ECE 6341

Spring 2016

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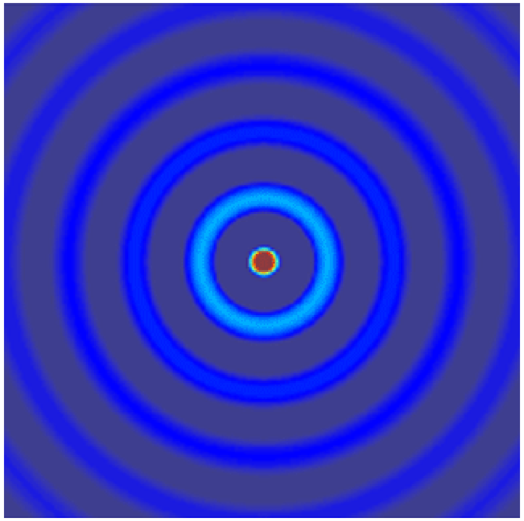
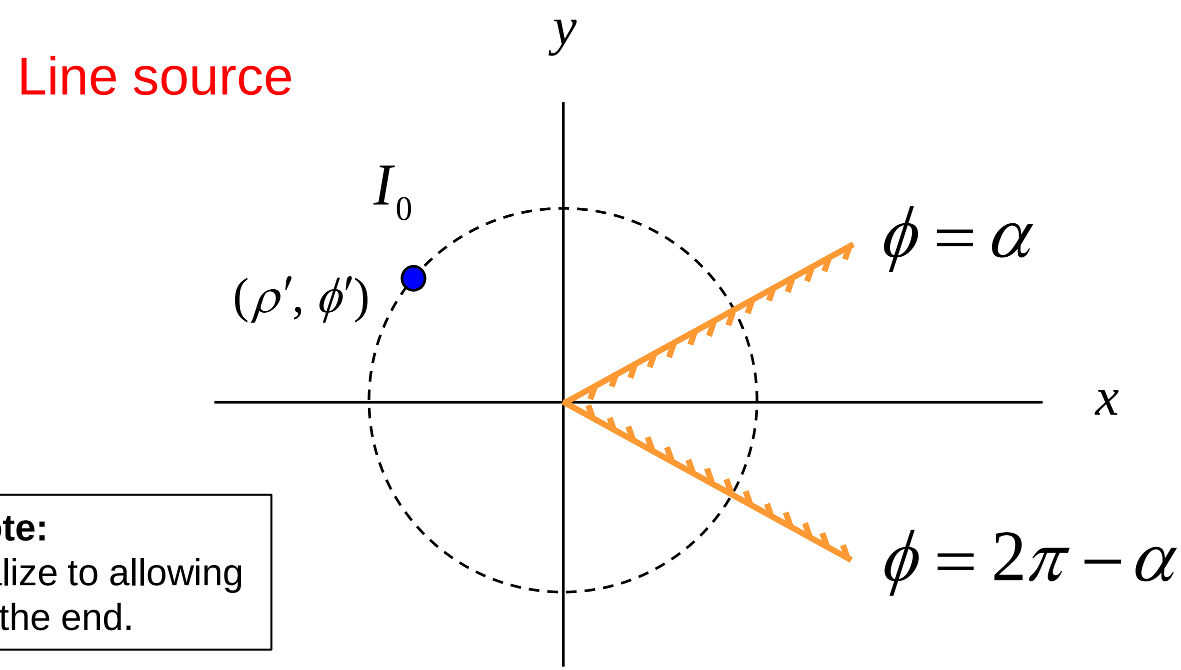
Notes 18



Scattering by Wedge



Note:
We will generalize to allowing for k_z at the end.

Assume TM_z: $A_z(\rho, \phi)$

Boundary conditions: $A_z = 0$ at $\phi = \alpha,\ 2\pi - \alpha$

Scattering by Wedge (cont.)

Let

$$\Phi(\phi) = h(\phi) = A\sin\upsilon(\phi-\alpha) + \cancel{B}\cos\upsilon(\phi-\alpha)$$

(B.C. at $\phi = \alpha$)

$\phi = 2\pi - \alpha$: $\sin\upsilon\left(\left(2\pi-\alpha\right)-\alpha\right)=0$

so $\upsilon\left(2\pi-2\alpha\right)=n\pi$

$$\Rightarrow \quad \upsilon = \upsilon_n = \frac{n\pi}{2\left(\pi-\alpha\right)}$$

Scattering by Wedge (cont.)

$$A_z = \sum_n B_{\upsilon_n}(k\rho)\sin\upsilon_n(\phi-\alpha)$$

Bessel function of order υ_n

Note: $n = 0 \;\Rightarrow\; \upsilon_n = 0$ (trivial solution)

Note: $n < 0 \;\Rightarrow\; \upsilon_n < 0$

$\Rightarrow$ $J_{\upsilon_n}(k\rho) \to \infty$ as $\rho \to 0$

since $J_\upsilon(x) \sim x^\upsilon \left(\dfrac{1}{2^\upsilon \Gamma(\upsilon+1)}\right)$ ($\upsilon \neq$ negative integer)

Scattering by Wedge (cont.)

Hence

$$\upsilon = \upsilon_n = \frac{n\pi}{2\left(\pi - \alpha\right)}$$

$$n = 1, 2, 3 \ldots$$

$$\upsilon_1 = \frac{\pi}{2\left(\pi - \alpha\right)}, \text{ etc.}$$

Scattering by Wedge (cont.)

$\rho < \rho'$ $$A_{z1} = \sum_{n=1}^{\infty} a_n \sin \upsilon_n \left(\phi - \alpha\right) J_{\upsilon_n}(k\rho)$$

For $\rho > \rho'$ assume $\upsilon_n > 0$ to match with the interior form.

$\rho > \rho'$ $$A_{z1} = \sum_{n=1}^{\infty} b_n \sin \upsilon_n \left(\phi - \alpha\right) H_{\upsilon_n}^{(2)}(k\rho)$$

Scattering by Wedge (cont.)

B.C.'s

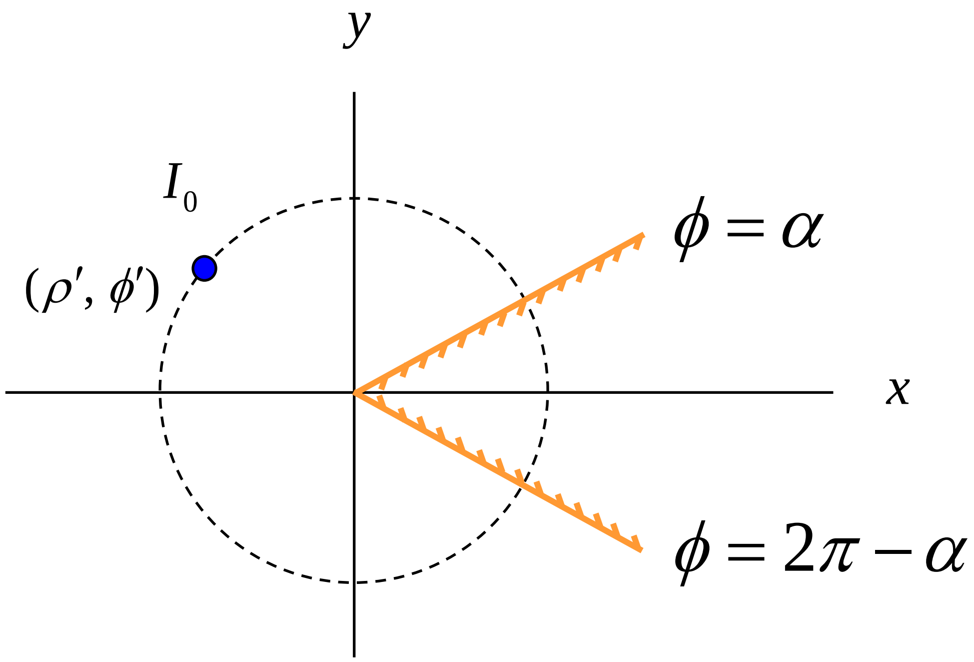
$$\rho = \rho'$$

$$E_{z1} = E_{z2}$$

$$H_{\phi 2} - H_{\phi 1} = J_{sz}\left(\phi\right) = \frac{I_0}{\rho'}\delta(\phi - \phi')$$

where

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_{z2}}{\partial \rho}$$



Scattering by Wedge (cont.)

Hence we have

$$\rho = \rho'$$

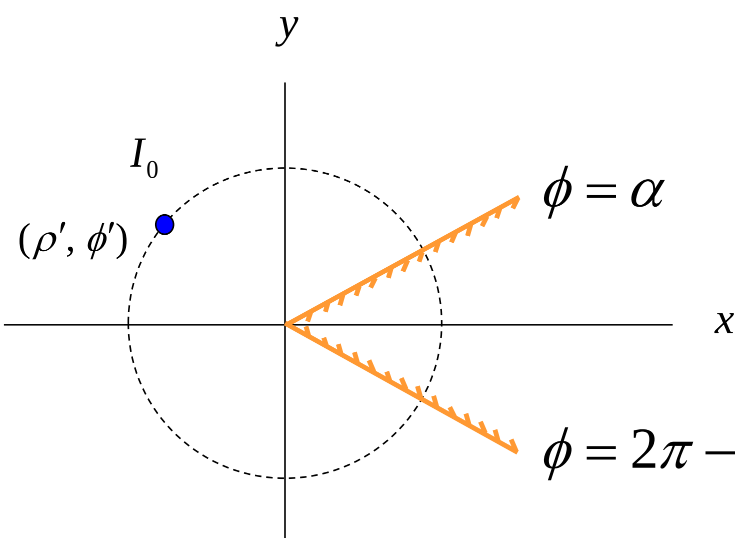
$$A_{z1} = A_{z2}$$

$$\left(-\frac{1}{\mu}\frac{\partial A_{z2}}{\partial \rho}\right) - \left(-\frac{1}{\mu}\frac{\partial A_{z1}}{\partial \rho}\right) = \frac{I_0}{\rho'}\delta(\phi - \phi')$$

Scattering by Wedge (cont.)

First B.C. :

$$A_{z1} = A_{z2}$$



$$\sum_{n=1}^{\infty} a_n J_{\upsilon_n}(k\rho') \sin\upsilon_n(\phi-\alpha) = \sum_{n=1}^{\infty} b_n H_{\upsilon_n}^{(2)}(k\rho') \sin\upsilon_n(\phi-\alpha)$$

Multiply by $\sin\upsilon_m(\phi-\alpha)$ and integrate over $\phi \in [\alpha, 2\pi-\alpha]$

Scattering by Wedge (cont.)

$$\int_{\alpha}^{2\pi-\alpha} \sin \upsilon_n(\phi-\alpha) \sin \upsilon_m(\phi-\alpha)\, d\phi = \begin{cases} 0, & m \neq n \\ \frac{1}{2}\left(2\pi - 2\alpha\right), & m = n \end{cases}$$

Note: To evaluate this integral, use $x = (\phi-\alpha)\left[\dfrac{\pi}{2\left(\pi-\alpha\right)}\right]$

Recall: $\upsilon_n = n\left(\dfrac{\pi}{2\left(\pi-\alpha\right)}\right)$

$$\int_{\alpha}^{2\pi-\alpha} \sin \upsilon_n(\phi-\alpha) \sin \upsilon_m(\phi-\alpha)\, d\phi = \frac{2\left(\pi-\alpha\right)}{\pi} \int_0^{\pi} \sin\left(nx\right) \sin\left(mx\right) dx$$

Hence $$a_m J_{\upsilon_m}(k\rho') = b_m H_{\upsilon_m}^{(2)}(k\rho')$$

Scattering by Wedge (cont.)

Second B.C. :

$$-\frac{1}{\mu}(k)\sum_{n=1}^{\infty}\sin\upsilon_n(\phi-\alpha)\left[b_n H_{\upsilon_n}^{(2)\prime}(k\rho')-a_n J_{\upsilon_n}'(k\rho')\right]$$

$$=\frac{I_0}{\rho'}\delta(\phi-\phi')$$

Multiply by $\sin\upsilon_m(\phi-\alpha)$ and integrate over $\phi\in\left[\alpha,2\pi-\alpha\right]$

$$-\frac{1}{\mu}(k)\left[\frac{1}{2}(2\pi-2\alpha)\right]\left[b_m H_{\upsilon_m}^{(2)\prime}(k\rho')-a_m J_{\upsilon_n}'(k\rho')\right]$$

$$=\frac{I_0}{\rho'}\sin\upsilon_m(\phi'-\alpha)$$

Scattering by Wedge (cont.)

Solution:

$$a_n = \frac{-\mu I_0}{k\rho'(\pi-\alpha)}\left(\frac{1}{DEN}\right)\sin \upsilon_n(\phi'-\alpha) H_{\upsilon_n}^{(2)}(k\rho')$$

$$b_n = \frac{-\mu_0}{k\rho'(\pi-\alpha)}\left(\frac{1}{DEN}\right)\sin \upsilon_n(\phi'-\alpha)\left[\frac{J_{\upsilon_n}(k\rho')}{H_{\upsilon_n}^{(2)}(k\rho')}\right] H_{\upsilon_n}^{(2)\prime}(k\rho')$$

where

$$DEN = H_{\upsilon_n}^{(2)\prime}(k\rho') J_{\upsilon_n}(k\rho') - H_{\upsilon_n}^{(2)}(k\rho') J'_{\upsilon_n}(k\rho')$$

$$= -j\left[\frac{2}{\pi k\rho}\right]$$

(Wronskian Identity)

Scattering by Wedge (cont.)

Generalization:

To generalize the solution for arbitrary k_z, we simply multiply the entire solution by $\exp(-jk_z z)$

and then make the substitution $k \rightarrow k_\rho$

The solution is then valid for a line source of the form:

$$I(z) = I_0\, e^{-jk_z z}$$

Edge Behavior

$$A_{z1} = \sum_{n=1}^{\infty} a_n \sin \upsilon_n \left(\phi - \alpha\right) J_{\upsilon_n}(k_\rho \rho)$$

As $\rho \to 0$, keep $n = 1$ term, since

$$J_\upsilon(x) \sim x^\upsilon \left(\frac{1}{2^\upsilon \Gamma(\upsilon + 1)} \right)$$

Hence

$$A_z \sim a_1 \sin \upsilon_1 \left(\phi - \alpha\right) J_{\upsilon_1}(k_\rho \rho) e^{-jk_z z}$$

so

$$A_z \propto \rho^{\upsilon_1} \qquad \upsilon_1 = \frac{\pi}{2\left(\pi - \alpha\right)}$$

Edge Behavior (cont.)

Therefore we have:

$$E_z = \frac{k_\rho^2}{j\omega\mu\varepsilon} A_z \propto \rho^{\upsilon_1}$$

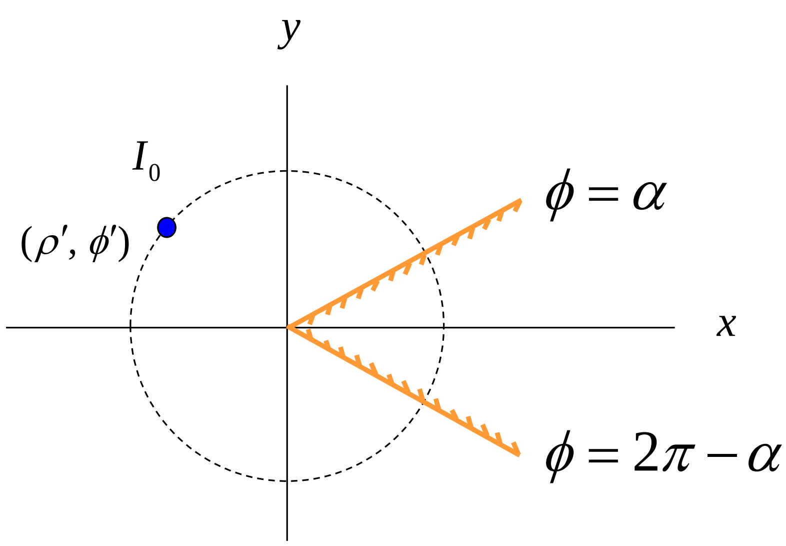
$$E_\rho = \frac{1}{j\omega\mu\varepsilon} \frac{\partial^2 A_z}{\partial\rho\,\partial z} \propto \rho^{\upsilon_1 - 1} \qquad (k_z \neq 0)$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon} \frac{1}{\rho} \frac{\partial^2 A_z}{\partial\phi\,\partial z} \propto \rho^{\upsilon_1 - 1} \qquad (k_z \neq 0)$$

Note: $k_z = 0$ corresponds to a uniform line current, where there is no charge density (and hence no normal electric field).

Edge Behavior (cont.)

$$E_z \to 0 \quad \text{as} \quad \rho \to 0$$



$$(E_\rho, E_\phi) \to \infty \quad \text{if} \quad \upsilon_1 - 1 < 0$$

$$\Rightarrow \upsilon_1 < 1$$

$$\Rightarrow \frac{\pi}{2(\pi - \alpha)} < 1$$

Hence $(E_\rho, E_\phi) \to \infty$ if

$$\frac{2(\pi - \alpha)}{\pi} > 1 \quad \Rightarrow \quad \pi - \alpha > \frac{\pi}{2} \quad \Rightarrow \quad \alpha < \frac{\pi}{2}$$

Therefore $(E_\rho, E_\phi) \to \infty$ if $\alpha < \dfrac{\pi}{2}$ (convex corner)

Knife Edge

Recall:

$$E_\rho, E_\phi \;\propto\; \rho^{\upsilon_1 - 1} \qquad \upsilon_1 = \frac{\pi}{2(\pi - \alpha)}$$

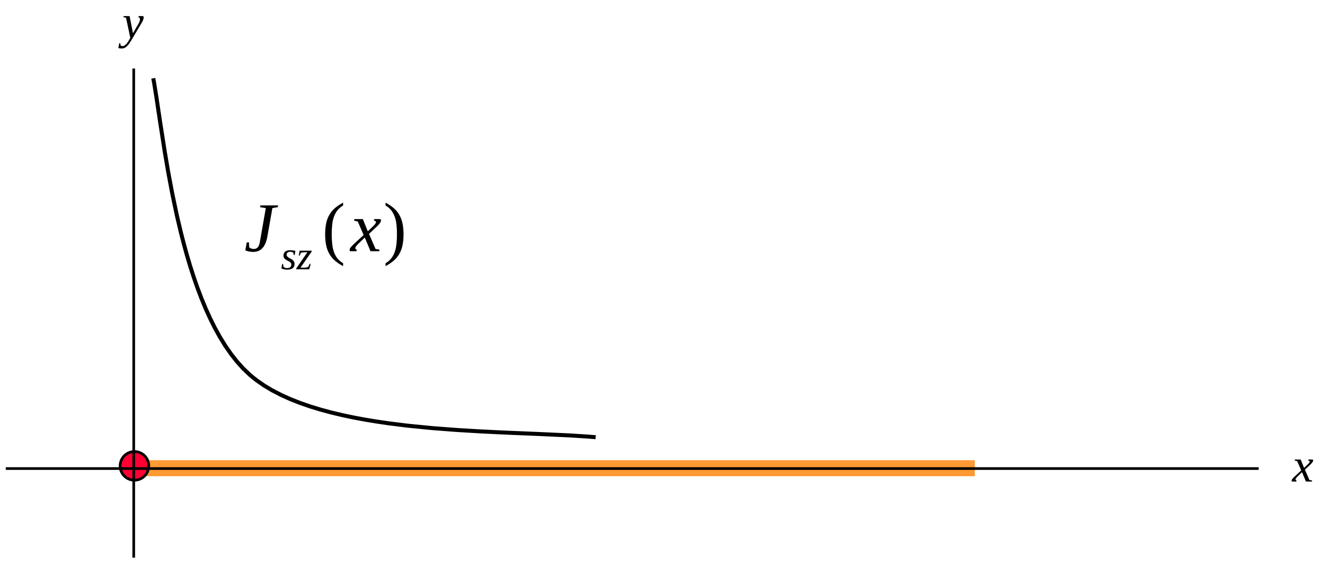
$$\alpha = 0 \;\Rightarrow\; \upsilon_1 = \frac{1}{2} \Rightarrow\; \upsilon_1 - 1 = -\frac{1}{2}$$

so

$$\left|\underline{E}\right| \propto \frac{1}{\sqrt{\rho}}$$

Knife Edge (cont.)

Parallel Current



At $\quad \phi = 0^{+}, \qquad x = \rho :$

$$J_{sz} = -H_x = -H_\rho$$

$$= -\frac{1}{\mu\rho}\frac{\partial A_z}{\partial \phi}$$

$$\approx -\frac{1}{\mu\rho} a_1 \upsilon_1 \cos\upsilon_1 \left(\phi - \alpha\right) J_{\upsilon_1}\left(k_\rho \rho\right) e^{-jk_z z}$$

Knife Edge (cont.)

so

$$J_{sz} \propto \frac{1}{\rho}\rho^{\upsilon_1} = \frac{1}{\rho}\rho^{1/2}$$

or

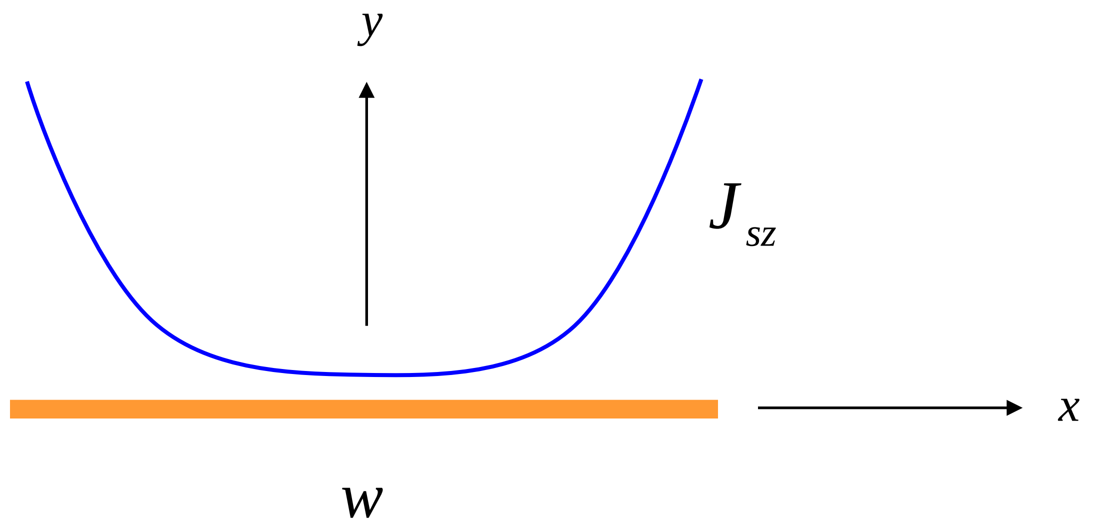
$$J_{sz} \propto \rho^{-1/2}$$

or

$$J_{sz} \propto \frac{1}{\sqrt{x}}$$

Strip in Free Space

Current on Strip



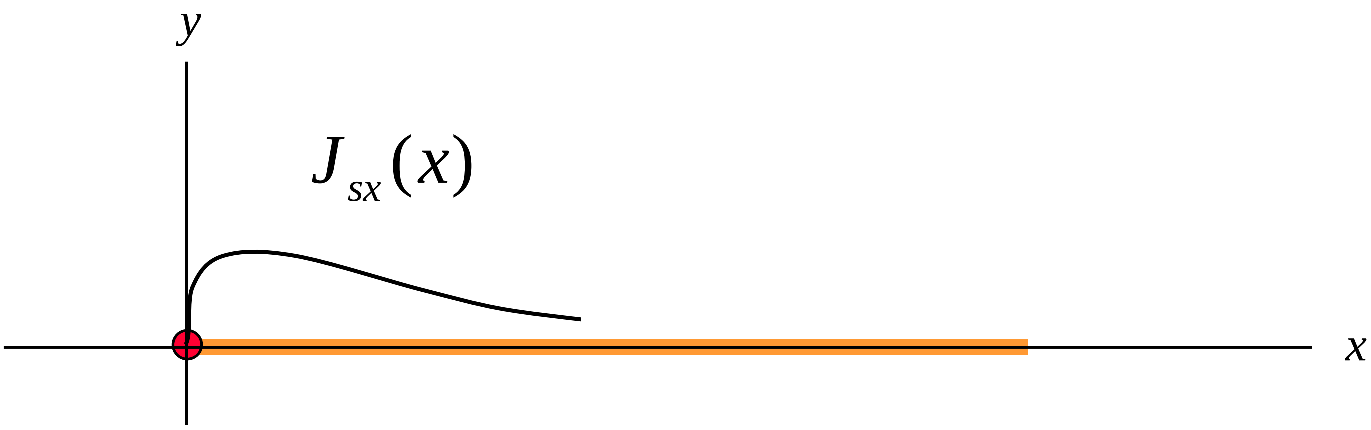
From conformal mapping:

$$J_{sz} = \frac{I_0 / \pi}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}}$$

"Maxwell function"

Knife Edge (cont.)

Perpendicular Current



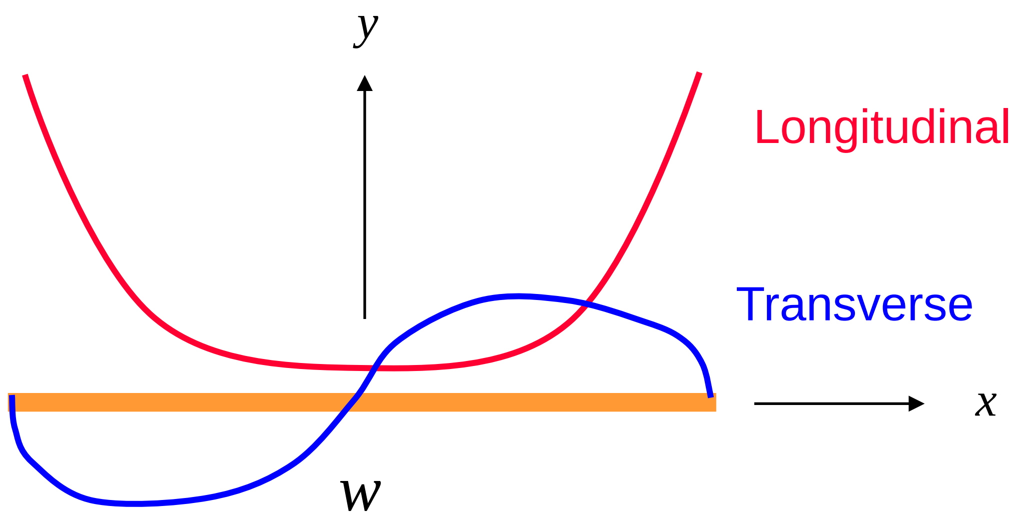
At $\phi = 0^{+}, \qquad J_{sx} = H_z$

Note: To have this component, we must use a TE_z solution (e.g., using a magnetic current source).

If we did the TE_z solution, the result would show that $J_{sx} \propto \sqrt{x}$

Microstrip line

Total Current Density on a Strip



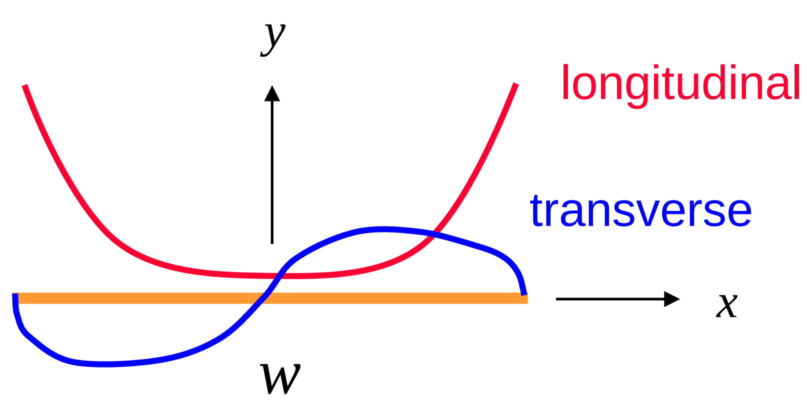
Note:
The current has both components, due to the fact that the mode is not exactly TEM (due to the substrate).

$$\nabla \cdot \underline{J}_s = -j\omega\rho_s$$

$$\frac{\partial J_{sx}}{\partial x} - jk_z J_{sz} = -j\omega\rho_s$$

The longitudinal current and the charge density are even functions, while the transverse current is an odd function.

Microstrip line (cont.)

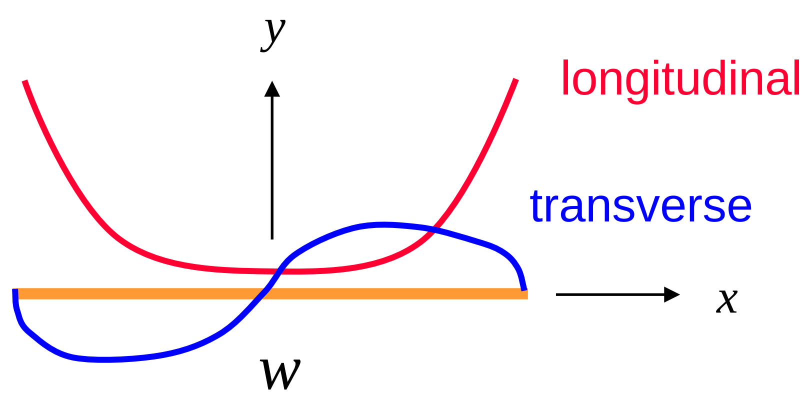


Fourier-Maxwell Basis Function Expansion:

$$J_{sz}\left(x,z\right)=e^{-jk_z z}\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2-x^2}}\left[\sum_{m=0}^{M-1}a_m\cos\left(\frac{2m\pi x}{w}\right)\right]$$

$$J_{sx}\left(x,z\right)=e^{-jk_z z}\sqrt{\left(\frac{w}{2}\right)^2-x^2}\left[\sum_{n=1}^{N}b_n\sin\left(\frac{\left(2n-1\right)\pi x}{w}\right)\right]$$

Microstrip line (cont.)



Chebyshev-Maxwell Basis Function Expansion:

$$J_{sz}\left(x,z\right)=e^{-jk_z z}\frac{1}{\sqrt{\left(\frac{w}{2}\right)^2-x^2}}\left[\sum_{m=0}^{M-1}a_m T_{2m}\left(\frac{2x}{w}\right)\right]\left(\frac{2\left(1+\delta_{m0}\right)}{\pi w}\right)$$

$$J_{sx}\left(x,z\right)=e^{-jk_z z}\sqrt{\left(\frac{w}{2}\right)^2-x^2}\left[\sum_{n=1}^{N}b_n U_{2n-1}\left(\frac{2x}{w}\right)\right]\left(\frac{j4}{\pi w}\right)$$

Meixner* Edge Condition

$$U_E < \infty$$

This condition must be satisfied at all edges. Mathematically, imposing this condition in the solution of a problem is necessary to ensure a unique solution.

C. J. Bouwkamp. "A note on singularities occurring at sharp edges in electromagnetic diffraction theory," Physica (Utrecht), vol. 12, pp. 467-474. Oct., 1946.

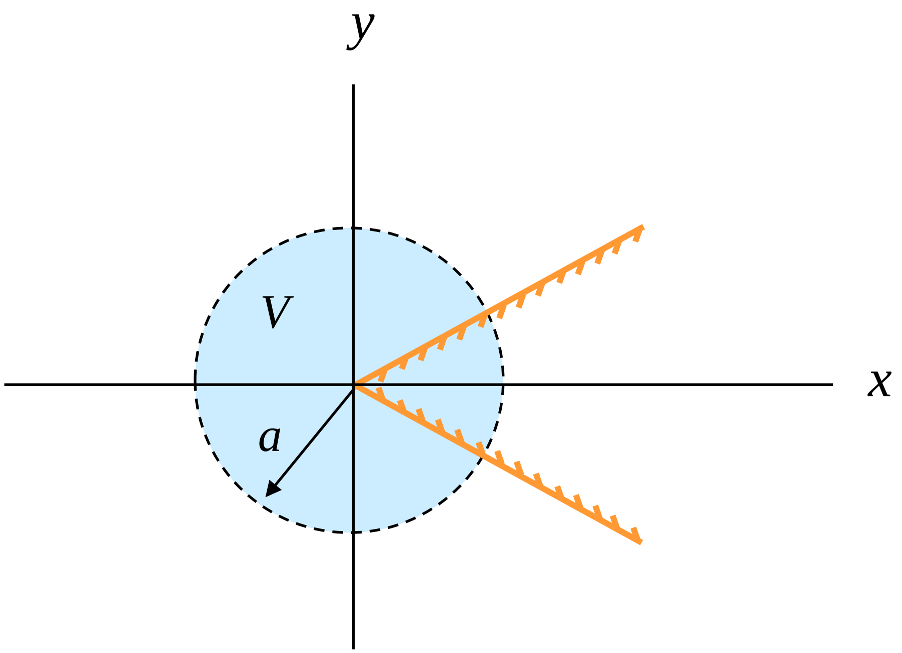
*J. Meixner, "DIe kantenbedingung in der theorie du beugung electromagnetischer wellen an vollkommen leitenden ebenen schirm," Ann. Phys., vol. 6, pp 1-9, 1949.

Meixner Edge Condition (cont.)

Meixner condition:

$$U_E < \infty$$

Let's verify this for the wedge:



$$U_E = \int_V \frac{1}{4} \varepsilon \left| \underline{E} \right|^2 dV$$

$$= \frac{1}{4} \varepsilon \int_V \rho^{2(\upsilon_1 - 1)} \rho \, d\rho \, d\phi \, dz$$

$$E_\rho \propto \rho^{\upsilon_1 - 1}$$

$$E_\phi \propto \rho^{\upsilon_1 - 1}$$

Meixner Edge Condition (cont.)

We require that

$$\int_{\delta}^{a} \rho^{2(\upsilon_1 - 1)} \rho \, d\rho < \infty \quad \text{as} \quad \delta \to 0$$

or

$$\int_{\delta}^{a} \rho^{2\upsilon_1 - 1} d\rho < \infty$$

or

$$\frac{1}{2\upsilon_1} \rho^{2\upsilon_1} \Big|_{\delta}^{a} < \infty$$

or

$$\rho^{2\upsilon_1} < \infty \quad \text{as} \quad \rho = \delta \to 0$$

This <u>will be satisfied</u> since $2\upsilon_1 > 0$

Recall: $\upsilon_1 = \dfrac{\pi}{2(\pi - \alpha)}$