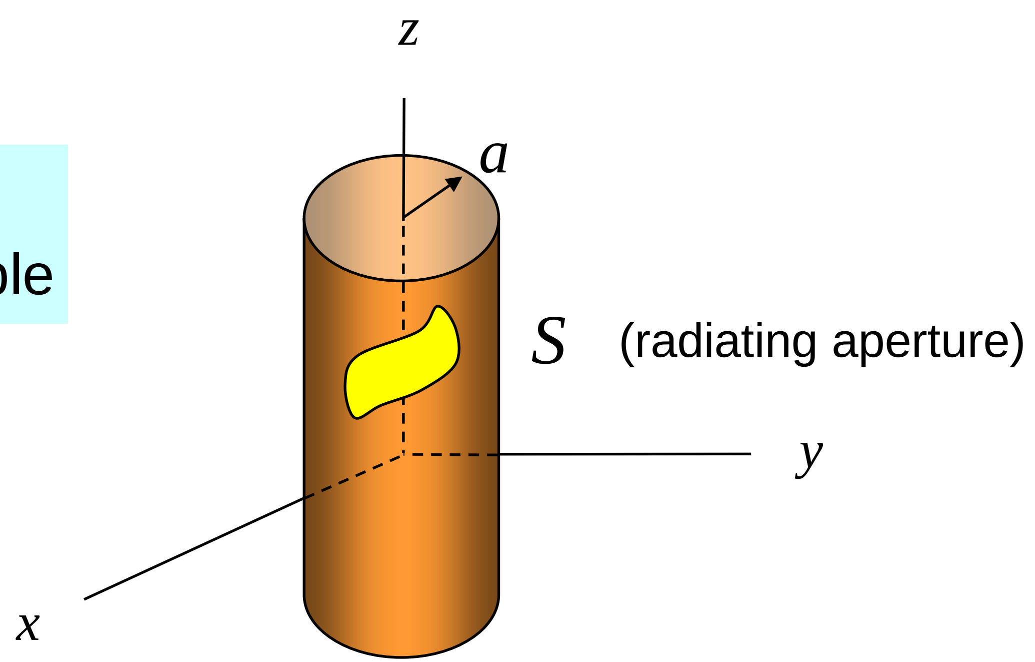


Notes 17

Radiation From Aperture in Cylinder

$\upsilon = n =$ integer

$k_z =$ continuous variable



$$A_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{+\infty} f_n(k_z) H_n^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

$$F_z = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{+\infty} g_n(k_z) H_n^{(2)}\left(k_\rho \rho\right) e^{-jk_z z} dk_z$$

Note: The factor of 2π is added for convenience.

Aperture in Cylinder (cont.)

From the TM_z/TE_z Tables:

$$E_z = \frac{1}{j\omega\mu_0\varepsilon_0}\left(k_0^2 + \frac{\partial^2}{\partial z^2}\right)A_z$$

$$E_\phi = \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z} + \frac{1}{\varepsilon_0}\frac{dF_z}{d\rho}$$

Hence, on the aperture we have

$$E_z(a,\phi,z) = \frac{1}{2\pi}\frac{1}{j\omega\mu_0\varepsilon_0}\sum_{n=-\infty}^{+\infty} e^{jn\phi}\int_{-\infty}^{+\infty} f_n(k_z)\,H_n^{(2)}\left(k_\rho a\right)\underbrace{\left(k_0^2 - k_z^2\right)}_{k_\rho^2} e^{-jk_z z}\,dk_z$$

$$E_\phi(a,\phi,z) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{jn\phi}\int_{-\infty}^{+\infty}\left\{\begin{array}{l}\dfrac{1}{j\omega\mu_0\varepsilon_0}\dfrac{1}{a}(jn)(-jk_z)\,f_n(k_z)\,H_n^{(2)}\left(k_\rho a\right)\\[2ex] +\dfrac{1}{\varepsilon_0}k_\rho\, g_n(k_z)H_n^{(2)\prime}\left(k_\rho a\right)\end{array}\right\}e^{-jk_z z}\,dk_z$$

Aperture in Cylinder (cont.)

First, we'll work with E_z.

To simplify the notation, let

$$a_n(k_z) \equiv \frac{1}{j\omega\mu_0\varepsilon_0} f_n(k_z)\left(k_\rho^2\right) H_n^{(2)}\left(k_\rho a\right)$$

Then

$$E_z\left(a,\phi,z\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{jn\phi}\int_{-\infty}^{+\infty} a_n(k_z)\, e^{-jk_z z}\, dk_z$$

Aperture in Cylinder (cont.)

Multiply by $e^{-jm\phi}$ and integrate over $[0, 2\pi]$

$$\int_0^{2\pi} E_z(a,\phi,z)\, e^{-jm\phi}\, d\phi = \left(\frac{1}{2\pi}\right)(2\pi)\int_{-\infty}^{+\infty} a_m(k_z)\, e^{-jk_z z}\, dk_z$$

(from orthogonality, with $m = n$)

Next, divide both sides by 2π:

$$\frac{1}{2\pi}\int_0^{2\pi} E_z(a,\phi,z)\, e^{-jm\phi}\, d\phi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} a_m(k_z)\, e^{-jk_z z}\, dk_z$$

$$= F^{-1} a_m(k_z)$$

Aperture in Cylinder (cont.)

Hence

$$a_m(k_z) = F\left\{\frac{1}{2\pi}\int_0^{2\pi} E_z(a,\phi,z)\,e^{-jm\phi}\,d\phi\right\}$$

$$= \int_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_0^{2\pi} E_z(a,\phi,z)\,e^{-jm\phi}\,d\phi\right]e^{jk_z z}\,dz$$

Define "cylindrical transform":

$$F_c\left\{f(\phi,z)\right\} = \overline{f}(m,k_z) \equiv \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} f(\phi,z)\,e^{-jm\phi}\,e^{+jk_z z}\,dz\,d\phi$$

Then

$$a_m(k_z) = \overline{E}_z(a,m,k_z)$$

Aperture in Cylinder (cont.)

Recall that

$$a_n(k_z) \equiv \frac{1}{j\omega\mu_0\varepsilon_0} f_n(k_z)\left(k_\rho^2\right) H_n^{(2)}\left(k_\rho a\right)$$

Hence

$$f_n(k_z) = \frac{j\omega\mu_0\varepsilon_0 \bar{E}_z(a,n,k_z)}{k_\rho^2 H_n^{(2)}(k_\rho a)}$$

Similarly, (details omitted) we have

$$g_n(k_z) = \frac{\varepsilon_0}{k_\rho H_n^{(2)\prime}\left(k_\rho a\right)}\left[\bar{E}_\phi\left(a,n,k_z\right) - \frac{nk_z}{ak_\rho^2}\bar{E}_z(a,n,k_z)\right]$$

Aperture in Cylinder (cont.)

Now use the far-field identity:

$$\begin{pmatrix} A_z \\ F_z \end{pmatrix} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\phi} \int_{-\infty}^{+\infty} \begin{pmatrix} f_n(k_z) \\ g_n(k_z) \end{pmatrix} H_n^{(2)}(k_\rho \rho)\, e^{-jk_z z} dk_z$$

$$\Big\Downarrow \quad r \to \infty$$

$$\begin{pmatrix} A_z \\ F_z \end{pmatrix} \sim \frac{1}{2\pi} \left(\frac{e^{-jk_0 r}}{r} \right) \sum_{n=-\infty}^{+\infty} e^{jn\phi} \left(2 j^{n+1} \right) \begin{pmatrix} f_n(k_0 \cos\theta) \\ g_n(k_0 \cos\theta) \end{pmatrix}$$

As before,

$$E_\theta \sim j\omega \sin\theta A_z$$

Aperture in Cylinder (cont.)

Also, from duality,

$$H_\theta \sim j\omega \sin\theta F_z$$

Hence

$$E_\phi \sim -\eta_0 H_\theta$$
$$\sim -j\omega\eta_0 \sin\theta F_z$$

or

$$E_\phi \sim -j\frac{k_0}{\varepsilon_0}\sin\theta F_z$$

Aperture in Cylinder (cont.)

Summary

$$E_\theta \sim j\omega \sin\theta A_z$$

$$E_\phi \sim -j\frac{k_0}{\varepsilon_0}\sin\theta F_z$$

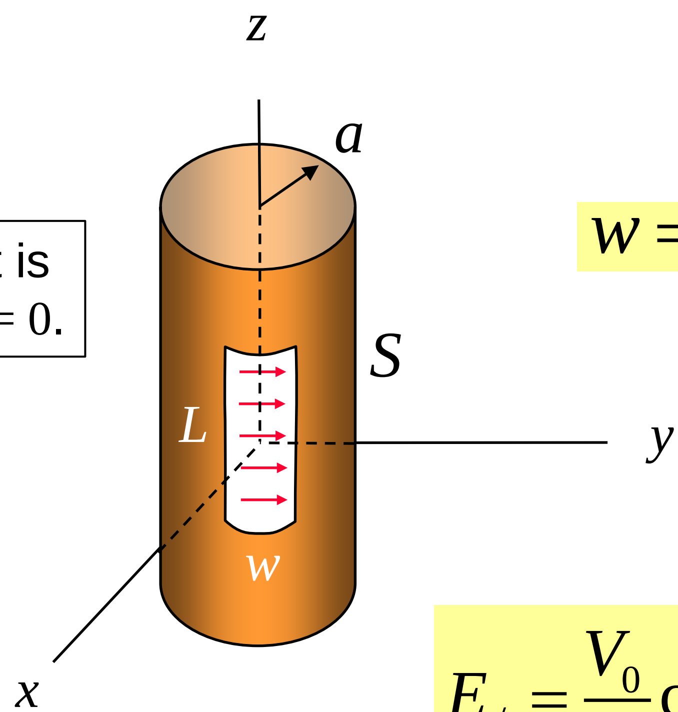
$$\begin{pmatrix} A_z \\ F_z \end{pmatrix} \sim \frac{1}{2\pi}\left(\frac{e^{-jk_0 r}}{r}\right)\sum_{n=-\infty}^{+\infty} e^{jn\phi}\left(2j^{n+1}\right)\begin{pmatrix} f_n(k_0\cos\theta) \\ g_n(k_0\cos\theta) \end{pmatrix}$$

$$f_n(k_z) = \frac{j\omega\mu_0\varepsilon_0 \bar{E}_z(a,n,k_z)}{k_\rho^2 H_n^{(2)}(k_\rho a)}$$

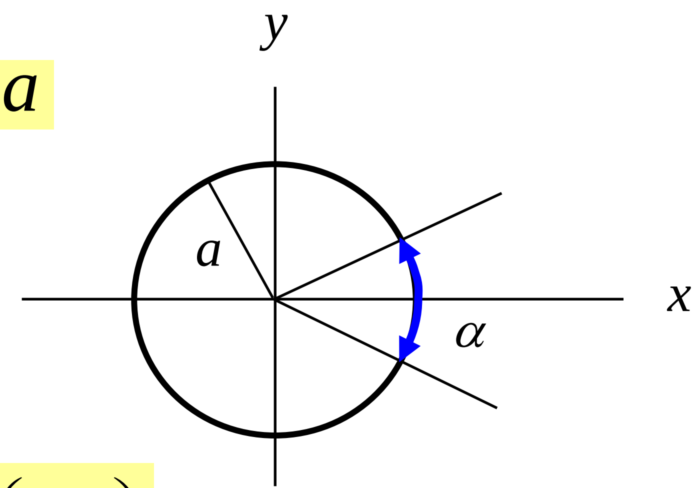
$$g_n(k_z) = \frac{\varepsilon_0}{k_\rho H_n^{(2)\prime}\left(k_\rho a\right)}\left[\bar{E}_\phi(a,n,k_z) - \frac{nk_z}{ak_\rho^2}\bar{E}_z(a,n,k_z)\right]$$

Example

A vertical slot is centered at $z = 0$.



$$w = \alpha a$$



$$E_\phi = \frac{V_0}{w}\cos\left(\frac{\pi z}{L}\right)$$

$$\overline{E}_z = 0$$

$$\overline{E}_\phi(a,n,k_z) = \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi(a,\phi,z)\,e^{-jn\phi}e^{+jk_z z}\,dz\,d\phi$$

Example (cont.)

$$\bar{E}_\phi(a,n,k_z) = \frac{1}{2\pi}\int_{-\alpha/2}^{\alpha/2}\left(\frac{V_0}{w}\right)e^{-jn\phi}d\phi\int_{-L/2}^{L/2}\cos\left(\frac{\pi z}{L}\right)e^{jk_z z}dz$$

$$= \left(\frac{1}{2\pi}\right)\left[\left(\frac{V_0}{w}\right)\frac{\alpha\sin\left(\frac{n\alpha}{2}\right)}{\left(\frac{n\alpha}{2}\right)}\right]\left[\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-(k_z L)^2}(2\pi L)\right]$$

Simplifying, we have

$$\bar{E}_\phi(a,n,k_z) = \frac{V_0 L}{a}\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\left[\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-(k_z L)^2}\right]$$

$$\operatorname{sinc}(x) \equiv \frac{\sin(x)}{x}$$

Example (cont.)

From previous analysis:

$$f_n(k_z) = \frac{j\omega\mu_0\varepsilon_0 \bar{E}_z(a,n,k_z)}{k_\rho^2 H_n^{(2)}(k_\rho a)}$$

$$g_n(k_z) = \frac{\varepsilon_0}{k_\rho H_n^{(2)\prime}\left(k_\rho a\right)}\left[\bar{E}_\phi\left(a,n,k_z\right) - \frac{nk_z}{a{k_\rho}^2}\bar{E}_z(a,n,k_z)\right]$$

Hence, we have

$$f_n(k_z) = 0$$

$$g_n(k_z) = \frac{\varepsilon_0 \bar{E}_\phi\left(a,n,k_z\right)}{k_\rho H_n^{(2)\prime}\left(k_\rho a\right)}$$

The field is TE_z polarized.

Example (cont.)

Recall that

$$E_\phi \sim -j\frac{k_0}{\varepsilon_0}\sin\theta\, F_z$$

$$F_z \sim \frac{1}{2\pi}\left(\frac{e^{-jk_0 r}}{r}\right)\sum_{n=-\infty}^{+\infty} e^{jn\phi}\left(2\,j^{n+1}\right)g_n\left(k_0\cos\theta\right)$$

where we now have

$$g_n(k_z) = \frac{\varepsilon_0 \bar{E}_\phi\left(a,n,k_z\right)}{k_\rho\, H_n^{(2)\prime}\left(k_\rho a\right)} \qquad \bar{E}_\phi\left(a,n,k_z\right) = \frac{VL}{a}\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\left[\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-\left(k_z L\right)^2}\right]$$

Example (cont.)

Hence, after simplifying, we have

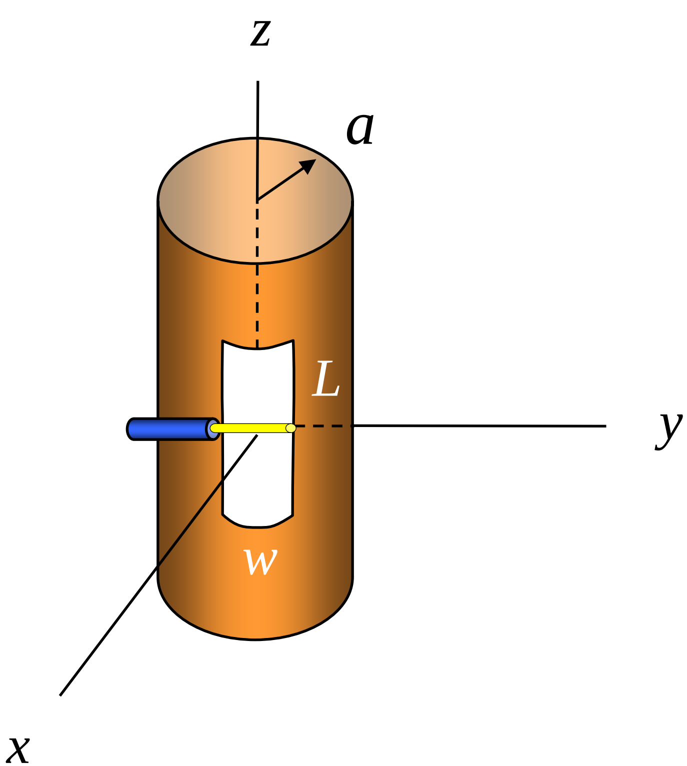
$$E_\theta = 0$$

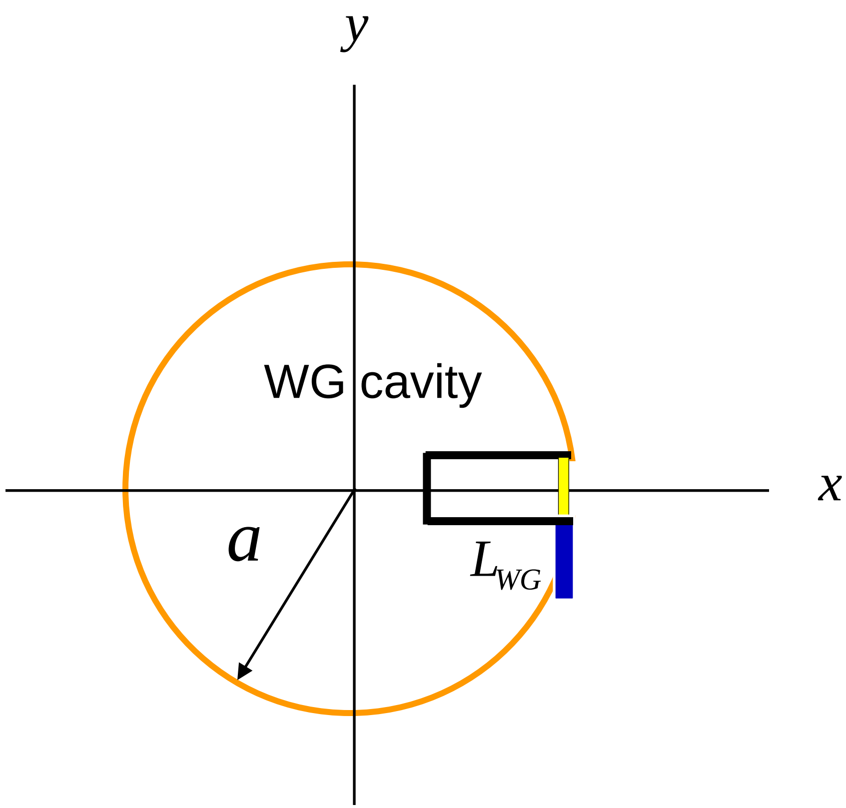
$$E_\phi \sim \left(\frac{V}{\pi}\right)\left(\frac{L}{a}\right)\left(\frac{e^{-jk_0 r}}{r}\right)\sum_{n=-\infty}^{+\infty} j^n e^{jn\phi}\left(\frac{1}{H_n^{(2)\prime}(k_0 a\sin\theta)}\right)\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\left(\frac{\cos\left(\frac{1}{2}k_0 L\cos\theta\right)}{\pi^2-\left(k_0 L\cos\theta\right)^2}\right)$$

Where we have used:

In far field: $k_z = k_0\cos\theta \quad \Rightarrow \quad k_\rho = \left(k_0^2 - k_z^2\right)^{1/2} = k_0\sin\theta$

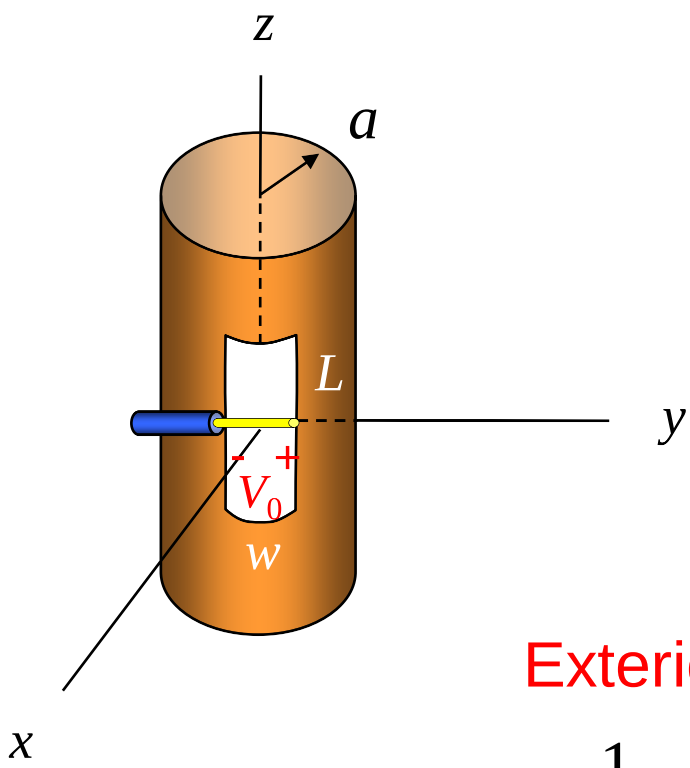
Input Impedance

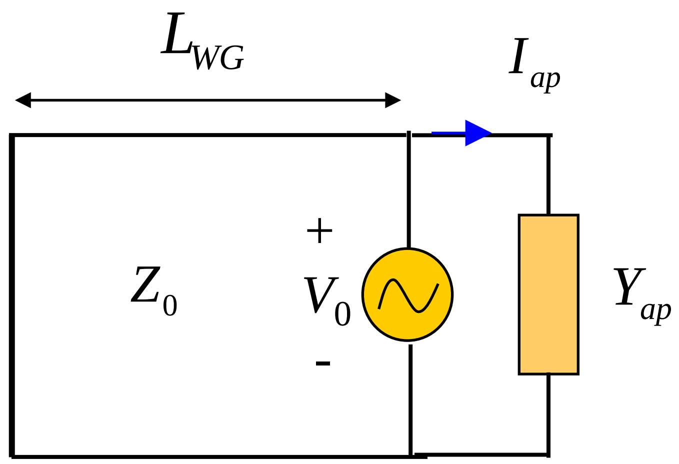




The slot is assumed to be backed by a cavity, which is a short-circuited length of rectangular waveguide ($L \times w$) operating in the TE_{10} mode.

Input Impedance (cont.)





Circuit Model

Exterior modeling

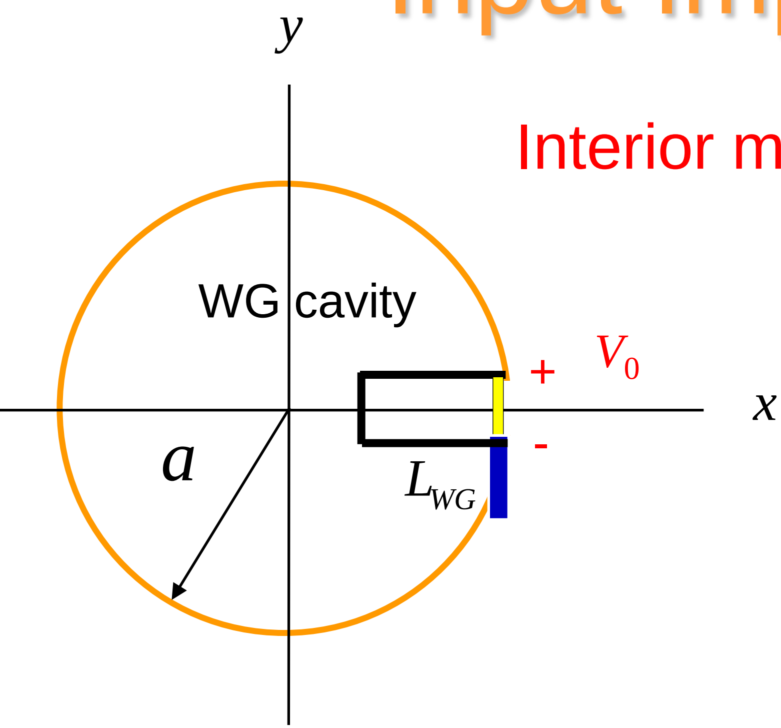
$$P_s = \frac{1}{2} V_0 I_{ap}^* = \frac{1}{2} Y_{ap}^* \left|V_0\right|^2$$

P_s = complex power radiated into exterior

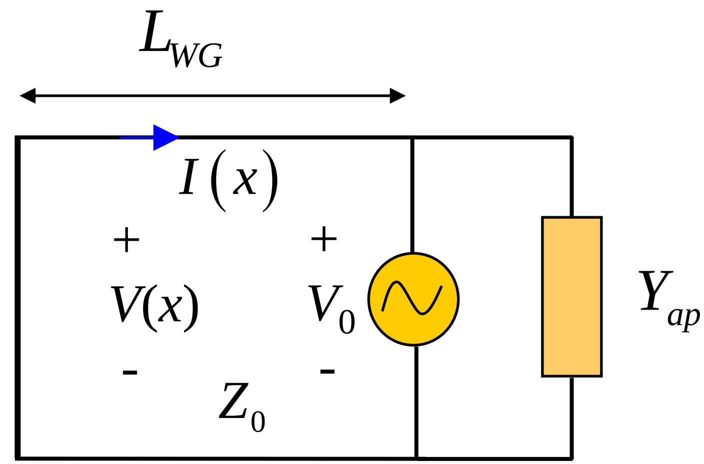
so

$$Y_{ap} = \frac{2P_s^*}{\left|V_0\right|^2}$$

Input Impedance (cont.)



Interior modeling

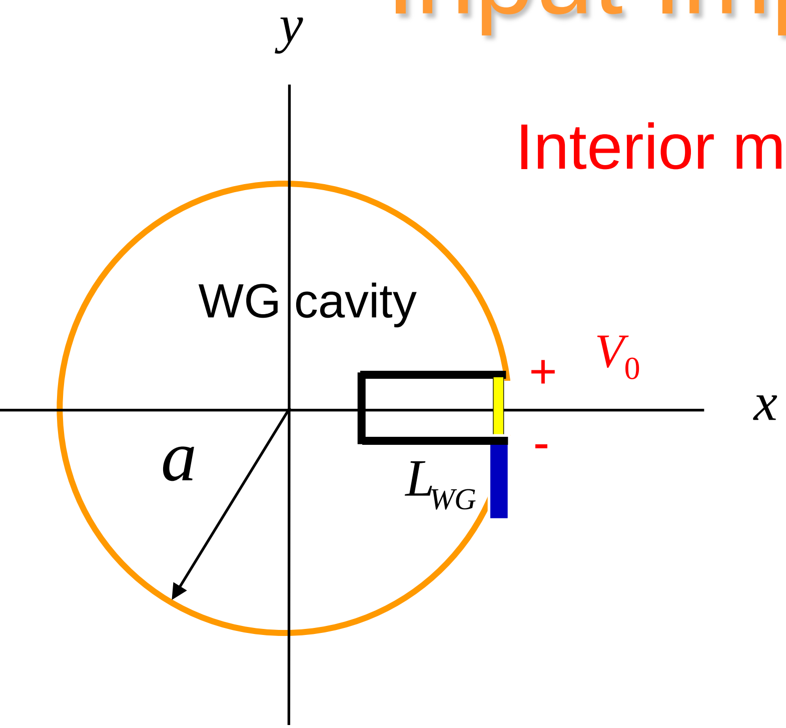


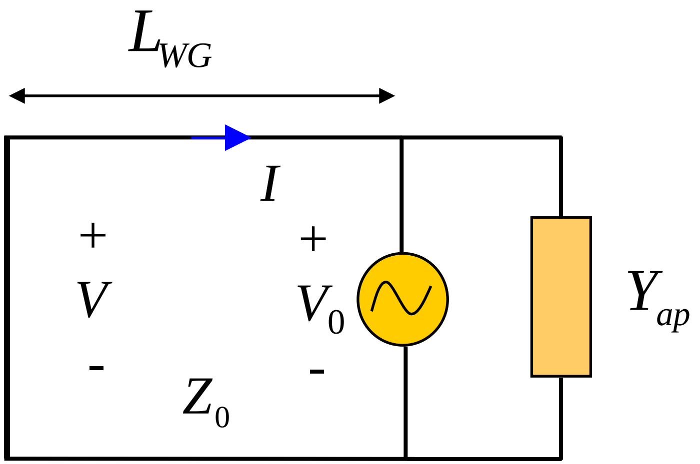
$V \equiv WG$ centerline voltage

The voltage on the TL is chosen as the actual voltage at the center of the WG.

$$V(x) = E_{yc}(x)\, w$$

Input Impedance (cont.)





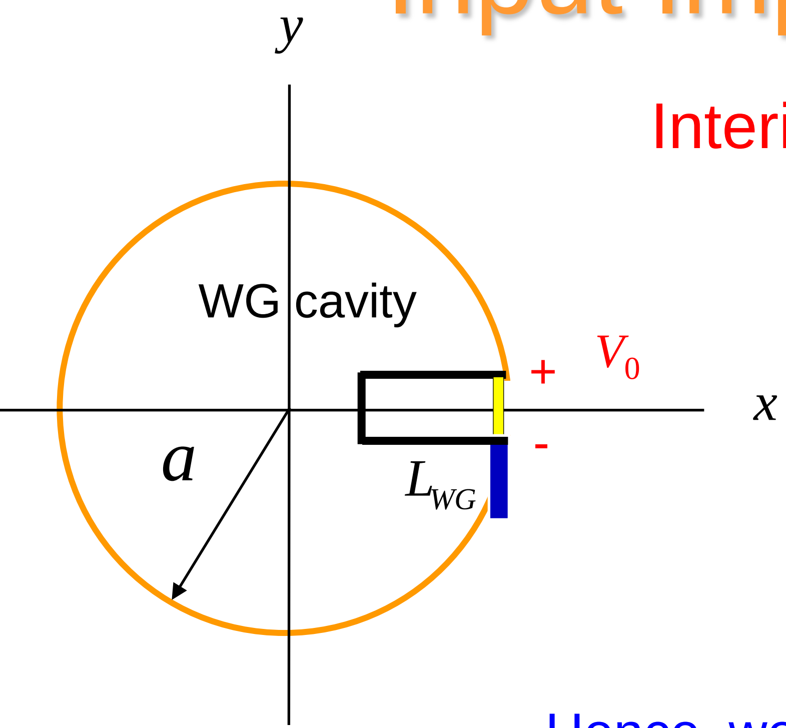
$V \equiv WG$ centerline voltage

To find Z_0, equate complex power flows on WG and TL:

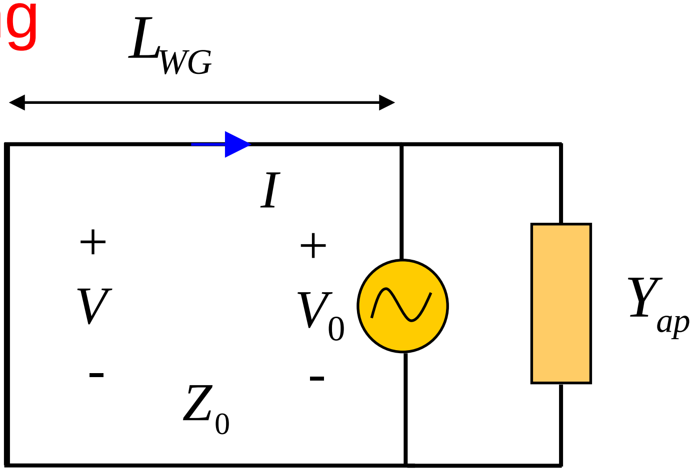
$$\frac{1}{2}VI^* = \frac{1}{2}\frac{|V|^2}{Z_0} = \int_0^w\int_0^L \frac{1}{2}E_y H_z^*\,dzdy = \int_0^w\int_0^L \frac{1}{2}E_y\left(\frac{E_y}{Z_{10}^{TE}}\right)^* dzdy = \int_0^w\int_0^L \frac{1}{2}\frac{|E_{yc}|^2}{Z_{10}^{TE}}\sin^2\left(\frac{\pi z}{L}\right)dzdy$$

$$= \frac{1}{2}w\left(\frac{L}{2}\right)\frac{|E_{yc}|^2}{Z_{10}^{TE}} = \left(\frac{wL}{4}\right)\frac{1}{Z_{10}^{TE}}\left|\frac{V}{w}\right|^2$$

Input Impedance (cont.)



Interior modeling



$V \equiv WG$ centerline voltage

Hence, we have

$$\frac{1}{2}\frac{|V|^2}{Z_0} = \left(\frac{wL}{4}\right)\frac{1}{Z_{10}^{TE}}\left|\frac{V}{w}\right|^2$$

or

$$Z_0 = 2\left(\frac{w}{L}\right) Z_{10}^{TE}$$

where

$$Z_{10}^{TE} = \eta_0 / \sqrt{1-\left(\pi / \left(k_0 L\right)\right)^2}$$

Input Impedance (cont.)

Exterior modeling of aperture

Aperture admittance: $Y_{ap} = \dfrac{2P_s^*}{|V_0|^2}$

Hence, we have

$$Y_{ap} = \left(\frac{2}{|V_0|^2}\right)\frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi^* H_z \, a \, dz \, d\phi$$

where

$$H_z = \frac{1}{j\omega\mu_0\varepsilon_0}\left(k_0^2 + \frac{\partial^2}{\partial z^2}\right)F_z$$

$$= \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{jn\phi}\int_{-\infty}^{+\infty} k_\rho^2 \, g_n(k_z) H_n^{(2)}(k_\rho a) e^{-jk_z z} dk_z$$

Input Impedance (cont.)

Hence

$$Y_{ap} = \left(\frac{2}{|V_0|^2}\right)\left(\frac{1}{2}\right)\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi^*(a,\phi,z)$$

$$\left[\left(\frac{1}{j\omega\mu_0\varepsilon_0}\right)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{jn\phi}\int_{-\infty}^{+\infty} k_\rho^2\, g_n(k_z)\, H_n^{(2)}(k_\rho a)\, e^{-jk_z z} dk_z\right] a\, dz\, d\phi$$

Ordering the spatial integrations first, we have

$$Y_{ap} = \left(\frac{a}{|V_0|^2}\right)\frac{1}{j\omega\mu_0\varepsilon_0}\left(\frac{1}{2\pi}\right)\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty} k_\rho^2\, g_n(k_z) H_n^{(2)}(k_\rho a)$$

$$\cdot\left[\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi^*(a,\phi,z)\, e^{jn\phi} e^{-jk_z z} dz d\phi\right] dk_z$$

Input Impedance (cont.)

For the term in brackets,

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi^*(a,\phi,z)\,e^{jn\phi}e^{-jk_z z}\,dz\,d\phi = \left\{\int_0^{2\pi}\int_{-\infty}^{+\infty} E_\phi(a,\phi,z)\,e^{-jn\phi}e^{+jk_z z}\,dz\,d\phi\right\}^*$$

$$= 2\pi\bar{E}_\phi^*(a,n,k_z)$$

Hence

$$Y_{ap} = \left(\frac{a}{|V_0|^2}\right)\left(\frac{1}{j\omega\mu_0\varepsilon_0}\right)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty} k_\rho^2\, g_n(k_z)\, H_n^{(2)}(k_\rho a)$$

$$\cdot 2\pi\left[\left(\frac{V_0 L}{a}\right)\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-(k_z L)^2}\right]^* dk_z$$

Input Impedance (cont.)

Substituting for g_n, we have

$$\text{Recall}: \; g_n(k_z) = \frac{\varepsilon_0 \bar{E}_\phi(a,n,k_z)}{k_\rho H_n^{(2)\prime}(k_\rho a)}$$

$$Y_{ap} = \left(\frac{a}{|V_0|^2}\right)\left(\frac{1}{j\omega\mu_0\varepsilon_0}\right)\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty} k_\rho^2 \left(\frac{\varepsilon_0}{k_\rho H_n^{(2)\prime}(k_\rho a)}\right) H_n^{(2)}(k_\rho a)$$

$$\cdot \left[\frac{V_0 L}{a}\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-(k_z L)^2}\right]\left[2\pi\frac{V_0 L}{a}\operatorname{sinc}\left(\frac{n\alpha}{2}\right)\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2-(k_z L)^2}\right]^* dk_z$$

Input Impedance (cont.)

Simplifying, we have:

$$Y_{ap} = \left(\frac{a}{j\omega\mu_0}\right)\left(\frac{L}{a}\right)^2 \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} k_\rho \left(\frac{H_n^{(2)}(k_\rho a)}{H_n^{(2)\prime}(k_\rho a)}\right)$$

$$\cdot \operatorname{sinc}^2\left(\frac{n\alpha}{2}\right)\left(\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2 - (k_z L)^2}\right)^2 dk_z$$

Input Impedance (cont.)

In normalized form:

$$Y_{ap} = -j\left(\frac{k_0 a}{\eta_0}\right)\left(\frac{L}{a}\right)^2 \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} \bar{k}_\rho \left(\frac{H_n^{(2)}(k_\rho a)}{H_n^{(2)\prime}(k_\rho a)}\right) \cdot \operatorname{sinc}^2\left(\frac{n\alpha}{2}\right)\left(\frac{\cos\left(\frac{k_z L}{2}\right)}{\pi^2 - (k_z L)^2}\right)^2 d\bar{k}_z$$

$$\bar{k}_z = k_z / k_0$$

$$d\bar{k}_z = dk_z / k_0$$

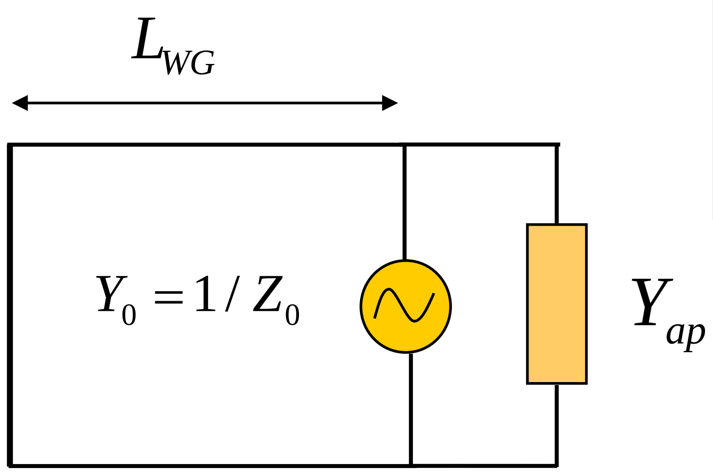
$$\bar{k}_\rho = k_\rho / k_0$$

Input Impedance (cont.)

The total input admittance seen by the feed is then:

$$Z_{in} = 1 / Y_{in} \qquad Y_{in} = Y_{stub} + Y_{ap}$$

$$Y_{stub} = -jY_0 \cot\left(k_z^{10} L_{WG}\right)$$



WG stub

$$Z_0 = 2\left(\frac{w}{L}\right) Z_{10}^{TE}$$

Note:
The model neglects the effects of higher-order waveguide modes.
(This could be accounted for approximately by adding a probe reactance to the input impedance.)

$$X_{probe} \approx \frac{\eta_0}{2\pi}(k_0 w)\left(-\gamma + \ln\left(\frac{2}{k_0 a_{probe}}\right)\right)$$