ECE 6341

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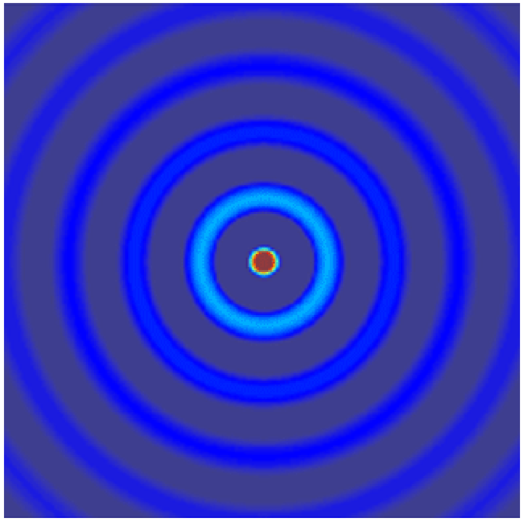


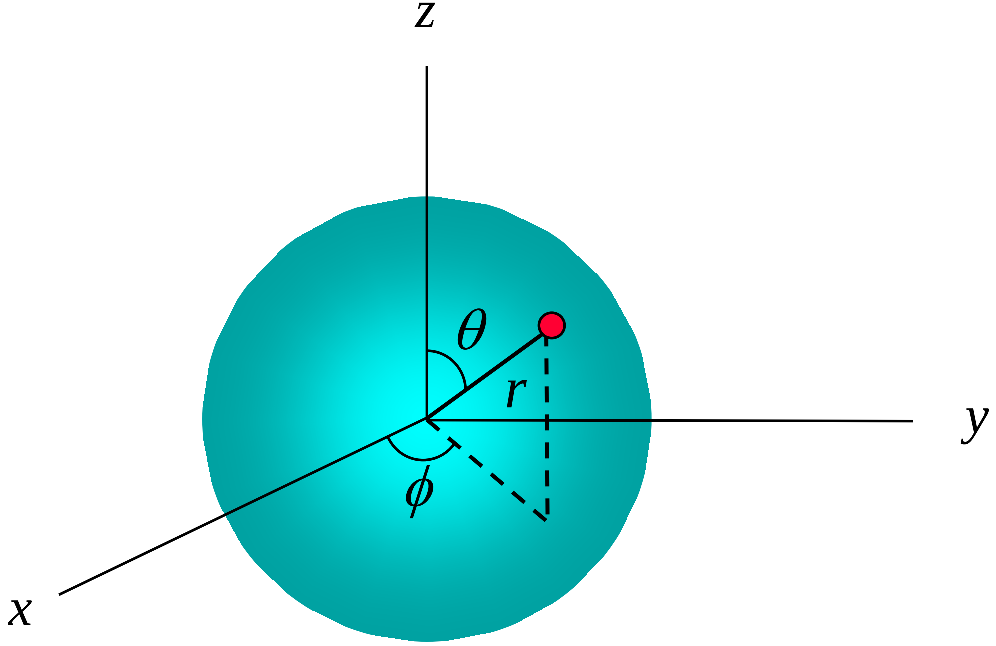
Notes 20

Spherical Wave Functions

Consider solving

$$\nabla^2\psi + k^2\psi = 0$$

in spherical coordinates



Spherical Wave Functions (cont.)

In spherical coordinates we have

$$\nabla^2\psi = \nabla \cdot \nabla\psi$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Hence we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0 \qquad (1)$$

Using separation of variables, let

$$\psi = R(r)\,H(\theta)\,\Phi(\phi) \qquad (2)$$

Spherical Wave Functions (cont.)

After substituting Eq. (2) into Eq. (1), divide by ψ:

$$\frac{1}{r^2 R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right)+\frac{1}{r^2 \sin\theta H}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial H}{\partial \theta}\right)$$
$$+\frac{1}{r^2 \sin^2\theta}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}+k^2=0$$

At this point, we cannot yet say that all of the dependence on any given variable is only within one term.

Spherical Wave Functions (cont.)

Next, multiply by $r^2 \sin^2\theta$:

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{H}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial H}{\partial\theta}\right)$$

$$+\underline{\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}}+k^2r^2\sin^2\theta=0 \qquad (3)$$

Since the underlined term is the only one which depends on ϕ ,
It must be equal to a constant,

Hence, set $$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2 \qquad (4)$$

Spherical Wave Functions (cont.)

Hence,

$$\Phi(\phi)=\begin{cases}\cos m\phi\\ \sin m\phi\end{cases} \qquad (5)$$

In general, $m \rightarrow w$ (not an integer).

Now divide Eq. (3) by $\sin^2\theta$ and use Eq. (4), to obtain

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{H\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial H}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}+k^2r^2=0 \qquad (6)$$

Spherical Wave Functions (cont.)

The underlined terms are the only ones that involve θ now.

This time, the separation constant is customarily chosen as $-n(n+1)$

$$\frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-n(n+1) \qquad (7)$$

In general, $n \rightarrow \upsilon$ (not an integer)

To simplify this, let $x=\cos\theta$

$$dx=-\sin\theta\, d\theta$$

$$\frac{d}{d\theta}=-\sin\theta\frac{d}{dx}$$

and denote $y(x)=H(\theta)$

Spherical Wave Functions (cont.)

Using $x = \cos\theta$ $\quad \frac{d}{d\theta} = -\sin\theta \frac{d}{dx}$ $\quad \sin\theta = \left(1-x^2\right)^{1/2}$

$$\frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-n\left(n+1\right)$$

$$\Downarrow$$

$$\frac{1}{y\left(1-x^2\right)^{1/2}}\left[-\left(1-x^2\right)^{1/2}\frac{d}{dx}\left[\left(1-x^2\right)^{1/2}\left(-1\right)\left(1-x^2\right)^{1/2}y'\right]\right]$$

$$-\frac{m^2}{\left(1-x^2\right)}+n(n+1)=0$$

Spherical Wave Functions (cont.)

Canceling terms,

$$\frac{1}{y\left(1-x^2\right)^{1/2}}\left[-\left(1-x^2\right)^{1/2}\frac{d}{dx}\left[\left(1-x^2\right)^{1/2}(-1)\left(1-x^2\right)^{1/2}y'\right]\right]$$

$$-\frac{m^2}{\left(1-x^2\right)}+n(n+1)=0$$

Multiplying by y, we have

$$\frac{d}{dx}\left[\left(1-x^2\right)y'\right]+\left[n(n+1)-\frac{m^2}{\left(1-x^2\right)}\right]y=0 \qquad (8)$$

Spherical Wave Functions (cont.)

Eq. (8) is the associated Legendre equation. The solutions are represented as

$$y(x) = \begin{cases} P_n^m(x) & \text{Associated Legendre function of the first kind.} \\ Q_n^m(x) & \text{Associated Legendre function of the second kind.} \end{cases}$$

n = "order", m = "degree"

If $m = 0$, Eq. (8) is called the Legendre equation, in which case

$$y(x) = \begin{cases} P_n^0(x) \\ Q_n^0(x) \end{cases} = \begin{cases} P_n(x) & \text{Legendre function of the first kind.} \\ Q_n(x) & \text{Legendre function of the second kind.} \end{cases}$$

Spherical Wave Functions (cont.)

Hence:

$$H(\theta) = \begin{cases} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{cases}$$

To be as general as possible:

$$n \to \upsilon$$
$$m \to w$$

$$H(\theta) = \begin{cases} P_\upsilon^w(\cos\theta) \\ Q_\upsilon^w(\cos\theta) \end{cases}$$

Spherical Wave Functions (cont.)

Substituting Eq. (7) into Eq. (6) now yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)+k^2r^2=0$$

Next, let $x = kr$

$$dx = k\,dr$$

$$\frac{d}{dr} = k\frac{d}{dx}$$

and denote $y(x) = R(r)$

Spherical Wave Functions (cont.)

We then have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)+k^2r^2=0$$

$$x = kr \qquad \frac{d}{dr} = k\frac{d}{dx}$$

$$\frac{1}{y}k\frac{d}{dx}\left(\frac{x^2}{k^2}k\frac{dy}{dx}\right)-n(n+1)+x^2=0$$

$$\frac{d}{dx}\left(x^2y'\right)+\left[x^2-n(n+1)\right]y=0$$

Spherical Wave Functions (cont.)

$$\frac{d}{dx}\left(x^2 y'\right)+\left[x^2-n\left(n+1\right)\right]y=0$$

or

$$x^2 y''+2xy'+\left[x^2-n\left(n+1\right)\right]y=0$$

"spherical Bessel equation"

Solution: $b_n(x)$

Note the lower case b.

Spherical Wave Functions (cont.)

Denote $y(x) = b_n(x)$

and let

$$b_n(x) = x^{-1/2} g_n(x)$$

$$b_n'(x) = x^{-1/2} g_n' - \frac{1}{2} x^{-3/2} g_n$$

$$b_n''(x) = x^{-1/2} g_n'' - \frac{1}{2} x^{-3/2} g_n' - \frac{1}{2} x^{-3/2} g_n' + \frac{3}{4} x^{-5/2} g_n$$

Hence

$$\left(x^{3/2} g_n'' - x^{1/2} g_n' + \frac{3}{4} x^{-1/2} g_n \right) + \left(2x^{1/2} g_n' - x^{-1/2} g_n \right)$$

$$+ \left[x^2 - n(n+1) \right] x^{-1/2} g_n(x) = 0$$

Spherical Wave Functions (cont.)

Multiply by $x^{1/2}$

$$\left(x^2 g_n'' - x g_n' + \frac{3}{4} g_n\right) + \left(2x g_n' - g_n\right) + \left[x^2 - n(n+1)\right] g_n(x) = 0$$

Combine these terms

or

$$x^2 g_n'' + x g_n' - \frac{1}{4} g_n + x^2 g_n - n(n+1) g_n = 0$$

Combine these terms

Use

$$-\left(\frac{1}{4} + n(n+1)\right) = -\left(n + \frac{1}{2}\right)^2$$

Define $\gamma \equiv n + \frac{1}{2}$

Spherical Wave Functions (cont.)

We then have $$x^2 g_n'' + x g_n' + \left(x^2 - \gamma^2\right) g_n = 0$$

This is Bessel's equation of order γ.

Hence $$g_n = \begin{pmatrix} J_\gamma(x) \\ Y_\gamma(x) \end{pmatrix}$$

so that

$$b_n = \frac{1}{\sqrt{x}} \begin{pmatrix} J_{n+1/2}(x) \\ Y_{n+1/2}(x) \end{pmatrix} \cdot \sqrt{\frac{\pi}{2}}$$

added for convenience

Spherical Wave Functions (cont.)

Define

$$j_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x)$$

$$y_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, Y_{n+1/2}(x)$$

Then

$$R(r) = b_n(kr) = \begin{pmatrix} j_n(kr) \\ y_n(kr) \end{pmatrix}$$

Summary

$$\nabla^2\psi + k^2\psi = 0$$

$$\psi = \begin{pmatrix} j_n\left(kr\right) \\ y_n\left(kr\right) \end{pmatrix} \begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix} \begin{pmatrix} \cos\left(m\phi\right) \\ \sin\left(m\phi\right) \end{pmatrix}$$

$$b_n(x) = \sqrt{\frac{\pi}{2x}}\, B_{n+1/2}(x)$$

In general,

$$m \to w$$

$$n \to \upsilon$$

Properties of Spherical Bessel Functions

$\upsilon = n$

$$b_n(x) = \sqrt{\frac{\pi}{2x}}\, B_{n+1/2}(x)$$

Bessel functions of half-integer order are given by closed-form expressions.

$$J_\upsilon(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\upsilon+k)!}\left(\frac{x}{2}\right)^{\upsilon+2k} \qquad z! = \Gamma(z+1)$$

This becomes a closed-form expression!

$$\upsilon = n + 1/2$$

Properties of Spherical Bessel Functions (cont.)

Examples:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin(x)}{x} - \cos(x) \right]$$

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left[\frac{\cos(x)}{x} + \sin(x) \right]$$

$$Y_{\upsilon}(x) \equiv \frac{J_{\upsilon}(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)}{\sin(\upsilon\pi)} \quad \Rightarrow \quad Y_{1/2}(x) = -J_{-1/2}(x)$$

$$Y_{3/2}(x) = J_{-3/2}(x)$$

Properties of Spherical Bessel Functions (cont.)

Proof for $\upsilon = 1/2$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$$

Start with:

$$J_\upsilon(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\upsilon+k)!}\left(\frac{x}{2}\right)^{\upsilon+2k} \Rightarrow J_{1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(1/2+k)!}\left(\frac{x}{2}\right)^{1/2+2k}$$

Hence

$$\sqrt{\frac{x}{2}}J_{1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(1/2+k)!}\left(\frac{x}{2}\right)^{2k+1}$$

Properties of Spherical Bessel Functions (cont.)

Examine the factorial expression:

$$(k+1/2)! = (k+1/2)(k-1/2)(k-3/2)\ldots(3/2)\left(\frac{1}{2}!\right)$$

$$= (k+1/2)(k-1/2)(k-3/2)\ldots(3/2)\left(\sqrt{\pi}/2\right)$$

$$= (2k+1)(2k-1)(2k-3)\ldots(3)\left(\sqrt{\pi}/2\right)\left(\frac{1}{2}\right)^k$$

$$= \frac{(2k+1)!\left(\sqrt{\pi}/2\right)\left(\frac{1}{2}\right)^k}{(2k)(2k-2)(2k-4)\ldots4\cdot2}$$

$$= \frac{(2k+1)!\left(\sqrt{\pi}/2\right)\left(\frac{1}{2}\right)^k}{2^k(k)(k-1)(k-2)\ldots2\cdot1}$$

$$= \frac{(2k+1)!\left(\sqrt{\pi}/2\right)\left(\frac{1}{2}\right)^k}{2^k k!}$$

Note:

$$x! = x(x-1)!$$

$$(1/2)! = \sqrt{\pi}/2$$

Properties of Spherical Bessel Functions (cont.)

Hence

$$\sqrt{\frac{x}{2}}J_{1/2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\left[\dfrac{(2k+1)!\left(\sqrt{\pi}/2\right)\left(\frac{1}{2}\right)^k}{2^k k!}\right]}\left(\frac{x}{2}\right)^{2k+1}$$

Hence, we have

$$\sqrt{\frac{\pi x}{2}}J_{1/2}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\left[\dfrac{(2k+1)!}{2^{2k+1}}\right]}\left(\frac{x}{2}\right)^{2k+1}$$

Properties of Spherical Bessel Functions (cont.)

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\left[\frac{(2k+1)!}{2^{2k+1}}\right]} \left(\frac{x}{2}\right)^{2k+1}$$

or

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

We then recognize that

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sin(x)$$

Properties of Legendre Functions

Relation to Legendre functions (when $w = m =$ integer):

$$P_n^m(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

$$Q_n^m(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} Q_n(x)$$

These also hold for $n \to \upsilon$.

For $m \to w$ (not an integer) the associated Legendre function is defined in terms of the hypergeometric function.

Properties of Legendre Functions (cont.)

Rodriguez's formula (for $\upsilon = n$):

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Legendre polynomial

(a polynomial of order n)

$$P_0(x) = \left(x^2 - 1\right)^0 = 1$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}\left(x^2 - 1\right) = x$$

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}\left(x^2 - 1\right)^2 = \frac{1}{2}\left(3x^2 - 1\right)$$

Properties of Legendre Functions (cont.)

Note: $$P_n^m\left(x\right)=0,\;\; m>n$$

This follows from these two relations:

$$P_n^m(x)=\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

$$P_n\left(x\right)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n=\text{polynomial of order } n$$

Properties of Legendre Functions (cont.)

$$w = m, \quad \upsilon = n$$

$$Q_n^m(x) = \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} Q_n(x)$$

$Q_n(x) =$ infinite series, not a polynomial (may blow up)

$$Q_n(\pm 1) = \infty$$ (see next slide)

The Q_n functions all tend to infinity as $x \to \pm 1$

$\Rightarrow \theta \to 0, \pi$ Recall: $x = \cos\theta$

Properties of Legendre Functions (cont.)

Lowest-order Q_n functions:

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

$$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

Properties of Legendre Functions (cont.)

Negative index identities:

$$P_{-(n+1)}(x) = P_n(x)$$

(This identity also holds for $\upsilon \neq n$.)

$$Q_{-(n+1)}(x) = -\pi(-1)^n P_n(x) + Q_n(x)$$

Properties of Legendre Functions (cont.)

$\upsilon \neq n$ (see Harrington, Appendix E)

$P_\upsilon(x) =$ infinite series

$Q_\upsilon(x) =$ infinite series

$$P_\upsilon(x) = \sum_{m=0}^{N} \frac{(-1)^m (\upsilon+m)!}{(m!)^2 (\upsilon-m)!} \left(\frac{1-x}{2}\right)^m - \frac{\sin(\upsilon\pi)}{\pi} \sum_{m=N+1}^{\infty} \frac{(m-1-\upsilon)!(m+\upsilon)!}{(m!)^2} \left(\frac{1-x}{2}\right)^m$$

$N =$ largest integer less than or equal to υ.

$$P_\upsilon(1) = 1$$

$$P_\upsilon(-1) = \infty$$

$$Q_\upsilon(\pm 1) = \infty$$

$$Q_\upsilon(x) = \frac{\pi}{2}\left[\frac{P_\upsilon(x)\cos(\upsilon\pi) - P_\upsilon(-x)}{\sin(\upsilon\pi)}\right]$$

Both are valid solutions, which are linearly independent for $\upsilon \neq n$

(see next slide)

Properties of Legendre Functions (cont.)

Proof that a valid solution is $P_\upsilon^m(-x)$

$$\frac{d}{dx}\left[\left(1-x^2\right)P_\upsilon^{m\prime}\left(x\right)\right]+\left[\upsilon(\upsilon+1)-\frac{m^2}{\left(1-x^2\right)}\right]P_\upsilon^m\left(x\right)=0$$

Let $t=-x$ $\qquad \frac{d}{dx}=-\frac{d}{dt}$

Then

$$-\frac{d}{dt}\left[\left(1-t^2\right)\left(-1\right)P_\upsilon^{m\prime}\left(-t\right)\right]+\left[\upsilon(\upsilon+1)-\frac{m^2}{\left(1-t^2\right)}\right]P_\upsilon^m\left(-t\right)=0$$

or $(t\rightarrow x)$

$$\frac{d}{dx}\left[\left(1-x^2\right)P_\upsilon^{m\prime}\left(-x\right)\right]+\left[\upsilon(\upsilon+1)-\frac{m^2}{\left(1-x^2\right)}\right]P_\upsilon^m\left(-x\right)=0$$

Hence, a valid solution is $P_\upsilon^m(-x)$

Properties of Legendre Functions (cont.)

$\upsilon \neq n$

$P_{\upsilon}^{m}(x)$ and $P_{\upsilon}^{m}(-x)$

are two linearly independent solutions.

Valid independent solutions:

$$\begin{pmatrix} P_{\upsilon}^{m}(\cos\theta) \\ Q_{\upsilon}^{m}(\cos\theta) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} P_{\upsilon}^{m}(\cos\theta) \\ P_{\upsilon}^{m}(-\cos\theta) \end{pmatrix}$$

We have a choice which set we wish to use.

Properties of Legendre Functions (cont.)

$\upsilon = n$

$$P_n(-x) = (-1)^n P_n(x)$$

(They are linearly dependent.)

In this case we <u>must</u> use

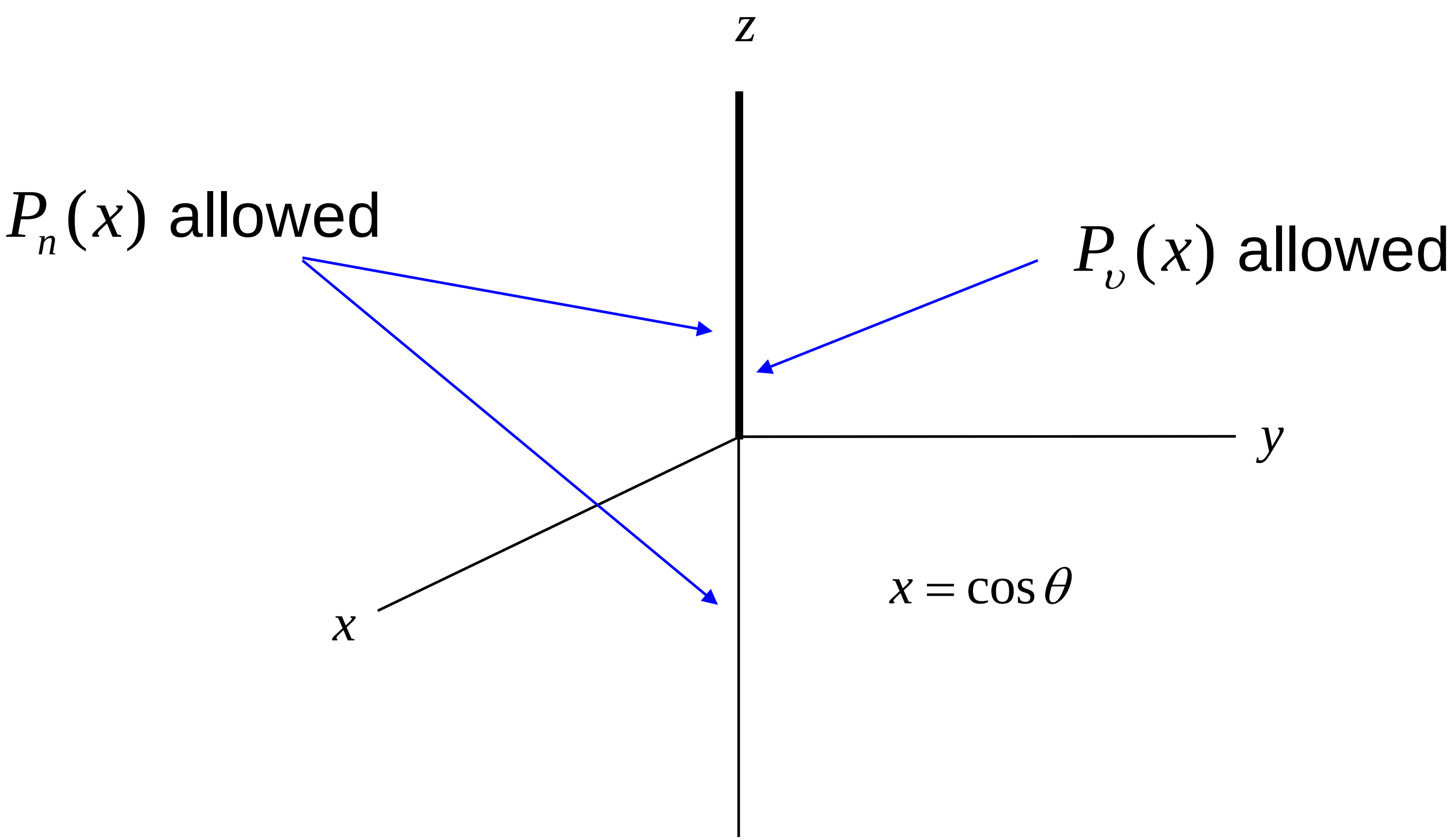
$$\begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix}$$

Properties of Legendre Functions (cont.)

Summary of z-axis properties ($x = \cos(\theta)$)

$\upsilon = n$	$\upsilon \neq n$
$P_n(1) = 1$	$P_\upsilon(1) = 1$
$Q_n(\pm 1) = \infty$	$Q_\upsilon(\pm 1) = \infty$
$P_n(-1) = (-1)^n$	$P_\upsilon(-1) = \infty$

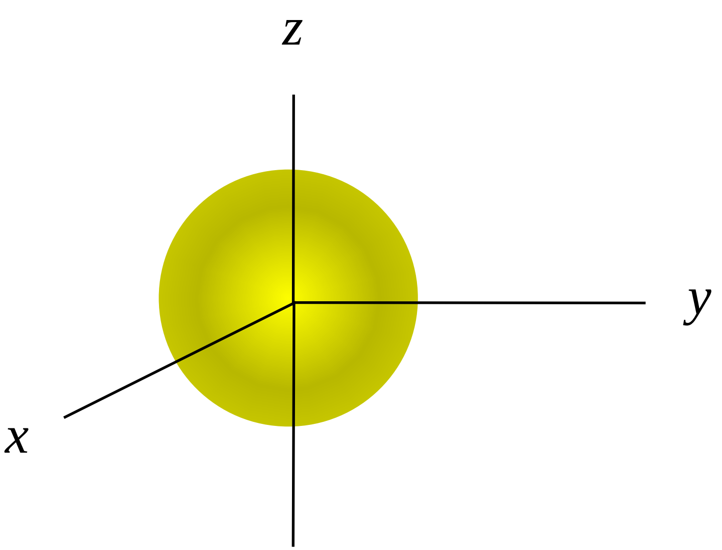
Properties of Legendre Functions (cont.)

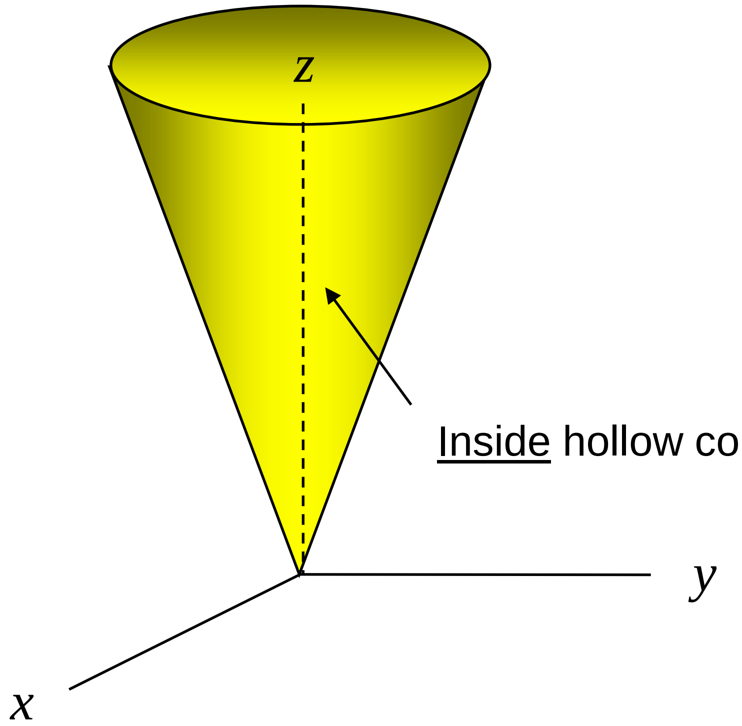


$Q_n(x)$ and $Q_\upsilon(x)$ are not allowed on $\pm z$ axis.

Properties of Legendre Functions (cont.)

Outside or inside sphere

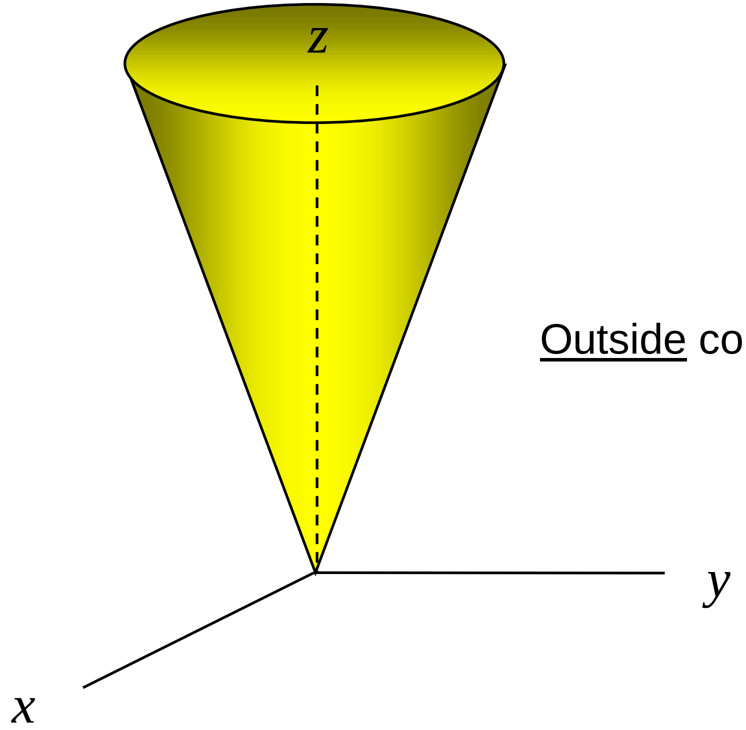




Only $P_n(x)$ is allowed

Both $P_n(x)$ and $P_\upsilon(x)$ are allowed

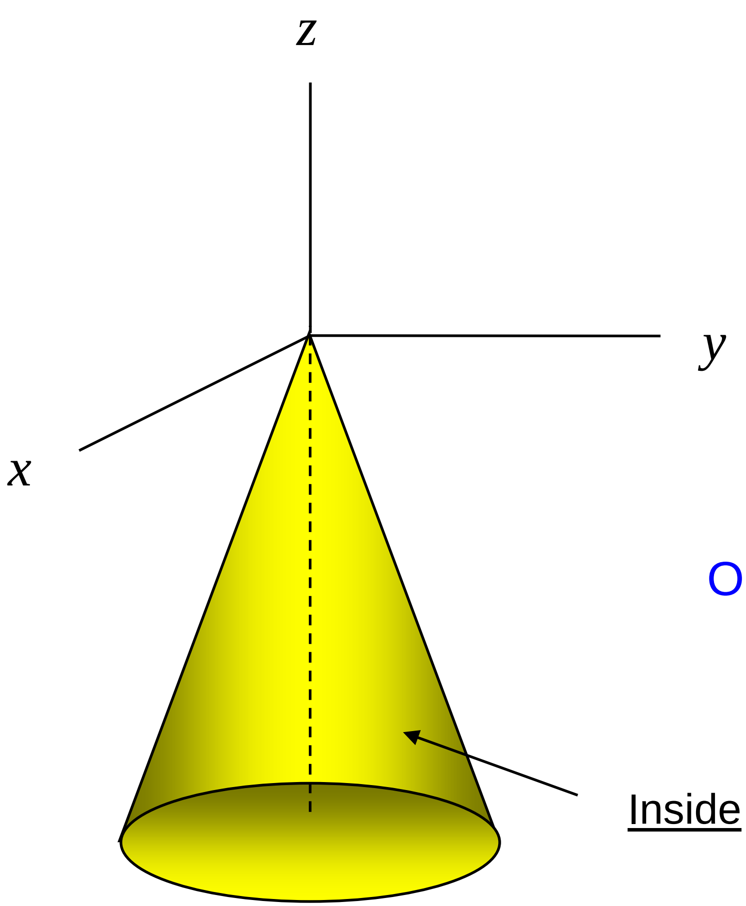
Properties of Legendre Functions (cont.)



Outside cone

Only $P_n(x)$ is allowed

Properties of Legendre Functions (cont.)

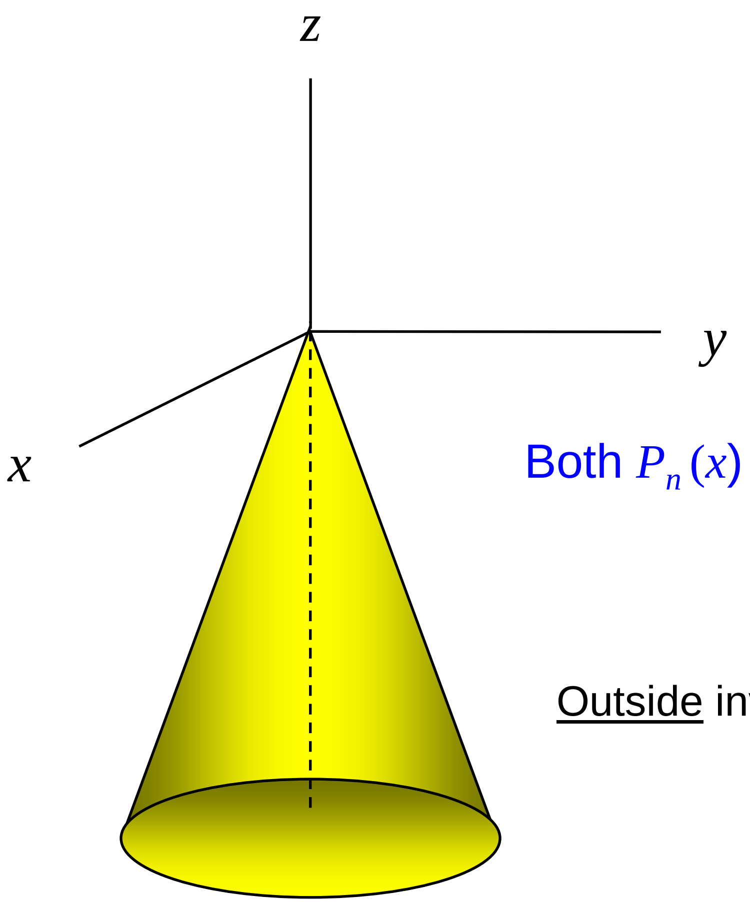


Only $P_n(x)$ is allowed

Inside inverted cone

Note: The physics is the same as for the upright cone, but the mathematical form of the solution is different!

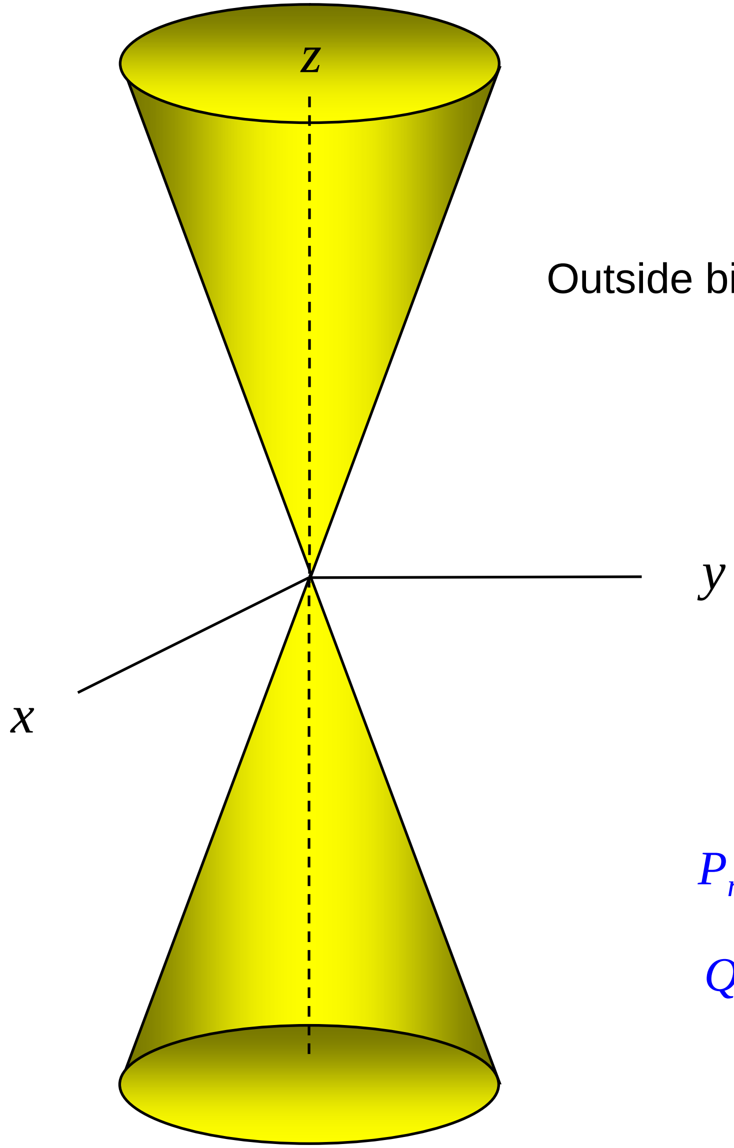
Properties of Legendre Functions (cont.)



Both $P_n(x)$ and $P_\upsilon(x)$ are allowed

Outside inverted cone

Properties of Legendre Functions (cont.)



Outside bicone

$P_n(x)$ and $P_\upsilon(x)$ are allowed

$Q_n(x)$ and $Q_\upsilon(x)$ are allowed