ECE 6341

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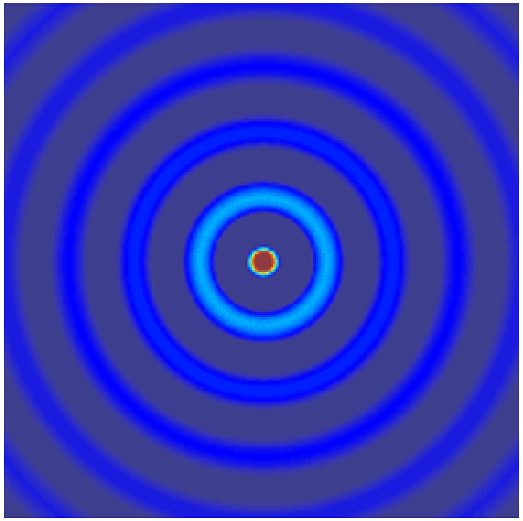
Prof. David R. Jackson
ECE Dept.



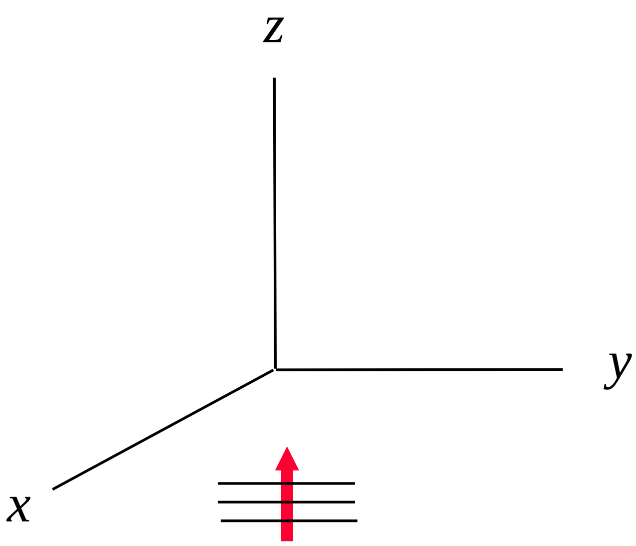
Notes 25

Scalar Wave Transformation

Incident wave:

$$\psi(x,y,z) = e^{-jkz}$$

This is a scalar function,
e.g. the pressure of a sound wave.



$$\psi = e^{-jkz} = e^{-jkr\cos\theta}$$

$$= \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta)$$

Notes:

- No ϕ variation $\rightarrow m = 0$
- Must be finite at the origin (only j_n)
- Must be finite on the z axis (only P_n)

Note: Spherical Bessel functions are used here, not Schelkunoff Bessel functions.

Scalar Wave Transformation (cont.)

Multiply both sides by $P_m(\cos\theta)\sin\theta$ and integrate.

Orientation:

$$\int_0^{\pi} P_n(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta = \begin{cases} \dfrac{2}{2n+1}, & n = m \\ 0 & n \neq m \end{cases}$$

(Harrington Eq. (6.41))

Hence

$$a_m\left(\frac{2}{2m+1}\right) j_m\left(kr\right) = \int_0^{\pi} e^{-jkr\cos\theta}\,P_m(\cos\theta)\,\sin\theta\,d\theta$$

We can now relabel $m \to n$.

Scalar Wave Transformation (cont.)

Let

$$x = kr$$

$$u = \cos\theta$$

$$du = -\sin\theta \, d\theta$$

We then have

$$a_n \left(\frac{2}{2n+1} \right) j_n(x) = -\int_{+1}^{-1} e^{-jxu} P_n(u) \, du$$

$$= \int_{-1}^{+1} e^{-jxu} P_n(u) \, du$$

Scalar Wave Transformation (cont.)

The coefficients are therefore determined from

$$a_n\left(\frac{2}{2n+1}\right)j_n(x)=\int_{-1}^{+1}e^{-jxu}\,P_n(u)\,du$$

To find the coefficients, take the limit as $x\to 0$.

Scalar Wave Transformation (cont.)

Recall that

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \qquad\qquad J_\upsilon(x) \sim \frac{x^\upsilon}{2^\upsilon \Gamma(\upsilon+1)} \qquad x << 1$$

Note:

$$\Gamma(n+1) = n!$$

Therefore, as $x \to 0$, we have

$$j_n(x) \sim \sqrt{\frac{\pi}{2x}} \left(\frac{x^{n+1/2}}{2^{n+1/2}\Gamma\left(n + \frac{1}{2} + 1\right)} \right)$$

Scalar Wave Transformation (cont.)

or

$$j_n(x) \sim \sqrt{\pi}\,\frac{x^n}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}$$

As $x \to 0$ we therefore have

$$a_n\left(\frac{2}{2n+1}\right) j_n(x) = \int_{-1}^{+1} e^{-jxu}\, P_n(u)\, du$$

$$a_n\left(\frac{2}{2n+1}\right)\left[\sqrt{\pi}\,\frac{x^n}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}\right] = \int_{-1}^{+1} e^{-jxu}\, P_n(u)\, du$$

Scalar Wave Transformation (cont.)

Note: If we now let $x \to 0$ we get zero on both sides (unless $n = 0$).

Solution: Take the derivative with respect to x (n times) before setting $x = 0$.

$$j_n(x) \sim \sqrt{\pi}\,\frac{x^n}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}$$

$$\frac{d^n}{dx^n} j_n(x) \sim \sqrt{\pi}\,\frac{n!}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}$$

Scalar Wave Transformation (cont.)

Hence

$$a_n\left(\frac{2}{2n+1}\right) j_n^{(n)}(0) = \int_{-1}^{+1} (-ju)^n P_n(u)\,du$$

$$a_n\left(\frac{2}{2n+1}\right)\left[\sqrt{\pi}\,\frac{n!}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}\right] = (-j)^n \int_{-1}^{+1} u^n P_n(u)\,du$$

$$= (-j)^n (2) \int_0^1 u^n P_n(u)\,du$$

Define

$$I_n \equiv \int_0^1 u^n P_n(u)\,du$$

Scalar Wave Transformation (cont.)

Hence

$$a_n = (-j)^n \, 2I_n \left[\left(\frac{2n+1}{2} \right) \left[2^{n+1} \Gamma \left(n + \frac{1}{2} + 1 \right) \frac{1}{n!\sqrt{\pi}} \right] \right]$$

Next, we try to evaluate I_n:

$$I_n = \int_0^1 u^n P_n(u) \, du$$

$$= \int_0^1 u^n \left[\frac{1}{2^n n!} \frac{d^n}{du^n} \left(u^2 - 1\right)^n \right] du$$

(Rodriguez's formula)

Scalar Wave Transformation (cont.)

Therefore
$$I_n = \frac{1}{2^n n!} \int_0^1 u^n \left[\frac{d^n}{du^n}\left(u^2 - 1\right)^n \right] du$$

Integrate by parts *n* times:
$$\int_0^1 f\left(\frac{dg}{du}\right) du = \left[fg \right]_0^1 - \int_0^1 g\left(\frac{df}{du}\right) du$$

$$I_n = \frac{1}{2^n n!}\left[(-1)^n n! \int_0^1 \left(u^2 - 1\right)^n du \right]$$

Notes:

$$\left[\frac{d^m}{du^m}\left(u^2 - 1\right)^n \right]_{u=1} = 0, \quad m < n \qquad \left[\frac{d^m}{du^m} u^n \right]_{u=0} = 0, \quad m < n \qquad \frac{d^n}{du^n} u^n = n!$$

Scalar Wave Transformation (cont.)

or

$$I_n = \frac{1}{2^n}\int_0^1 \left(1-u^2\right)^n du$$

Schaum's outline Mathematical Handbook Eq. (15.24):

$$\int_0^1 \left(1-x^2\right)^n dx = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+1\right)}{2\Gamma\left(n+\frac{1}{2}+1\right)}$$

Scalar Wave Transformation (cont.)

Hence

$$I_n = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}$$

Scalar Wave Transformation (cont.)

We then have

$$a_n = (-j)^n 2\left[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)}\right]\left(\frac{2n+1}{2}\right)\left[2^{n+1}\Gamma\left(n+\frac{1}{2}+1\right)\frac{1}{n!\sqrt{\pi}}\right]$$

Note: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Hence,

$$a_n = (-j)^n \Gamma(n+1)(2n+1)\frac{1}{n!}$$

Scalar Wave Transformation (cont.)

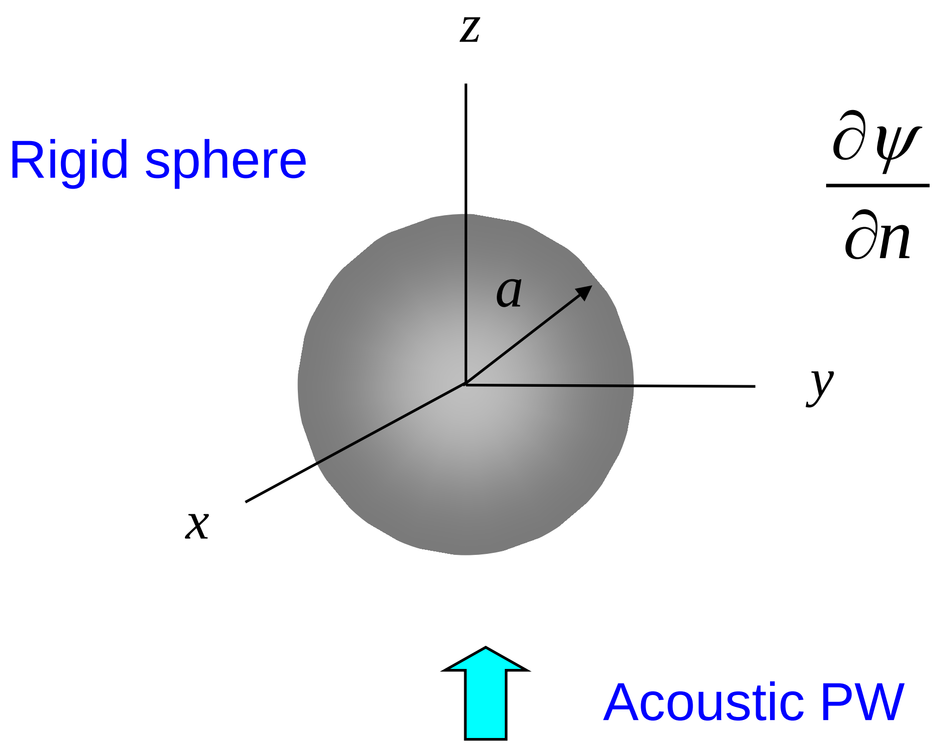
Now use $\Gamma(n+1) = n!$

so, $$a_n = (-j)^n (2n+1)$$

Hence

$$e^{-jkz} = \sum_{n=0}^{\infty} (-j)^n (2n+1) \, j_n(kr) P_n(\cos\theta)$$

Acoustic Scattering



$$\frac{\partial \psi}{\partial n} = 0 \Big|_{r=a}$$

so, $\psi(x, y, z) =$ pressure of sound wave

Acoustic Scattering (cont.)

$$\psi^i = e^{-jkz}$$

$$k = \frac{2\pi}{\lambda_s}$$

$\lambda_s = \lambda$ of sound wave

$$\psi = \psi^i + \psi^s$$

$$\frac{\partial \psi^s}{\partial n} = -\left.\frac{\partial \psi^i}{\partial n}\right|_{r=a}$$

Acoustic Scattering (cont.)

We have

$$\psi^i = \sum_{n=0}^{\infty} a_n \, j_n(kr) \, P_n(\cos\theta)$$

where $a_n = (-j)^n (2n+1)$

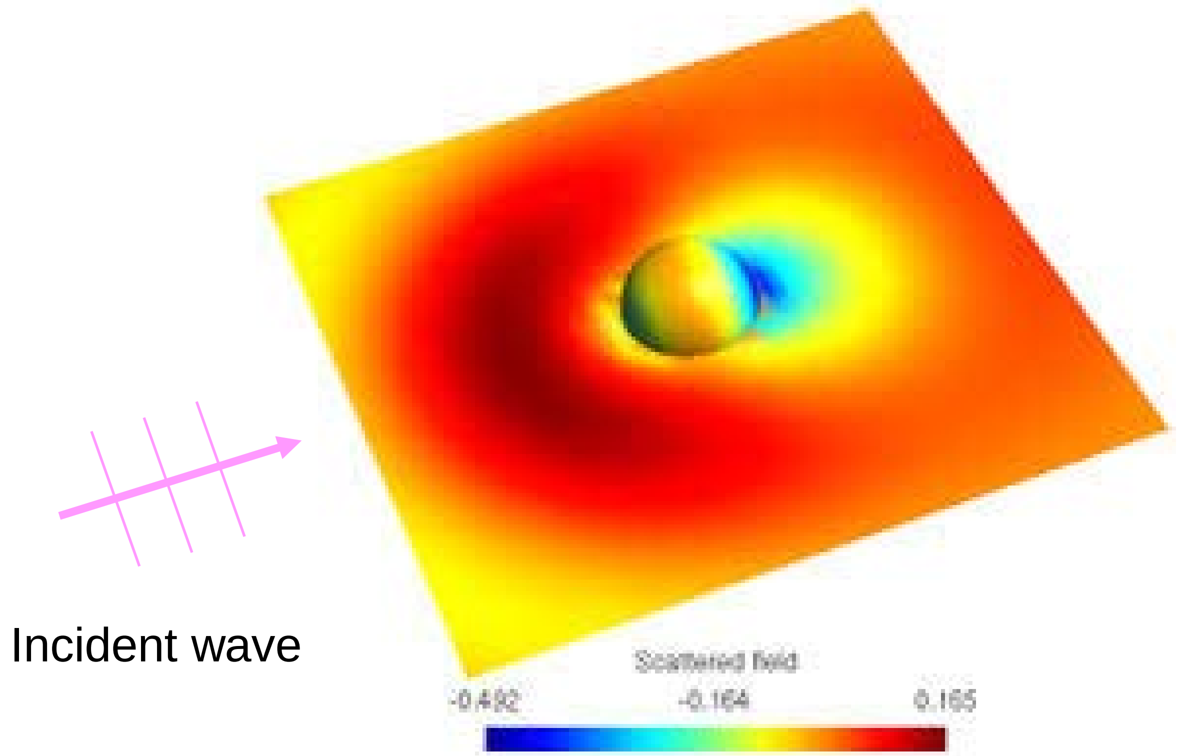
Choose

$$\psi^s = \sum_{n=0}^{\infty} b_n \, h_n^{(2)}(kr) \, P_n(\cos\theta)$$

Hence,

$$b_n = -a_n \left[\frac{j_n'(ka)}{h_n'^{(2)}(ka)} \right]$$

Acoustic Scattering (cont.)



Real part of pressure scattered by a sphere for $ka = 1.0$

http://www.paraffinalia.co.uk/Software/examples.shtml