ECE 6341

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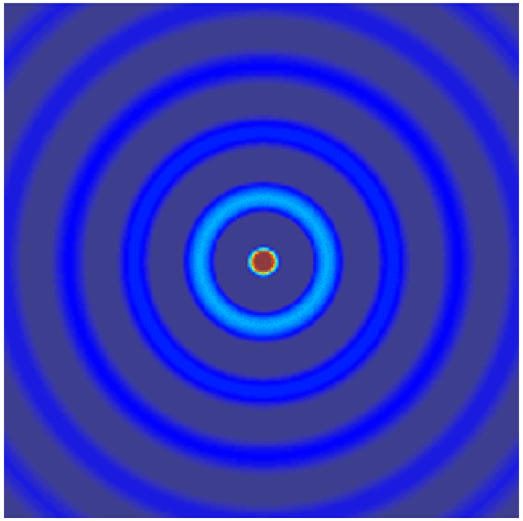


Notes 8

Cylindrical Wave Functions

Helmholtz equation: $$\nabla^2\psi + k^2\psi = 0$$

$$\psi(\rho,\phi,z) = A_z \text{ or } F_z$$

$$\left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right) + k^2\psi = 0$$

Separation of variables:

$$\psi(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$$

Substitute into previous equation and divide by ψ.

Cylindrical Wave Functions (cont.)

$$\left(\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}\right)+k^2\psi=0$$

⇩ $\psi(\rho,\phi,z)=R(\rho)\Phi(\phi)Z(z)$

$$\left(\frac{\partial^2 R}{\partial\rho^2}\Phi Z+\frac{1}{\rho}\frac{\partial R}{\partial\rho}\Phi Z+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}RZ+\frac{\partial^2 Z}{\partial z^2}R\Phi\right)+k^2 R\Phi Z=0$$

⇩ Divide by ψ

$$\left(\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}+\frac{Z''}{Z}\right)+k^2=0$$

Cylindrical Wave Functions (cont.)

$$\left(\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} \right) + k^2 = 0 \qquad (1)$$

or

$$\underbrace{\frac{Z''}{Z}}_{f(z)} = \underbrace{-k^2 - \frac{R''}{R} - \frac{1}{\rho}\frac{R'}{R} - \frac{1}{\rho^2}\frac{\Phi''}{\Phi}}_{g(\rho,\phi)}$$

Hence, $f(z)$ = constant = - k_z^2

Cylindrical Wave Functions (cont.)

Hence

$$\frac{Z''}{Z} = -k_z^{\ 2}$$

$$Z(z) = h(k_z z) = \left\{ e^{\pm j k_z z}, \sin(k_z z), \cos(k_z z) \right\}$$

Next, to isolate the ϕ -dependent term, multiply Eq. (1) by ρ^2 :

$$\rho^2 \left(\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + -k_z^2 \right) + k^2 \rho^2 = 0$$

Cylindrical Wave Functions (cont.)

Hence

$$\underbrace{\frac{\Phi''}{\Phi}}_{f(\phi)} = \underbrace{\rho^2\left(-k^2 + k_z^{\;2} - \frac{1}{\rho}\frac{R'}{R} - \frac{R''}{R}\right)}_{g(\rho)} \qquad (2)$$

Hence,

$$\frac{\Phi''}{\Phi} = \text{constant} = -\upsilon^2$$

so

$$\Phi = h(\upsilon\phi) = \left\{e^{\pm j\upsilon\phi}, \sin(\upsilon\phi), \cos(\upsilon\phi)\right\}$$

Cylindrical Wave Functions (cont.)

From Eq. (2) we now have

$$-\upsilon^2 = \rho^2 \left(-k^2 + k_z{}^2 - \frac{1}{\rho}\frac{R'}{R} - \frac{R''}{R} \right)$$

The next goal is to solve this equation for $R(\rho)$.

First, multiply by R and collect terms:

$$\rho^2 R'' + \rho R' + \rho^2 \left(k^2 - k_z^2 \right) R - \upsilon^2 R = 0$$

Cylindrical Wave Functions (cont.)

Define $k_\rho^{\,2} \equiv k^2 - k_z^{\,2}$

Then, $$\rho^2 R'' + \rho R' + \left[\left(k_\rho \rho \right)^2 - \upsilon^2 \right] R = 0$$

Next, define $$\begin{cases} x = k_\rho \rho \\ y(x) = R(\rho) \end{cases}$$

Note that $$R'(\rho) = \frac{dR}{d\rho} = \frac{dy}{dx}\frac{dx}{d\rho} = y'(x) k_\rho$$

and $$R''(p) = y''(x) k_\rho^2$$

Cylindrical Wave Functions (cont.)

Then we have

$$x^2 y'' + xy' + \left[x^2 - \upsilon^2 \right] y = 0$$

Bessel equation of order υ

Two independent solutions: $J_\upsilon(x), Y_\upsilon(x)$

Hence

$$y(x) = AJ_\upsilon(x) + BY_\upsilon(x)$$

Therefore

$$R(\rho) = \left\{ J_\upsilon(k_\rho \rho),\, Y_\upsilon(k_\rho \rho) \right\}$$

Cylindrical Wave Functions (cont.)

Summary

$$\psi\left(\rho,\phi,z\right)=R\left(\rho\right)\Phi(\phi)Z(z)$$

$$Z(z)=\left\{e^{\pm jk_z z},\sin(k_z z),\cos(k_z z)\right\}$$

$$\Phi=\left\{e^{\pm j\upsilon\phi},\sin(\upsilon\phi),\cos(\upsilon\phi)\right\}$$

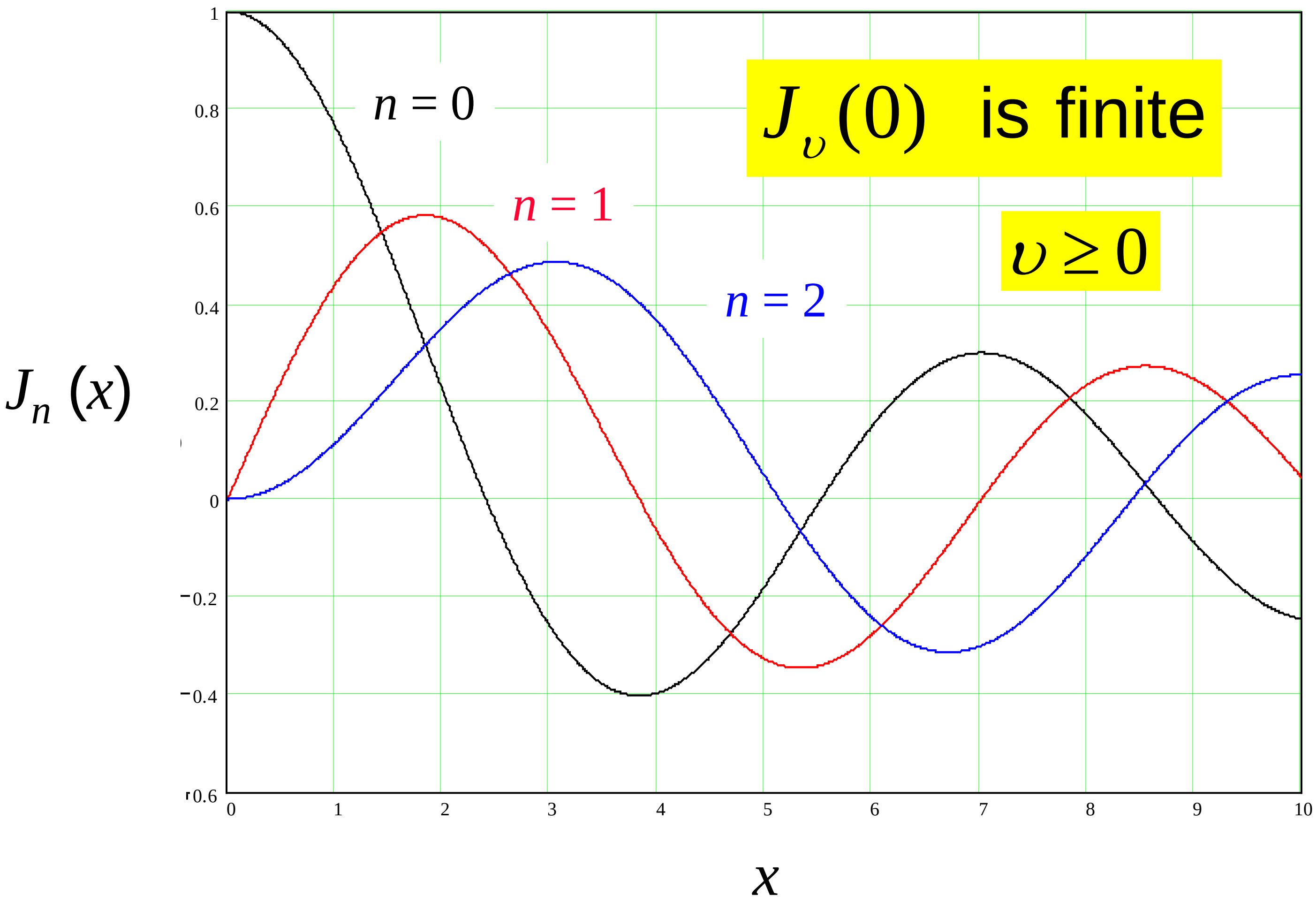
$$R(\rho)=\left\{J_\upsilon(k_\rho\rho),\,Y_\upsilon(k_\rho\rho)\right\}$$

$$k_\rho^2=k^2-k_z^{\;2}$$

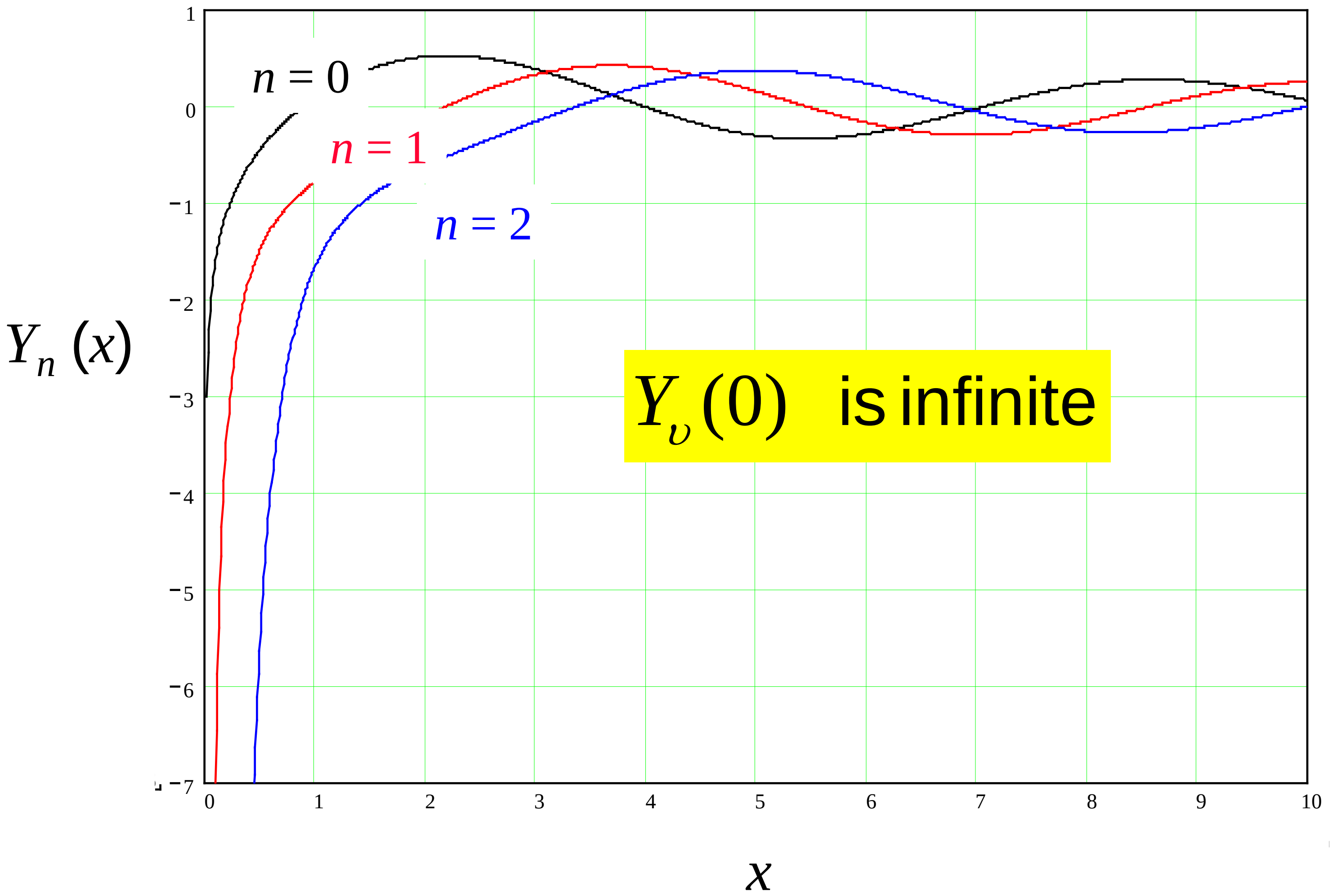
References for Bessel Functions

- M. R. Spiegel, *Schaum's Outline Mathematical Handbook*, McGraw-Hill, 1968.
- M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Government Printing Office, Tenth Printing, 1972.
- N. N. Lebedev, *Special Functions & Their Applications*, Dover Publications, New York, 1972.

Properties of Bessel Functions



Bessel Functions (cont.)



Bessel Functions (cont.)

Small-Argument Properties ($x \to 0$):

$$J_\upsilon(x) \approx Ax^{\upsilon}, \quad \upsilon \neq -1, -2, \ldots$$

$$J_\upsilon(x) \approx Ax^{-\upsilon}, \quad \upsilon = -1, -2, \ldots$$

$$Y_\upsilon(x) \approx Bx^{-|\upsilon|}, \quad \upsilon \neq 0$$

$$Y_0(x) \approx C \ln(x), \quad \upsilon = 0$$

For order zero, the Bessel function of the second kind Y_0 behaves as $\ln(x)$ rather than algebraically.

Bessel Functions (cont.)

Non-Integer Order:

$$\upsilon \neq n$$

$$y(x) = \left\{ J_{\upsilon}(x), J_{-\upsilon}(x) \right\}$$

Two linearly independent solutions

Note:

- Bessel equation is unchanged by $\upsilon \rightarrow -\upsilon$
- $J_{-\upsilon}(x)$ is a always a valid solution

These are linearly independent when υ is not an integer.

$$J_{\upsilon}(x) \approx A_1 x^{\upsilon}, \quad J_{-\upsilon}(x) \approx A_2 x^{-\upsilon} \quad \text{as} \quad x \rightarrow 0$$

Bessel Functions (cont.)

Symmetry property

$$\upsilon = n$$

$$\left.\begin{aligned} J_{-n}(x) &= (-1)^n J_n(x) \\ Y_{-n}(x) &= (-1)^n Y_n(x) \end{aligned}\right\}$$

The functions J_n and J_{-n} are no longer linearly independent.

Bessel Functions (cont.)

Frobenius solution[†]:

$$J_\upsilon(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\upsilon+k)!} \left(\frac{x}{2}\right)^{\upsilon+2k}$$

$$z! = \Gamma(z+1)$$

This is valid for any υ (including $\upsilon = n$).

[†]**Ferdinand Georg Frobenius** (October 26, 1849 – August 3, 1917) was a German mathematician, best known for his contributions to the theory of differential equations and to group theory (Wikipedia).

Bessel Functions (cont.)

Definition of Y_υ

$$Y_\upsilon(x) \equiv \frac{J_\upsilon(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)}{\sin(\upsilon\pi)}$$

$\upsilon \neq \ldots -2, -1, 0, 1, 2 \ldots$

(This definition gives a "nice" asymptotic behavior as $x \to \infty$.)

For integer order: $Y_n(x) = \lim_{\upsilon \to n} Y_\upsilon(x)$

Bessel Functions (cont.)

From the limiting definition, we have, as $\upsilon \to n$:

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\left[\Phi(k)+\Phi(n+k)\right]\frac{1}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}$$

(Schaum's Outline Mathematical Handbook, Eq. (24.9))

where $\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} \quad (p > 0) \qquad \Phi(0) = 0$

Bessel Functions (cont.)

Example

Prove: $J_{-n}(x) = (-1)^n J_n(x)$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(-n+k)!} \left(\frac{x}{2}\right)^{-n+2k}$$

Denote: $k = n + k'$

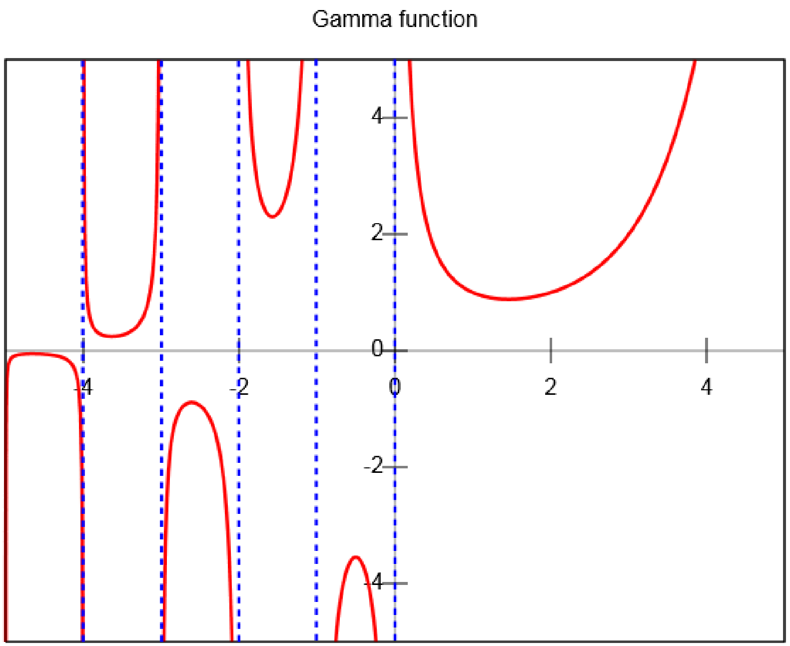
$$J_{-n}(x) = \sum_{k'=-n}^{\infty} \frac{(-1)^{n+k'}}{(n+k')!(k')!} \left(\frac{x}{2}\right)^{n+2k'}$$

Bessel Functions (cont.)

Example (cont.)

$$J_{-n}(x) = \sum_{k=-n}^{\infty} \frac{(-1)^{n+k}}{(k)!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

Plot of Γ function
(from Wikipedia)



Note that

$$\upsilon! = \Gamma(\upsilon+1) = \infty, \quad \upsilon = -1, -2, -3\ldots$$

Bessel Functions (cont.)

Example (cont.)

Hence

$$J_{-n}(x) = \sum_{k=-n}^{\infty} \frac{(-1)^{n+k}}{(k)!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

⬇

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(k)!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

Bessel Functions (cont.)

Example (cont.)

Hence, we have

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2} \right)^{n+2k}$$

$$J_{-n}(x) = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k)!(n+k)!} \left(\frac{x}{2} \right)^{n+2k}$$

so

$$J_{-n}(x) = (-1)^n J_n(x)$$

Bessel Functions (cont.)

J_υ as $x \to 0$

From the Frobenius solution and the symmetry property, we have that

$$J_\upsilon(x) \sim x^\upsilon \left(\frac{1}{2^\upsilon \upsilon!} \right) \qquad \upsilon \neq -1, -2, -3, \ldots$$

$\Downarrow$

$$J_n(x) \sim x^n \left(\frac{1}{2^n n!} \right) \qquad n = 0, 1, 2, \ldots.$$

$\Downarrow$ $\quad J_{-n}(x) = (-1)^n J_n(x)$

$$J_{-n}(x) \sim (-1)^n x^n \left(\frac{1}{2^n n!} \right) \qquad n = 0, 1, 2, \ldots.$$

Bessel Functions (cont.)

Y_υ as $x \to 0$

$$Y_0(x) \sim \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right)+\gamma\right], \qquad \gamma = 0.5772156$$

$$Y_\upsilon(x) \sim -\frac{1}{\pi}(\upsilon-1)!\left(\frac{2}{x}\right)^{\upsilon}, \ \upsilon > 0 \quad \Rightarrow \quad Y_n(x) \sim -\frac{1}{\pi}(n-1)!\left(\frac{2}{x}\right)^{n}$$

$$n = 1, 2, 3, \ldots\ldots$$

$$Y_\upsilon(x) \sim \left(\frac{\cos\upsilon\pi}{\sin\upsilon\pi}\right)\frac{1}{\upsilon!}\left(\frac{x}{2}\right)^{\upsilon}, \ \upsilon < 0$$

$$\upsilon \neq -(2n+1)/2$$

$$\upsilon \neq -n$$

$$\Downarrow \quad Y_{-n}(x) = (-1)^n Y_n(x)$$

$$Y_{-n}(x) \sim (-1)^n\left(-\frac{1}{\pi}\right)(n-1)!\left(\frac{2}{x}\right)^{n}$$

$$n = 1, 2, 3, \ldots\ldots$$

Bessel Functions (cont.)

Asymptotic Formulas

$$x \to \infty$$

$$J_\upsilon(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\upsilon\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\upsilon(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\upsilon\pi}{2} - \frac{\pi}{4}\right)$$

Hankel Functions

$$H_{\upsilon}^{(1)}(x) \equiv J_{\upsilon}(x) + jY_{\upsilon}(x)$$
$$H_{\upsilon}^{(2)}(x) \equiv J_{\upsilon}(x) - jY_{\upsilon}(x)$$

As $x \to \infty$

$$H_{\upsilon}^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{+j\left(x - \upsilon\frac{\pi}{2} - \frac{\pi}{4}\right)}$$ Incoming wave

$$H_{\upsilon}^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-j\left(x - \upsilon\frac{\pi}{2} - \frac{\pi}{4}\right)}$$ Outgoing wave

These are valid for arbitrary order υ.

Fields In Cylindrical Coordinates

$$\underline{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{A}\right) - \frac{1}{\varepsilon}\nabla\times\underline{F}$$

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{F}\right)$$

$\underline{A} = \underline{\hat{z}}\, A_z$ or $\underline{F} = \underline{\hat{z}}\, F_z$

We expand the curls in cylindrical coordinates to get the following results.

TM_z Fields

$$\psi = A_z$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi \qquad H_z = 0$$

$$E_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial\rho\partial z} \qquad H_\rho = \frac{1}{\mu\rho}\frac{\partial\psi}{\partial\phi}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon\rho}\frac{\partial^2\psi}{\partial\phi\partial z} \qquad H_\phi = -\frac{1}{\mu}\frac{\partial\psi}{\partial\rho}$$

TE_z Fields

$$\psi = F_z$$

$$E_\rho = -\frac{1}{\varepsilon\rho}\frac{\partial\psi}{\partial\phi}$$

$$H_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2\psi}{\partial\rho\partial z}$$

$$E_\phi = \frac{1}{\varepsilon}\frac{\partial\psi}{\partial\rho}$$

$$H_\phi = \frac{1}{j\omega\mu\varepsilon\rho}\frac{\partial^2\psi}{\partial\phi\partial z}$$

$$E_z = 0$$

$$H_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$