ECE 6341

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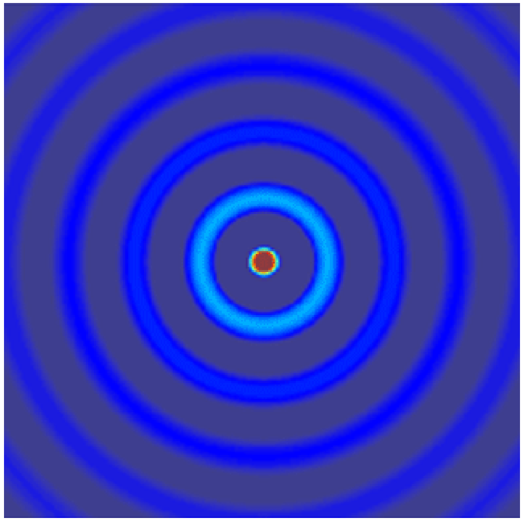

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Vector Potentials

If we choose $\psi = A_z, F_z$

then $\nabla^2 \psi + k^2 \psi = 0$

and $\psi = b_n(kr) \begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$

However, the $\underline{E}$ and $\underline{H}$ fields in spherical coordinates are complicated! (see Prob. 6.1 in Harrington).

This is because we have two components in spherical coordinates:

$$\underline{A} = \underline{\hat{r}}\left(A_z \cos\theta\right) + \underline{\hat{\theta}}\left(-A_z \sin\theta\right)$$

Vector Potentials (cont.)

A better choice is:

$$\underline{A} = \underline{\hat{r}}\, A_r$$

$$\underline{F} = \underline{\hat{r}}\, F_r$$

"Debye Potentials"

The electric and magnetic fields are given in terms of these potentials in a fairly simple manner (please see the next two slides).

Peter Joseph William Debye (March 24, 1884 – November 2, 1966) was a Dutch physicist and physical chemist, and Nobel laureate in Chemistry.

"…he studied under the theoretical physicist Arnold Sommerfeld, who later claimed that his most important discovery was Peter Debye."

http://en.wikipedia.org/wiki/Peter_Debye

Vector Potentials (cont.)

$$\underline{A} = \underline{\hat{r}}\, A_r$$

$$\underline{F} = \underline{\hat{r}}\, F_r$$

Use this choice of potentials together with the basic field equations:

$$\underline{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{A}\right) - \frac{1}{\varepsilon}\nabla\times\underline{F}$$

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\times\left(\nabla\times\underline{F}\right)$$

Vector Potentials (cont.)

$$\psi = A_r$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\psi$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 \psi}{\partial r \partial \theta}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r \sin\theta}\frac{\partial^2 \psi}{\partial r \partial \phi}$$

$$H_r = 0$$

$$H_\theta = \frac{1}{\mu r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$H_\phi = \frac{-1}{\mu r}\frac{\partial \psi}{\partial \theta}$$

$$\psi = F_r$$

$$E_r = 0$$

$$E_\theta = \frac{-1}{\varepsilon r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$E_\phi = \frac{1}{\varepsilon r}\frac{\partial \psi}{\partial \theta}$$

$$H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\psi$$

$$H_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 \psi}{\partial r \partial \theta}$$

$$H_\phi = \frac{1}{j\omega\mu\varepsilon r \sin\theta}\frac{\partial^2 \psi}{\partial r \partial \phi}$$

Vector Potentials (cont.)

How do we represent the solution for A_r and F_r in spherical coordinates?

First, let's assume that we have the "usual" solution (from ECE 6340) which has enforced the Lorenz Gauge:

$$\nabla \cdot \underline{A} = -j\omega\mu\varepsilon\,\Phi$$

We then have $\nabla^2 \underline{A} + k^2 \underline{A} = \underline{0}$ (vector Helmholtz equation)

$$\Rightarrow \begin{cases} \nabla^2 A_x + k^2 A_x = 0 \\ \nabla^2 A_y + k^2 A_y = 0 \\ \nabla^2 A_z + k^2 A_z = 0 \end{cases}$$

since $\nabla^2 \underline{A} = \underline{\hat{x}}\,\nabla^2 A_x + \underline{\hat{y}}\,\nabla^2 A_y + \underline{\hat{z}}\,\nabla^2 A_z$

Vector Potentials (cont.)

$$\nabla^2 \underline{A} + k^2 \underline{A} = \underline{0}$$

However, $$\left(\nabla^2 \underline{A}\right)_r \neq \nabla^2 A_r$$

since $$\nabla^2 \underline{A} \neq \underline{\hat{r}}\, \nabla^2 A_r + \underline{\hat{\theta}}\, \nabla^2 A_\theta + \underline{\hat{\phi}}\, \nabla^2 A_\phi$$

Hence $$\nabla^2 A_r + k^2 A_r \neq 0$$

$$A_r \neq b_n(kr) \begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

Vector Potentials (cont.)

Observations

- When using the Lorenz gauge (so that we have the vector Helmholtz equation), A_r and F_r do not satisfy the scalar Helmholtz equation.
- Furthermore, we can show from Maxwell's equations that using A_r and F_r implies that we cannot have the Lorenz gauge (proof given next).

Vector Potentials (cont.)

From Maxwell's Equations:

$$\nabla \cdot \underline{H} = 0$$

$$\Rightarrow \quad \underline{H} = \frac{1}{\mu} \nabla \times \underline{A}$$

$$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$

$$\Rightarrow \quad \nabla \times \underline{E} = -j\omega \nabla \times \underline{A}$$

$$\Rightarrow \quad \nabla \times \left(\underline{E} + j\omega \underline{A} \right) = \underline{0}$$

$$\underline{E} = -j\omega \underline{A} - \nabla \Phi$$

Vector Potentials (cont.)

$$\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$$ (source-free region)

$$\Rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \underline{A} \right) = j\omega\varepsilon \left(-j\omega \underline{A} - \nabla \Phi \right)$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times A \right) - k^2 \underline{A} = -j\omega\mu\varepsilon \nabla \Phi$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \underline{A} \right) - k^2 \underline{A} = -j\omega\mu\varepsilon \nabla \Phi$$

(vector wave equation in mixed-potential form)

Note: We have not assumed any "gauge" here.

Vector Potentials (cont.)

Assume $\underline{A} = \underline{\hat{r}}\, A_r$

Take $\underline{\hat{\theta}}, \underline{\hat{\phi}}$ components of the vector wave equation:

$$\frac{1}{r}\frac{\partial^2 A_r}{\partial r \partial \theta} = -j\omega\mu\varepsilon\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

$$\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r \partial \phi} = -j\omega\mu\varepsilon\left(\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right)$$

Both are satisfied if we choose

$$\frac{\partial A_r}{\partial r} = -j\omega\mu\varepsilon\,\Phi$$

"Debye Gauge"

Vector Potentials (cont.)

Compare with Lorenz Gauge:

$$\nabla \cdot \underline{A} = -j\omega\mu\varepsilon\Phi$$

$$\Rightarrow \quad \nabla \cdot \left(\underline{\hat{r}} A_r \right) = -j\omega\mu\varepsilon\Phi$$

$$\Rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) = -j\omega\mu\varepsilon\Phi$$

$$\Rightarrow \quad \frac{\partial A_r}{\partial r} + \frac{2}{r} A_r = -j\omega\mu\varepsilon\Phi$$

Not the same as the Debye Gauge !

When we use the Debye potentials, we must have the Debye gauge, which is different than the Lorenz gauge.

Vector Potentials (cont.)

When we use the Debye potentials:

$$\nabla \cdot \underline{A} \neq -j\omega\mu\varepsilon\Phi$$

Hence

$$\nabla^2 \underline{A} + k^2 \underline{A} \neq 0$$

When we use the Debye potentials, we no longer have the vector Helmholtz equation.

Vector Potentials (cont.)

Next step:

Take the radial component of the vector wave equation, to obtain a differential equation for A_r.

Vector Potentials (cont.)

$$\nabla \times \left(\nabla \times \underline{A} \right) - k^2 \underline{A} = -j\omega\mu\varepsilon \nabla \Phi$$

so

$$\underline{\hat{r}} \cdot \left[\nabla \times \left(\nabla \times \left(\underline{\hat{r}} A_r \right) \right) \right] - k^2 A_r = -j\omega\mu\varepsilon \frac{\partial \Phi}{\partial r}$$

$$= \frac{\partial^2 A_r}{\partial r^2}$$ (using Debye gauge)

Hence

$$\frac{\partial^2 A_r}{\partial r^2} - \underline{\hat{r}} \cdot \left[\nabla \times \left(\nabla \times \left(\underline{\hat{r}} A_r \right) \right) \right] + k^2 A_r = 0$$

Expanding, we have

$$\frac{\partial^2 A_r}{\partial r^2} + \left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial A_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial \phi^2} \right) + k^2 A_r = 0$$

Vector Potentials (cont.)

The potential A_r therefore satisfies

$$\frac{\partial^2 A_r}{\partial r^2} + \left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial A_r}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial\phi^2} \right) + k^2 A_r = 0$$

Compare with $\nabla^2 \psi + k^2 \psi = 0$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) + \left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \right) + k^2 \psi = 0$$

Not the same! $\nabla^2 A_r + k^2 A_r \neq 0$

Even when using the Debye Gauge, we don't get the scalar Helmholtz equation for A_r.

Vector Potentials (cont.)

Try this: $A_r = r\psi$

$$\frac{\partial A_r}{\partial r} = r\frac{\partial \psi}{\partial r} + \psi$$

$$\frac{\partial^2 A_r}{\partial r^2} = r\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{\partial r} + \frac{\partial \psi}{\partial r}$$

$$= r\frac{\partial^2 \psi}{\partial r^2} + 2\frac{\partial \psi}{\partial r}$$

$$= r\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r}\right]$$

$$= r\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)\right]$$

Vector Potentials (cont.)

Hence

$$\frac{\partial^2 A_r}{\partial r^2} + \left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial A_r}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 A_r}{\partial\phi^2} \right) + k^2 A_r = 0$$

$$r \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\psi}{\partial r} \right) \right] + r \left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \right) + k^2 r\psi = 0$$

Vector Potentials (cont.)

Divide by r:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)\right]+\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right)+k^2\psi=0$$

Now compare with $\nabla^2\psi+k^2\psi=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right)+\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right)+k^2\psi=0$$

They are the same!

Vector Potentials (cont.)

Hence $$A_r = r\psi$$

$$\nabla^2\psi + k^2\psi = 0$$

$$\psi = b_n(kr)\begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix}\begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

(The same holds for F_r.)

Vector Potentials (cont.)

Hence

$$A_r = (kr)\, b_n(kr) \begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

Define "Schelkunoff Spherical Bessel function"

$$\hat{B}_n(x) \equiv x\, b_n(x) = x\sqrt{\frac{\pi}{2x}}\, B_{n+1/2}(x)$$

Note: The Schelkunoff Bessel Functions are all given in closed form (for n an integer).

Then we have

$$A_r = \hat{B}_n(kr) \begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

(The same holds for F_r.)

Vector Potentials (cont.)

Summary

$$A_r = \hat{B}_n(kr)\begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix}\begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

$$\hat{B}_n(x) = x\, b_n(x) = x\sqrt{\frac{\pi}{2x}} B_{n+1/2}(x)$$

In general,

$$m \rightarrow w$$
$$n \rightarrow \upsilon$$

(The same holds for F_r.)

Vector Potentials (cont.)

Examine a typical Schelkunoff Bessel function for large x:

$$\hat{H}_n^{(2)}(x) = x\sqrt{\frac{\pi}{2x}}H_{n+1/2}^{(2)}(x)$$

$$\sim x\sqrt{\frac{\pi}{2x}}\left[\sqrt{\frac{2j}{\pi x}}e^{-jx}\, j^{n+1/2}\right]$$

$$= j^{n+1}\, e^{-jx}$$

The Schelkunoff Bessel functions do not go to zero as $x \to \infty$!

Hence A_r and F_r do not go to zero at infinity!

Vector Potentials (cont.)

Example: Calculate $\hat{H}_0^{(2)}(x)$

$$\hat{H}_0^{(2)}(x) = x\,h_0^{(2)}(x) = x\left[\sqrt{\frac{\pi}{2x}}\,H_{1/2}^{(2)}(x)\right]$$

$$H_{1/2}^{(2)}(x) = J_{1/2}(x) - jY_{1/2}(x)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x)$$

$$Y_\upsilon(x) \equiv \frac{J_\upsilon(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)}{\sin(\upsilon\pi)} \Rightarrow Y_{1/2}(x) = -J_{-1/2}(x)$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$$

Vector Potentials (cont.)

Hence

$$H_{1/2}^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\left(\sin x + j\cos x\right) = \sqrt{\frac{2}{\pi x}}\, j\, e^{-jx}$$

$$\hat{H}_0^{(2)}(x) = x\left[\sqrt{\frac{\pi}{2x}}\, H_{1/2}^{(2)}(x)\right] = x\left[\sqrt{\frac{\pi}{2x}}\left(\sqrt{\frac{2}{\pi x}}\, j\, e^{-jx}\right)\right]$$

so

$$\hat{H}_0^{(2)}(x) = j\, e^{-jx}$$

Similarly,

$$\hat{H}_0^{(1)}(x) = -\, j\, e^{+jx}$$