ECE 6341

Spring 2016

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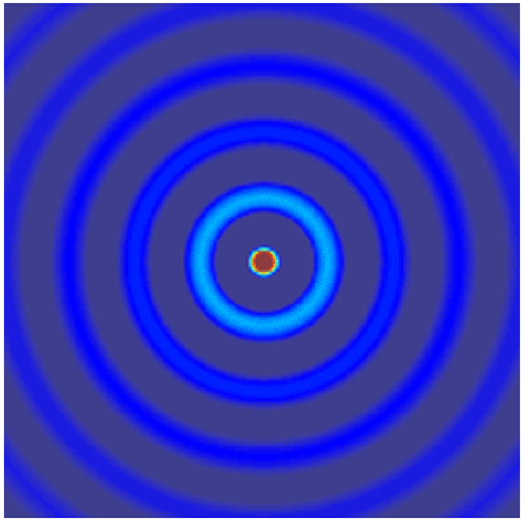
ECE Dept.



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Summary of Potentials

$$\begin{pmatrix} A_r \\ F_r \end{pmatrix} = \hat{B}_n(kr)\begin{pmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{pmatrix}\begin{pmatrix} \cos(m\phi) \\ \sin(m\phi) \end{pmatrix}$$

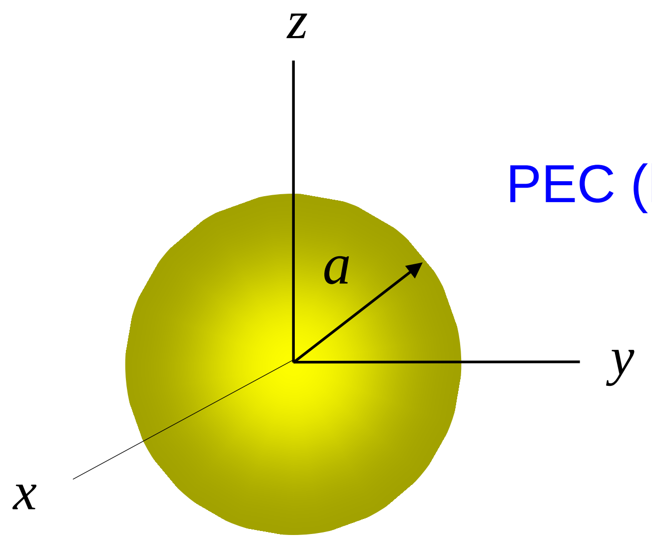
$$\hat{B}_n(x) = x\,b_n(x) = x\sqrt{\frac{\pi}{2x}}B_{n+1/2}(x)$$

$$B_\upsilon(x) = \begin{pmatrix} J_\upsilon(x) \\ Y_\upsilon(x) \\ H_\upsilon^{(1)}(x) \\ H_\upsilon^{(2)}(x) \end{pmatrix}$$

In general,

$$m \to w$$
$$n \to \upsilon$$

Spherical Resonator



TM$_r$ modes

$$A_r(r,\theta,\phi) = \hat{J}_n(kr)\,P_n^m(\cos\theta)\cos(m\phi)$$

$w = m =$ integer $(360^\circ$ included$)$

No Q_n^m (finite field at $\theta = 0, \pi$)

$\upsilon = n =$ integer $(\theta = \pi$ included$)$

No $\hat{Y}_n$ (finite field at $r = 0$)

Spherical Resonator (cont.)

$$E_\theta = 0\Big|_{r=a}$$

$$E_\phi = 0\Big|_{r=a}$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 A_r}{\partial r\partial\theta}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r}\frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi}$$

Spherical Resonator (cont.)

Hence we choose

$$\left.\frac{\partial A_r}{\partial r} = 0\right|_{r=a}$$

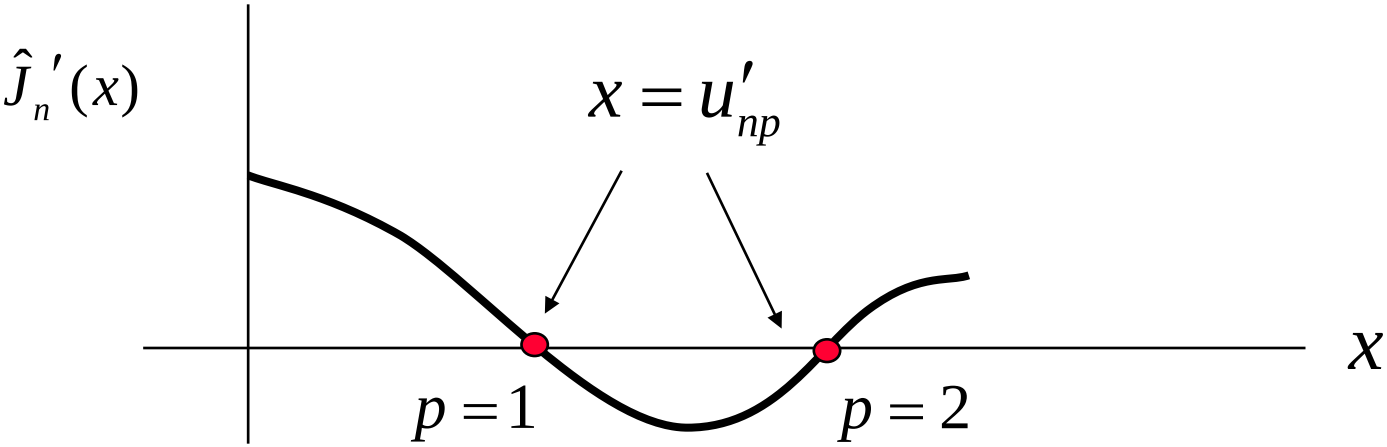
Therefore $\hat{J}_n{}'(ka) = 0$

Denote $\hat{J}_n{}'(u'_{np}) = 0$

$$u'_{np} = p^{th} \text{ root of } \hat{J}_n{}'(x)$$

Spherical Resonator (cont.)

$$u'_{np} = p^{th} \text{ root of } \hat{J}_n{}'(x)$$



Spherical Resonator (cont.)

$$ka = u'_{np}$$

Then $$\left(2\pi f_r^{TM}\sqrt{\mu\varepsilon}\right)a = u'_{np}$$

so $$f_r^{TM} = \frac{u'_{np}}{2\pi a\sqrt{\mu\varepsilon}}$$

Note: f_r is independent of m

For a non-trivial solution: $m \le n$

Spherical Resonator (cont.)

u'_{np} values

$p \setminus n$	1	2	3	4	5	6
1	2.744	3.870	4.973	6.062	7.140	8.211
2	6.117	7.443	8.722	9.968	11.189	12.391
3	9.317	10.713	12.064	13.380	14.670	15.939
4	12.486	13.921	15.314	16.674	18.009	19.321

Spherical Resonator (cont.)

TM$_{mnp}$ mode:

$$A_r(r,\theta,\phi) = \hat{J}_n\left(u'_{np}\frac{r}{a}\right)P_n^m(\cos\theta)\cos(m\phi)$$

Index m: controls oscillations in ϕ

Index n: controls oscillations in θ

Index p: controls oscillations in r

Spherical Resonator (cont.)

Note on $n = 0$:

If $n = 0$, we must also have $m = 0$ $(m \le n)$.

⇒ The TM_{00p} mode has a <u>trivial</u> field.

Proof:

$$\cos(0\phi) = 1 \qquad P_0^0(x) = 1 \qquad \hat{J}_0(x) = \cos x$$

$$\Rightarrow \quad A_r(r,\theta,\phi) = \cos(kr)$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\psi = 0$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2\psi}{\partial r\partial\theta} = 0$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r\sin\theta}\frac{\partial^2\psi}{\partial r\partial\phi} = 0$$

Spherical Resonator (cont.)

TE_r :
$$F_r = \hat{J}_n(kr)\,P_n^m(\cos\theta)\cos(m\phi)$$

$$E_\theta = -\frac{1}{\varepsilon r \sin\theta}\frac{\partial F_r}{\partial \phi}$$

$$E_\phi = \frac{1}{\varepsilon r}\frac{\partial F_r}{\partial \theta}$$

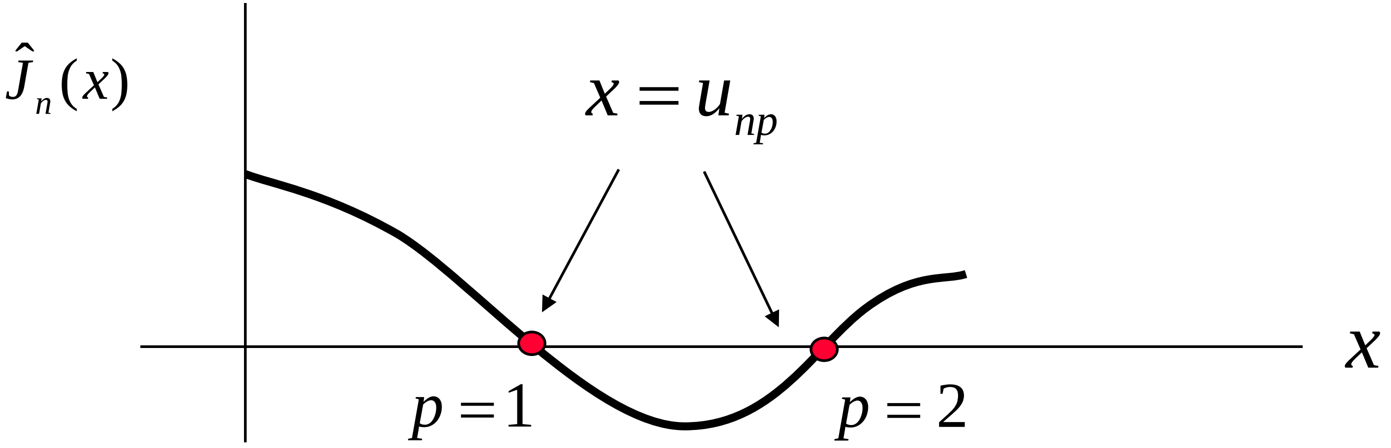
So choose

$$F_r = 0\Big|_{r=a}$$

Denote $\hat{J}_n(u_{np}) = 0$ $\qquad u_{np} = p^{th}$ root of $\hat{J}_n(x)$

Spherical Resonator (cont.)

$$u_{np} = p^{th} \text{ root of } \hat{J}_n(x)$$



Spherical Resonator (cont.)

u_{np} values

$p \backslash n$	1	2	3	4	5	6
1	4.493	5.763	6.988	8.163	9.356	10.513
2	7.725	9.095	10.417	11.705	12.967	14.207
3	10.904	12.323	13.698	15.050	16.35	17.648
4	14.066	15.515	16.924	18.301	19.653	20.983

Spherical Resonator (cont.)

$$ka = u_{np}$$

Then

$$\left(2\pi f_r^{TE}\sqrt{\mu\varepsilon}\right)a = u_{np}$$

so

$$f_r^{TE} = \frac{u_{np}}{2\pi a\sqrt{\mu\varepsilon}}$$

Note: $u_{np} = x_{n+1/2,\,p}$ since $\hat{B}_n(x) = x\,b_n(x) = x\sqrt{\dfrac{\pi}{2x}}B_{n+1/2}(x)$

Here we are using the notation from the cylindrical chapter.

Spherical Resonator (cont.)

Mode ordering (by cutoff frequency)

Notation: (m, n, p)

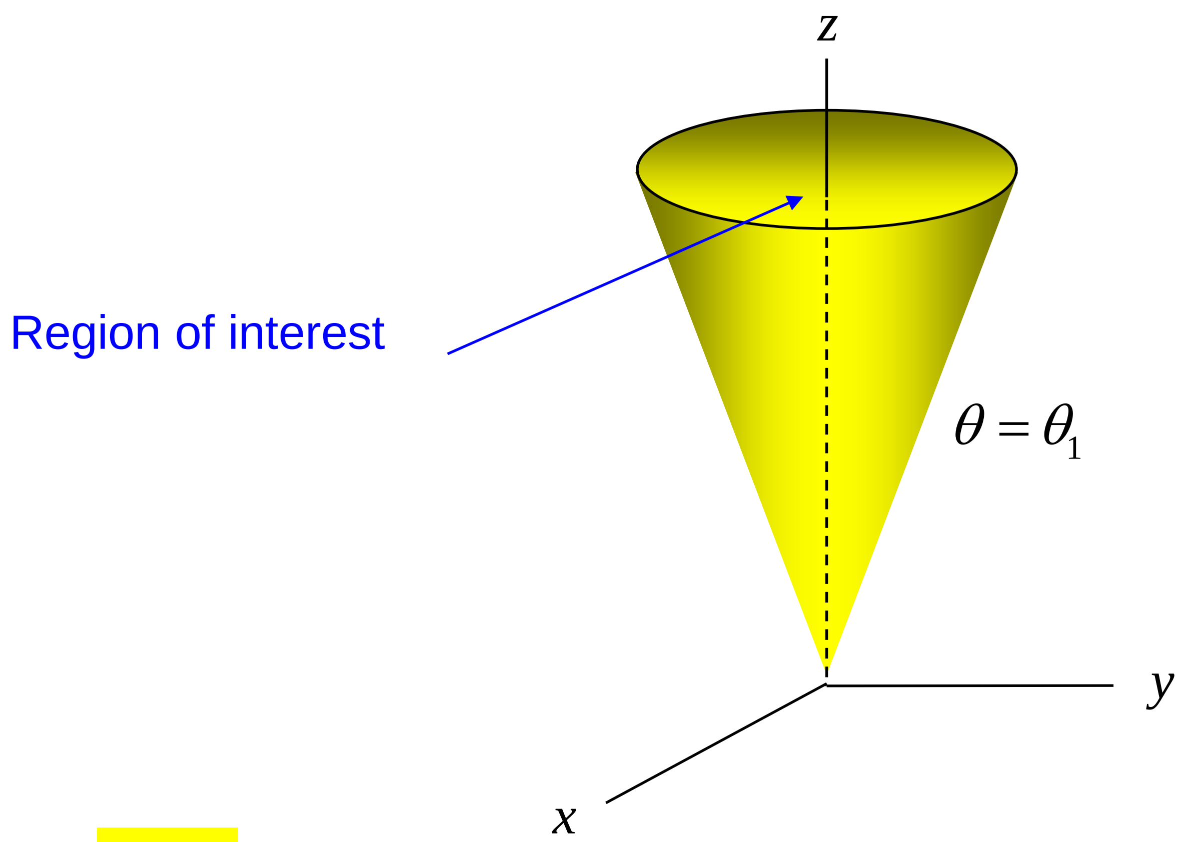
$$m \leq n$$

$$\left(TM_{011}, TM_{111}\right), \quad \left(TM_{021}, TM_{121}, TM_{221}\right), \quad \left(TE_{011}, TE_{111}\right)$$

$$n = 1,\ p = 1 \qquad n = 2,\ p = 1 \qquad n = 1,\ p = 1$$

$$u'_{11} = 2.744 \qquad u'_{21} = 3.870 \qquad u_{11} = 4.493$$

Hollow Conical PEC Horn



TM$_r$

$$A_r = \cos(m\phi)\, P_{\upsilon}^{m}(\cos\theta)\, \hat{H}_{\upsilon}^{(2)}(kr)$$

Note:

$$P_{\upsilon}^{m}(1) \neq \infty$$

$$P_{\upsilon}^{m}(-1) = \infty$$

Conical Horn (cont.)

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r$$

$$= 0\Big|_{\theta=\theta_1}$$

so

$$A_r = 0\Big|_{\theta=\theta_1}$$

Hence

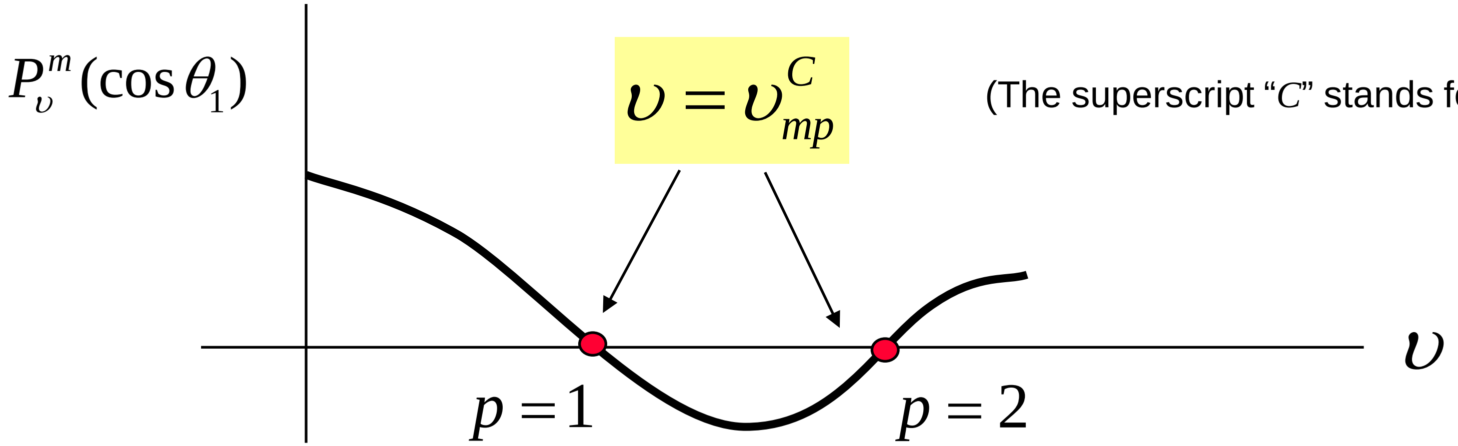
$$P_{\upsilon}^{m}(\cos\theta_1) = 0$$

Note: The integer index m is arbitrary.

(This is a transcendental equation for υ.)

Conical Horn (cont.)

Plot of function:



(The superscript "C" stands for "cone.")

$$A_r = \cos\left(m\phi\right) P_{\upsilon_{mp}^C}^m \left(\cos\theta\right) \hat{H}_{\upsilon_{mp}^C}^{(2)}(kr)$$

Conical Horn (cont.)

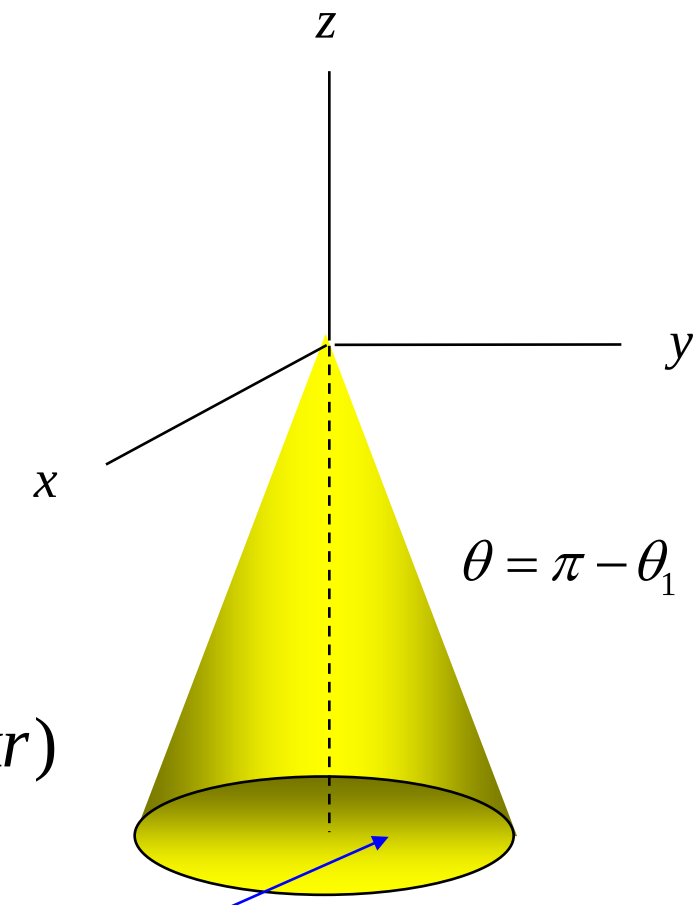
Note: An upside-down horn would use

$$\theta \to \pi - \theta$$
$$\cos\theta \to -\cos\theta$$

$$A_r = \cos(m\phi)\, P_{\upsilon_{mp}}^{m}(-\cos\theta)\, \hat{H}_{\upsilon_{mp}}^{(2)}(kr)$$

Alternatively,

$$A_r = \cos(m\phi) \sum_{n=m}^{\infty} A_n P_n^m(\cos\theta) \hat{H}_n^{(2)}(kr)$$

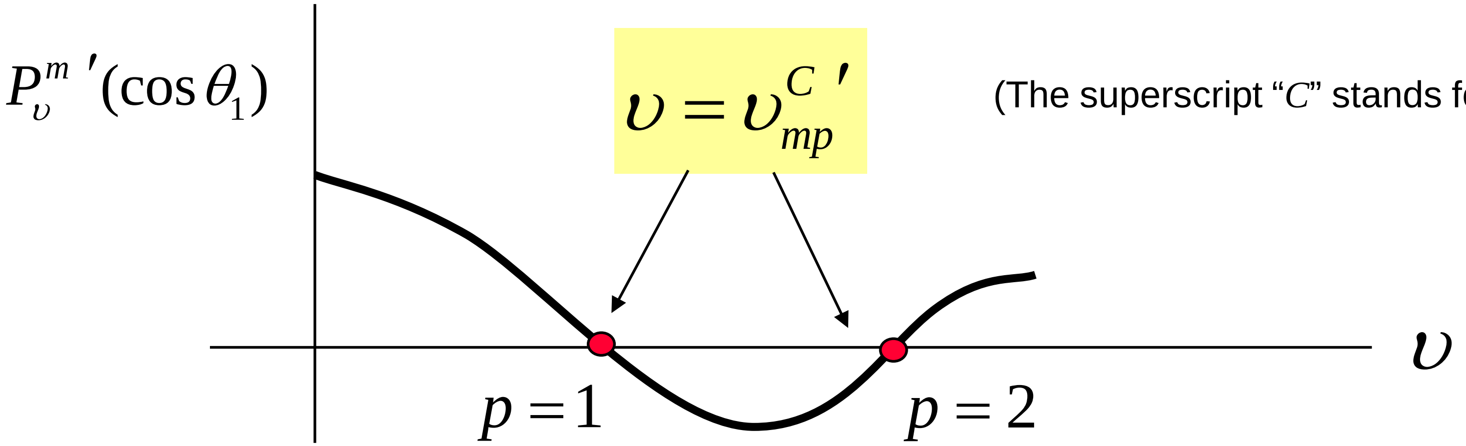


Region of interest

Conical Horn (cont.)

TE$_r$

$$\left.\frac{\partial F_r}{\partial \theta} = 0\right|_{\theta=\theta_1}$$



(The superscript “C” stands for “cone.”)

$$F_r = \sin(m\phi)\, P^m_{\upsilon^{C\,\prime}_{mp}}(\cos\theta)\, \hat{H}^{(2)}_{\upsilon^{C\,\prime}_{mp}}(kr)$$

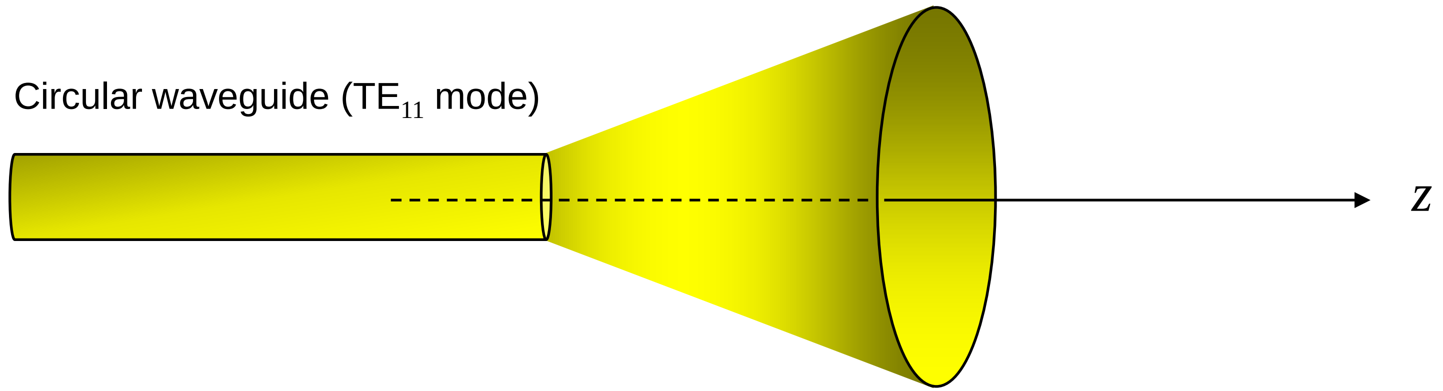
Conical Horn (cont.)

Feeding a conical horn from the TE_{11} mode of circular waveguide

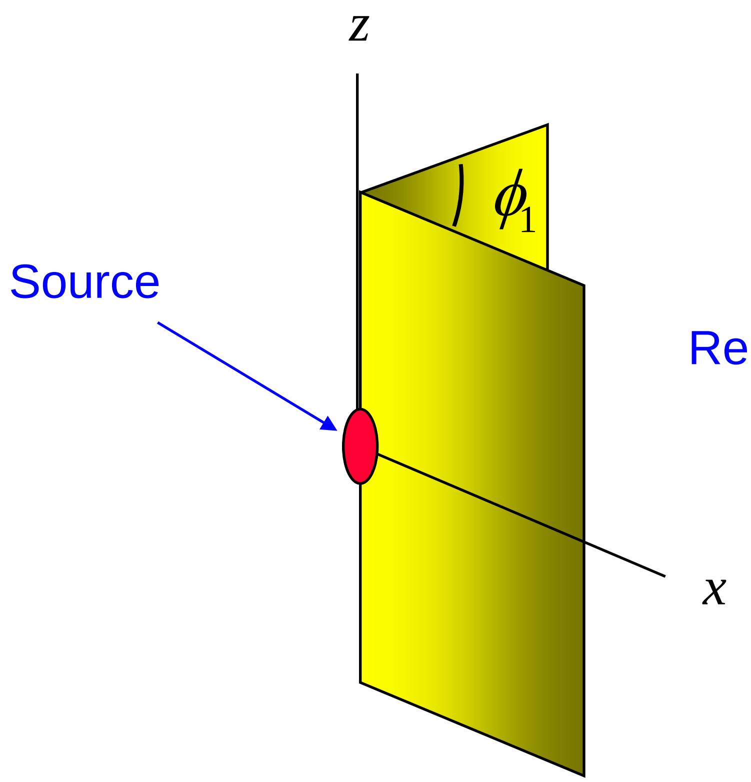
This can be approximately modeled as the TE_{11} mode of the spherical conical horn.

$$F_r = \sin\phi \, P^{(1)}_{\upsilon^C_{11}{}'}\left(\cos\theta\right) \hat{H}^{(2)}_{\upsilon^C_{11}{}'}(kr)$$

Circular waveguide (TE_{11} mode)



Wedge



Region of interest:

$$0 < \phi < \phi_1$$

Notes:

$$w \neq \text{integer} \left(360^\circ \text{ not included} \right)$$

$$\upsilon = n = \text{integer} \ \left(-z \ \text{ axis included} \right)$$

Wedge (cont.)

TM_r :

$$A_r = \sin\left(w\phi\right) P_n^w(\cos\theta)\, \hat{H}_n^{(2)}\left(kr\right)$$

$$A_r = 0 \;\; @ \;\; \phi = 0, \phi_1 \quad \Longrightarrow \quad w\phi_1 = m\pi$$

so

$$w = w_m = \frac{m\pi}{\phi_1}, \quad m = 1, 2 \ldots$$

$$A_r = \sin\left(w_m\phi\right) P_n^{w_m}(\cos\theta)\, \hat{H}_n^{(2)}\left(kr\right)$$

Wedge (cont.)

TE$_r$:

$$F_r = \cos\left(w\phi\right) P_n^{w}(\cos\theta)\, \hat{H}_n^{(2)}\left(kr\right)$$

$$\frac{\partial F_r}{\partial r} = 0 \quad @ \quad \phi = 0, \phi_1 \quad \Rightarrow \quad w\phi_1 = m\pi$$

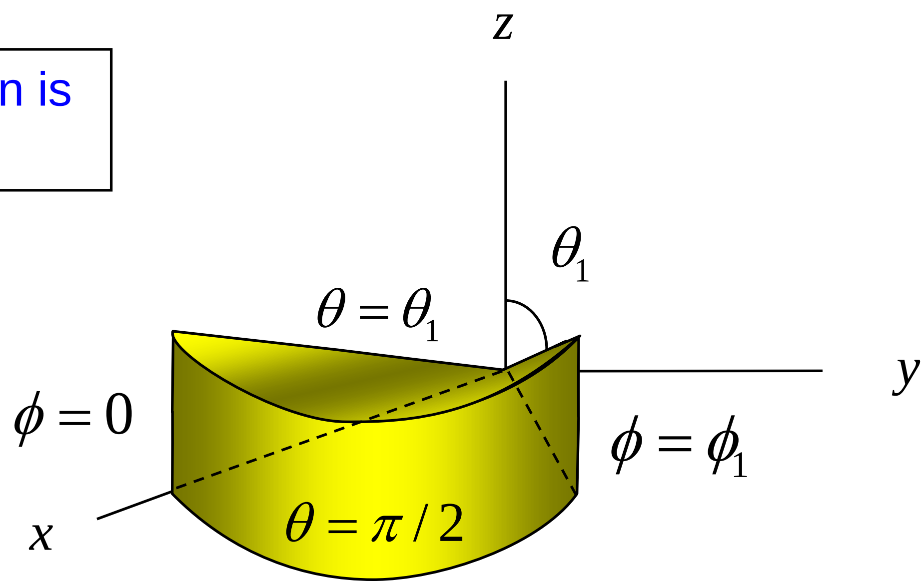
so

$$w = w_m = \frac{m\pi}{\phi_1}, \quad m = 1, 2 \ldots$$

$$A_r = \cos\left(w_m\phi\right) P_n^{w_m}(\cos\theta)\, \hat{H}_n^{(2)}\left(kr\right)$$

Spherical Horn

The bottom of the horn is on the $z = 0$ plane.



$w \neq$ integer

$\upsilon \neq$ integer $\left(-z \text{ axis not included}\right)$

Spherical Horn (cont.)

TM_r :

$$A_r = \hat{H}_\upsilon^{(2)}(kr)\sin(w\phi)\left[P_\upsilon^w(\cos\theta) + AP_\upsilon^w(-\cos\theta)\right]$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r$$

$$\Rightarrow \quad A_r = 0 \quad @ \quad \phi = 0, \phi_1$$

$$\Rightarrow \quad \sin w\phi_1 = 0$$

so

$$w = w_m = \frac{m\pi}{\phi_1}, \quad m = 1, 2 \ldots$$

Spherical Horn (cont.)

$\theta = \pi / 2:$

$$P_{\upsilon}^{w_m}(0) + A\,P_{\upsilon}^{w_m}(0) = 0$$

$$\Rightarrow \quad A = -1$$

$\theta = \theta_1:$

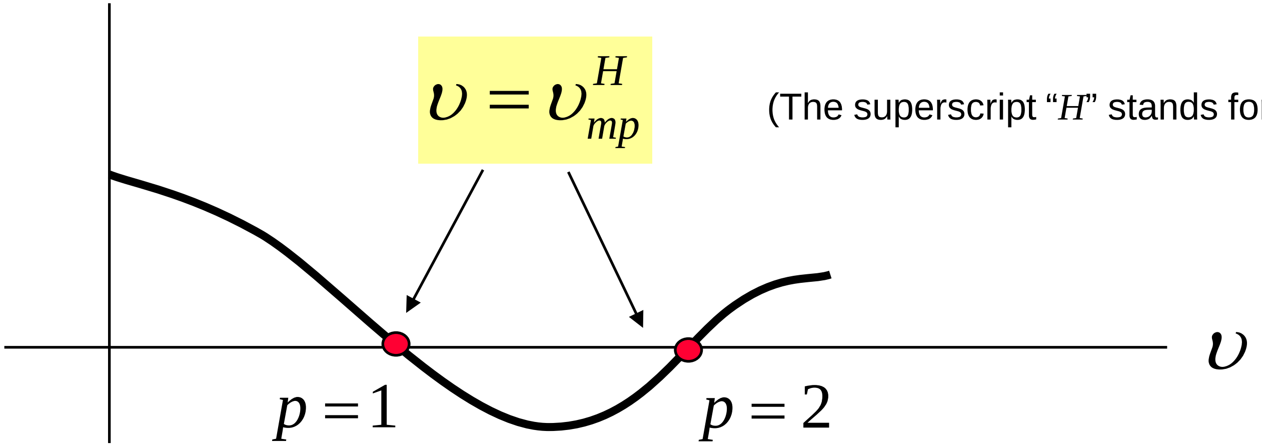
$$P_{\upsilon}^{w_m}(\cos\theta_1) - P_{\upsilon}^{w_m}(-\cos\theta_1) = 0$$

$$\Rightarrow \quad \upsilon = \upsilon_{mp}^{H}$$ (must be found numerically)

Spherical Horn (cont.)

Plot of function:

$$P_{\upsilon}^{w_m}\left(\cos\theta_1\right) - P_{\upsilon}^{w_m}\left(-\cos\theta_1\right)$$

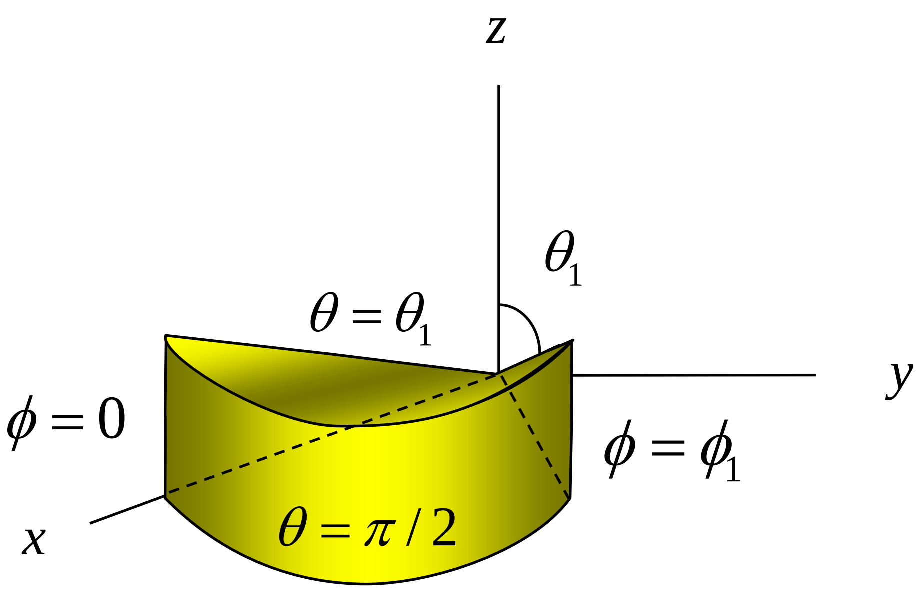


(The superscript “H” stands for “horn.”)

Spherical Horn (cont.)

TM$_{mp}$ mode:

$$A_r = \hat{H}^{(2)}_{\upsilon^H_{mp}}(kr)\,\sin(w_m\phi)\left[P^{w_m}_{\upsilon^H_{mp}}(\cos\theta) - P^{w_m}_{\upsilon^H_{mp}}(-\cos\theta)\right]$$



Spherical Horn (cont.)

TE_r :

$$F_r = \hat{H}_\upsilon^{(2)}(kr)\cos(w\phi)\left[P_\upsilon^w(\cos\theta) + AP_\upsilon^w(-\cos\theta)\right]$$

$$H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)F_r$$

$$\Rightarrow \quad \frac{\partial F_r}{\partial \phi} = 0 \quad @ \quad \phi = 0, \phi_1$$

$$\Rightarrow \quad \sin w\phi_1 = 0$$

so

$$w = w_m = \frac{m\pi}{\phi_1}, \quad m = 1, 2 \ldots$$

Spherical Horn (cont.)

$\theta = \pi / 2:$ $$\left(-\sin\theta\right)_{\theta=\pi/2}\left[P_{\upsilon}^{w_m\,\prime}(0) - A\,P_{\upsilon}^{w_m\,\prime}(0)\right] = 0$$

$\Rightarrow$ $A = +1$

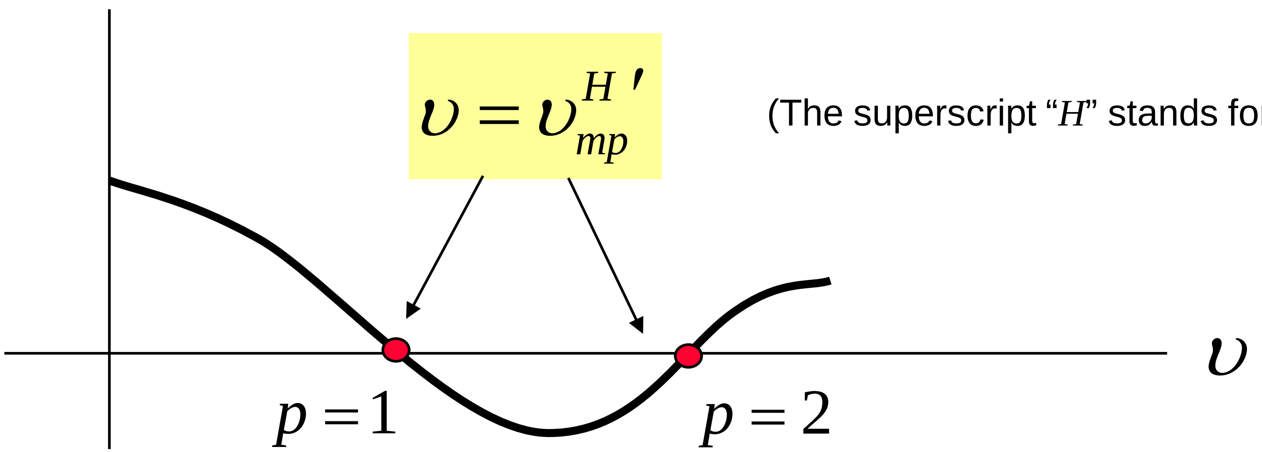
$\theta = \theta_1:$ $$P_{\upsilon}^{w_m\,\prime}\left(\cos\theta_1\right) + P_{\upsilon}^{w_m\,\prime}\left(-\cos\theta_1\right) = 0$$

$\Rightarrow$ $\upsilon = \upsilon_{mp}^{H\,\prime}$ (must be found numerically)

Spherical Horn (cont.)

Plot of function:

$$P_{\upsilon}^{w_m\,\prime}\left(\cos\theta_1\right)+P_{\upsilon}^{w_m\,\prime}\left(-\cos\theta_1\right)$$



(The superscript "*H*" stands for "horn.")

Spherical Horn (cont.)

TE$_{mp}$ mode:

$$F_r = \hat{H}^{(2)}_{\upsilon^{H\,\prime}_{mp}}(kr)\cos(w_m\phi)\left[P^{w_m}_{\upsilon^{H\,\prime}_{mp}}(\cos\theta)+P^{w_m}_{\upsilon^{H\,\prime}_{mp}}(-\cos\theta)\right]$$