ECE 6341

Spring 2016

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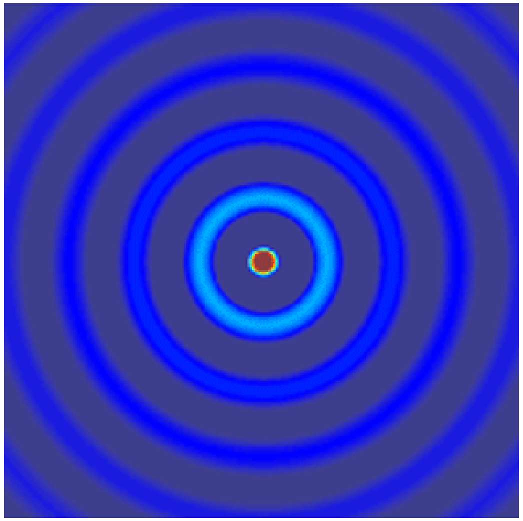
ECE Dept.



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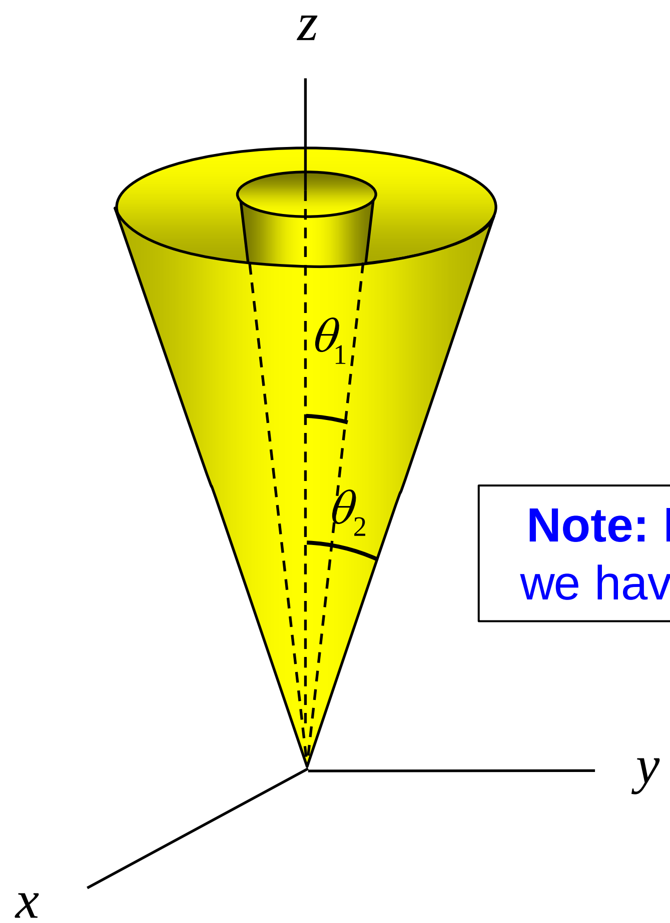
Spherical Coax

$w = m = \text{integer}\ (360^{\circ}\ \text{allowed})$

$\upsilon \neq \text{integer}$

Q is allowed



Note: If $\theta_2 > \pi / 2$, we have a bicone.

$$\begin{pmatrix} P_{\upsilon}^{m}(x) \\ Q_{\upsilon}^{m}(x) \end{pmatrix} \Rightarrow \begin{pmatrix} P_{\upsilon}^{m}(x) \\ P_{\upsilon}^{m}(-x) \end{pmatrix}$$

TM_r

$$A_r = \cos(m\phi)\,\hat{H}_{\upsilon}^{(2)}(kr)\left[P_{\upsilon}^{m}(\cos\theta) + A P_{\upsilon}^{m}(-\cos\theta)\right]$$

Spherical Coax (cont.)

Boundary conditions:

$$E_r = 0\Big|_{\theta=\theta_1,\theta_2}$$

$$\Rightarrow \quad A_r = 0\Big|_{\theta=\theta_1,\theta_2}$$

$$A_r = \cos(m\phi)\,\hat{H}_{\upsilon}^{(2)}(kr)\left[\underline{P_{\upsilon}^{m}(\cos\theta) + A P_{\upsilon}^{m}(-\cos\theta)}\right]$$

set to zero

Spherical Coax (cont.)

$$\theta = \theta_1$$

$$P_\upsilon^m(\cos\theta_1) + AP_\upsilon^m(-\cos\theta_1) = 0$$

$$\Rightarrow \quad A = -\frac{P_\upsilon^m(\cos\theta_1)}{P_\upsilon^m(-\cos\theta_1)}$$

Hence

$$H(\theta) = P_\upsilon^m(\cos\theta) - \left(\frac{P_\upsilon^m(\cos\theta_1)}{P_\upsilon^m(-\cos\theta_1)}\right) P_\upsilon^m(-\cos\theta)$$

Normalizing by multiplying by $P_\upsilon^m(-\cos\theta_1)$, we have

$$H(\theta) = P_\upsilon^m(\cos\theta)P_\upsilon^m(-\cos\theta_1) - P_\upsilon^m(\cos\theta_1)P_\upsilon^m(-\cos\theta)$$

Spherical Coax (cont.)

$$\theta = \theta_2$$

$$P_\upsilon^m\left(\cos\theta_2\right)P_\upsilon^m\left(-\cos\theta_1\right) - P_\upsilon^m\left(\cos\theta_1\right)P_\upsilon^m\left(-\cos\theta_2\right) = 0$$

This is a transcendental equation for υ: $\upsilon = \upsilon_{mp}^D$

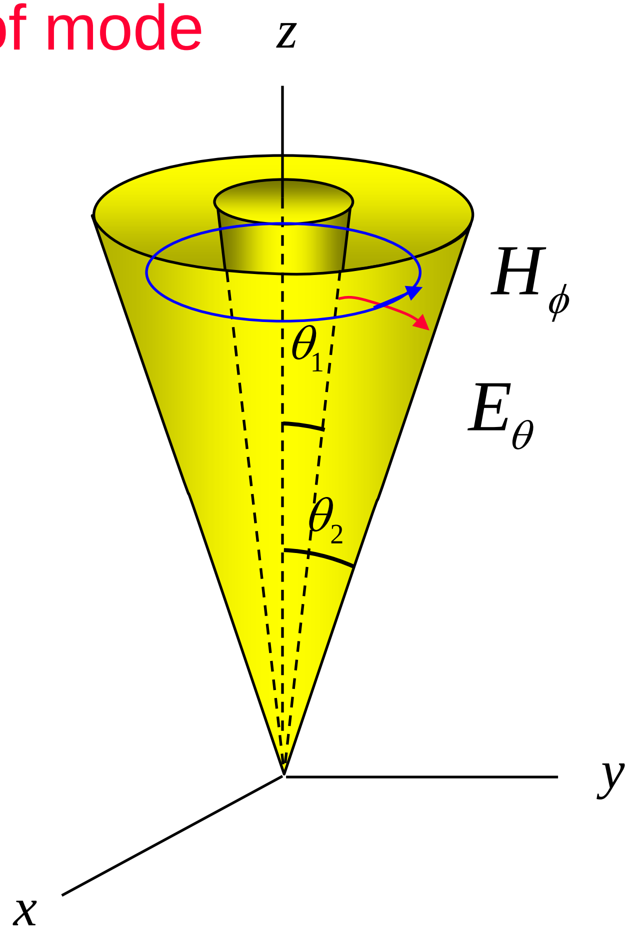
(The superscript "D" stands for "double cone.")

(These values must be found numerically.)

$$A_r = \cos\left(m\phi\right)\hat{H}^{(2)}_{\upsilon_{mp}^D}\left(kr\right)$$
$$\cdot\left[P_{\upsilon_{mp}^D}^m\left(\cos\theta\right)P_{\upsilon_{mp}^D}^m\left(-\cos\theta_1\right) - P_{\upsilon_{mp}^D}^m\left(\cos\theta_1\right)P_{\upsilon_{mp}^D}^m\left(-\cos\theta\right)\right]$$

Spherical Coax (cont.)

TEM_r transmission-line type of mode



Start by assuming A_r with $m = 0$:

$$\left(E_r, E_\theta, H_\phi\right)$$

From TM_r table:

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} = 0$$

$$H_\theta = \frac{1}{\mu r\sin\theta}\frac{\partial A_r}{\partial\phi} = 0$$

$$A_r = \hat{H}_\upsilon^{(2)}(kr)\left[AP_\upsilon(\cos\theta) + BQ_\upsilon(\cos\theta)\right]$$

Now it is more convenient to use $Q_\upsilon(x)$ instead of $P_\upsilon(-x)$.

Spherical Coax (cont.)

TEM property

Choose $\upsilon = n = 0: \quad \hat{H}_0^{(2)}(kr) = j\,e^{-jkr}$

$$A_r = e^{-jkr}\left[AP_0\left(\cos\theta\right) + BQ_0\left(\cos\theta\right)\right]$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2 A_r}{\partial r^2} + k^2 A_r\right) = 0 \quad (\text{TEM}_r)$$

Note: This potential satisfies the boundary conditions on both cones, since $E_r = E_\phi = 0$.

Spherical Coax (cont.)

$$A_r = e^{-jkr}\left[AP_0\left(\cos\theta\right) + BQ_0\left(\cos\theta\right)\right]$$

Examine the fields that come from the term $P_0(\cos\theta)$

Note that $P_0(x) = 1$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 A_r}{\partial r\,\partial\theta} = \frac{-jk}{j\omega\mu\varepsilon r}\frac{\partial A_r}{\partial\theta}$$

$$\frac{\partial}{\partial\theta}P_0\left(\cos\theta\right) = -\left(\sin\theta\right)P_0'\left(\cos\theta\right) = 0$$

⇨ $E_\theta = 0 \Rightarrow \underline{E} = \underline{0}$

⇨ $\underline{H} = \underline{0}$

Hence, we don't need to consider P_0.

Spherical Coax (cont.)

Choose $$A_r = e^{-jkr} Q_0(\cos\theta)$$

$$Q_0(x) = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \Rightarrow Q_0(\cos\theta) = \frac{1}{2}\ln\left[\frac{1+\cos\theta}{1-\cos\theta}\right]$$

$$= \frac{1}{2}\ln\left[\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}\right]$$

$$= \ln\cot\left(\frac{\theta}{2}\right)$$

Spherical Coax (cont.)

Hence

$$A_r = e^{-jkr} \ln \cot\left(\frac{\theta}{2}\right)$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 A_r}{\partial r\,\partial\theta} = \frac{-jk}{j\omega\mu\varepsilon r}\frac{\partial A_r}{\partial\theta} = \frac{-k}{\omega\mu\varepsilon r}\frac{\partial A_r}{\partial\theta}$$

$$H_\phi = -\frac{1}{\mu r}\frac{\partial A_r}{\partial\theta}$$

$$\Rightarrow \quad \frac{E_\theta}{H_\phi} = \frac{k}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon} = \eta$$

Spherical Coax (cont.)

$$E_\theta = \frac{-k}{\omega\mu\varepsilon r}\frac{\partial A_r}{\partial\theta}$$

$$= \frac{-k}{\omega\mu\varepsilon r}\frac{\partial}{\partial\theta}\left(\ln\left[\cot\left(\frac{\theta}{2}\right)\right]\right)e^{-jkr}$$

$$= \frac{-k}{\omega\mu\varepsilon r}e^{-jkr}\left[\frac{1}{\cot\left(\frac{\theta}{2}\right)}\left(-\csc^2\left(\frac{\theta}{2}\right)\right)\left(\frac{1}{2}\right)\right]$$

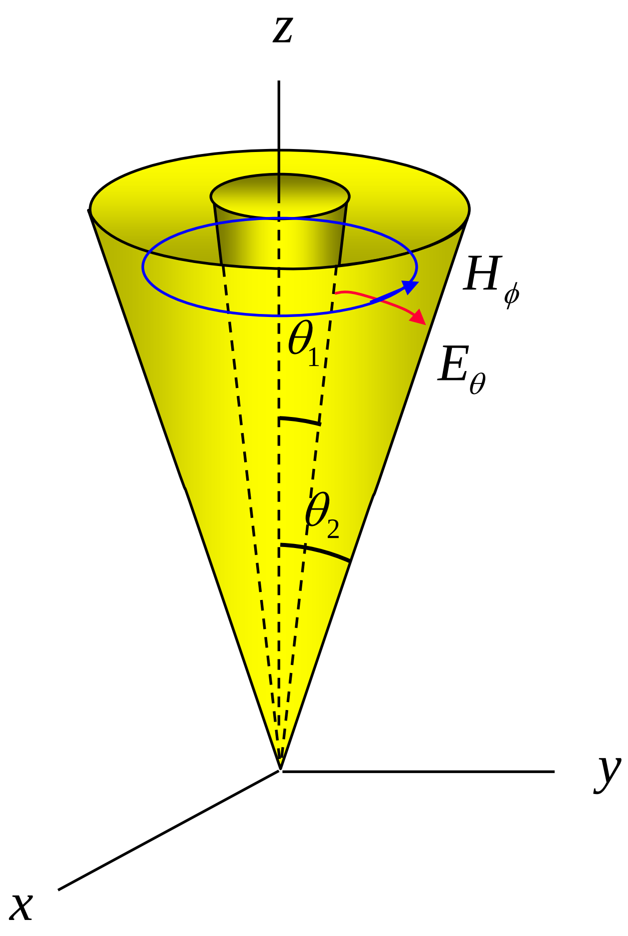
$$= \frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{r}e^{-jkr}\left[\frac{1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right] = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{r}e^{-jkr}\left(\frac{1}{\sin\left(2\left(\frac{\theta}{2}\right)\right)}\right)$$

Note: The constant in front will be ignored.

Spherical Coax (cont.)

$$E_\theta = \frac{1}{r} e^{-jkr} \left(\frac{1}{\sin\theta} \right)$$

$$H_\phi = \frac{E_\theta}{\eta}$$



Note: To compare with cylindrical coordinates, we can use

$$r \sin\theta = \rho$$

Spherical Coax (cont.)

Voltage on the coax:

$$V(r)=\int_{A}^{B}\underline{E}\cdot d\underline{r}$$

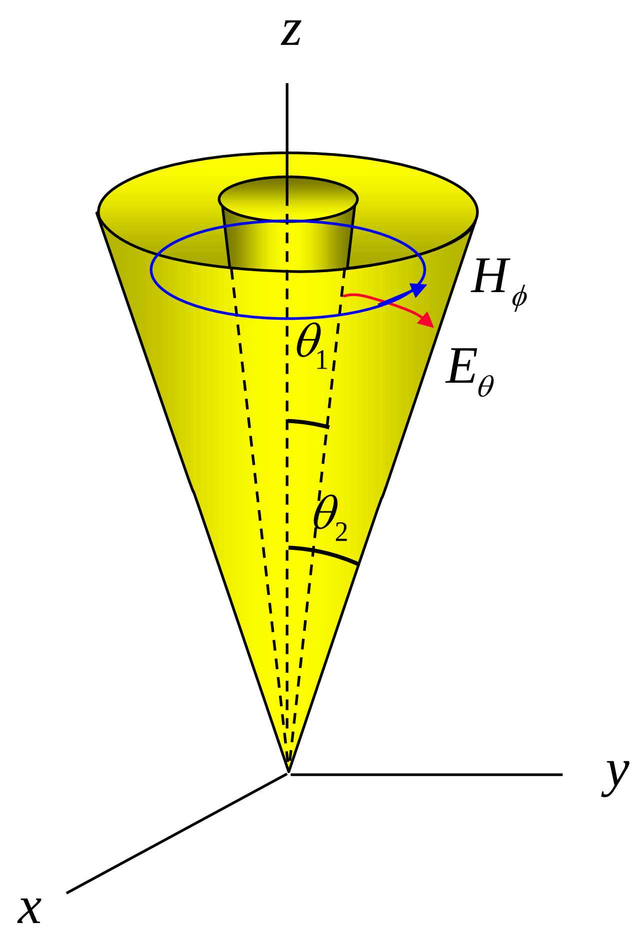
$$=\int_{A}^{B}E_\theta\, r d\theta + E_\phi(\sin\theta)\,d\phi + E_r dr$$

$$=\int_{\theta_1}^{\theta_2}E_\theta\, r d\theta$$

$$=\int_{\theta_1}^{\theta_2}\frac{1}{r}e^{-jkr}\left(\frac{1}{\sin\theta}\right)r d\theta$$

$$=e^{-jkr}\int_{\theta_1}^{\theta_2}\left(\frac{1}{\sin\theta}\right)d\theta$$

$$V(r)=e^{-jkr}\left[\ln\tan\left(\frac{\theta}{2}\right)\right]_{\theta_1}^{\theta_2}$$



$$V(r)=e^{-jkr}\left[\ln\tan\left(\frac{\theta}{2}\right)\right]_{\theta_1}^{\theta_2}$$