ECE 6341

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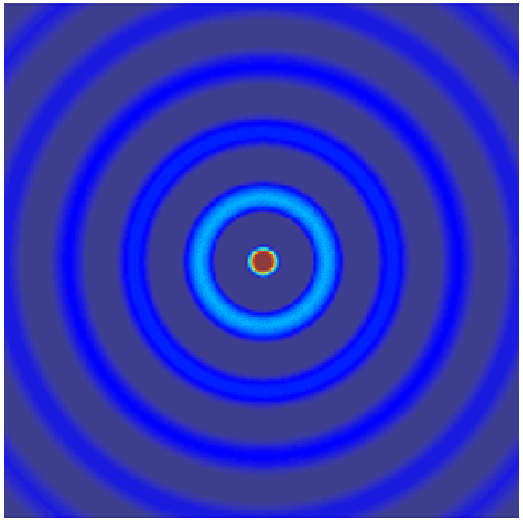

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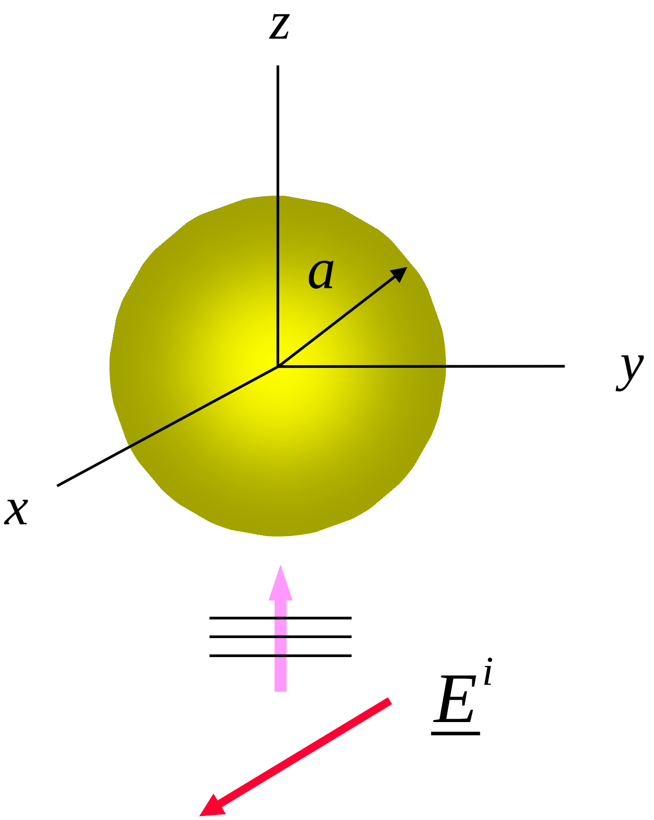
EM Plane-Wave Transformation

$$\underline{E}^i = \underline{\hat{x}}\, E_0\, e^{-jkz}$$

$$\underline{E}^i = \underline{\hat{x}}\, E_0 \sum_{n=0}^{\infty} a_n\, j_n(kr)\, P_n(\cos\theta)$$

$$a_n = (-j)^n (2n+1)$$



Note: The incident field will be represented using both A_r and F_r.

EM Plane-Wave Transformation

E_r directly corresponds to A_r
H_r directly corresponds to F_r

$$E_r^i = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2 A_r^i}{\partial r^2} + k^2 A_r^i\right)$$

$$H_r^i = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2 F_r^i}{\partial r^2} + k^2 F_r^i\right)$$

Any other field component will involve both A_r and F_r
(please see the next slide).

So, it is nice to work with these two radial components (E_r and H_r).

EM Plane-Wave Transformation (cont.)

$$\psi = A_r$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\psi$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 \psi}{\partial r \partial \theta}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r \sin\theta}\frac{\partial^2 \psi}{\partial r \partial \phi}$$

$$H_r = 0$$

$$H_\theta = \frac{1}{\mu r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$H_\phi = \frac{-1}{\mu r}\frac{\partial \psi}{\partial \theta}$$

$$\psi = F_r$$

$$E_r = 0$$

$$E_\theta = \frac{-1}{\varepsilon r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$E_\phi = \frac{1}{\varepsilon r}\frac{\partial \psi}{\partial \theta}$$

$$H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)\psi$$

$$H_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 \psi}{\partial r \partial \theta}$$

$$H_\phi = \frac{1}{j\omega\mu\varepsilon r \sin\theta}\frac{\partial^2 \psi}{\partial r \partial \phi}$$

EM Plane-Wave Transformation (cont.)

$$E_r^i = E_x^i \left(\underline{\hat{x}} \cdot \underline{\hat{r}} \right) = E_x^i \left(\sin\theta \cos\phi \right)$$

$$E_r^i = E_0 \sin\theta \cos\phi \sum_{n=0}^{\infty} a_n \, j_n\left(kr\right) P_n\left(\cos\theta\right)$$

Not in the form of a Debye spherical-wave expansion!

We need to put this in a form so that is matches with what we get from the Debye potential representation.

EM Plane-Wave Transformation (cont.)

Try this:

$$E_r^i = E_0 \sin\theta \cos\phi \, e^{-jkr\cos\theta}$$

$$= -E_0 \cos\phi \frac{\partial}{\partial\theta}\left(e^{-jkr\cos\theta}\right)\left(\frac{1}{-jkr}\right)$$

$$= \left(\frac{E_0}{jkr}\right)\cos\phi \frac{\partial}{\partial\theta}\left[\sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta)\right]$$

EM Plane-Wave Transformation (cont.)

Now use the "integration formula"

(Schaum's outline Eq. (26.2))

Harrington notation

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \left[(-1)^m \right]$$

Note: The $(-1)^m$ term is added to agree with the Harrington notation.

Hence

$$P_n^1(x) = -\sqrt{1-x^2}\, P_n'(x)$$ Eq. (E.16) in Harrington

or

$$P_n'(x) = -\frac{P_n^1(x)}{\sqrt{1-x^2}}$$

EM Plane-Wave Transformation (cont.)

Thus we have

$$\frac{\partial}{\partial\theta}P_n(\cos\theta) = -\sin\theta P_n'(\cos\theta)$$

$$= -\sin\theta\left(\frac{-1}{\sqrt{1-\cos\theta^2}}\right)P_n^1(\cos\theta)$$

$$= P_n^1(\cos\theta)$$

Hence

$$E_r^i = \left(\frac{E_0}{jkr}\right)\cos\phi\sum_{n=0}^{\infty}a_n j_n(kr)P_n^1(\cos\theta)$$

EM Plane-Wave Transformation (cont.)

Next, use $\hat{J}_n(kr) = kr\, j_n(kr)$

so $$E_r^i = \left(\frac{-jE_0}{(kr)^2}\right)\cos\phi \sum_{n=0}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta)$$

$$a_n = (-j)^n (2n+1)$$

Now let

$$A_r^i = E_0 \cos\phi \sum_{n=0}^{\infty} c_n \hat{J}_n(kr) P_n^1(\cos\theta)$$

$$E_r^i = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2 A_r^i}{\partial r^2} + k^2 A_r^i\right)$$

Goal: solve for c_n

EM Plane-Wave Transformation (cont.)

Compare these two equations for the incident radial field component:

E_r that is known for the incident plane wave

$$E_r^i = \left(\frac{-jE_0}{(kr)^2} \right) \cos\phi \sum_{n=0}^{\infty} a_n \hat{J}_n(kr) P_n^1(\cos\theta)$$

$$a_n = (-j)^n (2n+1)$$

We need to put these in the same form so we can solve for c_n.

$$E_r^i = \frac{1}{j\omega\mu\varepsilon} k^2 E_0 \cos\phi \sum_{n=0}^{\infty} c_n \left[\hat{J}_n''(kr) + \hat{J}_n(kr) \right] P_n^1(\cos\theta)$$

E_r from Debye potential

EM Plane-Wave Transformation (cont.)

We now need to evaluate

$$F = \hat{J}_n''\left(kr\right) + \hat{J}_n\left(kr\right)$$

To evaluate this, use

$$\hat{J}_n\left(x\right) = x\, j_n(x)$$

$$\hat{J}_n'\left(x\right) = j_n(x) + x\, j_n'(x)$$

$$\hat{J}_n''\left(x\right) = j_n'(x) + j_n'(x) + x j_n''(x)$$

$$= x\, j_n''(x) + 2\, j_n'(x)$$

EM Plane-Wave Transformation (cont.)

Hence

$$\hat{J}_n''(x)+\hat{J}_n(x)=\left(x\,j_n''(x)+2\,j_n'(x)\right)+x\,j_n(x)$$

To simplify this, use the spherical Bessel Eq:

$$x^2y''+2xy'+\left[x^2-n\left(n+1\right)\right]y=0$$

or

$$xy''+2y'+\left(\frac{1}{x}\right)\left[x^2-n\left(n+1\right)\right]y=0$$

Hence:

$$x\,j_n''(x)+2\,j_n'(x)=-\frac{1}{x}\left[x^2-n\left(n+1\right)\right]j_n\left(x\right)$$

EM Plane-Wave Transformation (cont.)

Therefore:

$$
\begin{aligned}
\hat{J}_n''(x) + \hat{J}_n(x) &= -\frac{1}{x}\left[x^2 - n(n+1)\right] j_n(x) + x\, j_n(x) \\
&= \frac{1}{x} n(n+1)\, j_n(x) \\
&= \frac{1}{x^2} n(n+1)\, \hat{J}_n(x)
\end{aligned}
$$

Hence:

$$
E_r^i = \frac{1}{j\omega\mu\varepsilon} k^2 E_0 \cos\phi \sum_{n=0}^{\infty} c_n \left[\hat{J}_n''(kr) + \hat{J}_n(kr)\right] P_n^1(\cos\theta)
$$

$$
\Downarrow
$$

$$
E_r^i = \frac{1}{j\omega\mu\varepsilon} k^2 E_0 \cos\phi \sum_{n=0}^{\infty} c_n P_n^1(\cos\theta) \left[\frac{1}{(kr)^2} n(n+1)\, \hat{J}_n(kr)\right]
$$

EM Plane-Wave Transformation (cont.)

or

E_r from Debye potential

$$E_r^i = \left(\frac{E_0}{j\omega\mu\varepsilon}\right)\frac{1}{r^2}\cos\phi\sum_{n=0}^{\infty} c_n\, n(n+1)\,\hat{J}_n(kr)\,P_n^1(\cos\theta)$$

Compare with the known expansion for E_r^i :

E_r that is known for the incident plane wave

$$E_r^i = \left(\frac{-jE_0}{(kr)^2}\right)\cos\phi\sum_{n=0}^{\infty} a_n \hat{J}_n(kr)\,P_n^1(\cos\theta)$$

$$a_n = (-j)^n(2n+1)$$

We see that

$$\left(\frac{E_0}{j\omega\mu\varepsilon}\right)c_n n(n+1) = \frac{-jE_0}{k^2}a_n$$

EM Plane-Wave Transformation (cont.)

$$\left(\frac{E_0}{j\omega\mu\varepsilon}\right)c_n n(n+1) = \frac{-jE_0}{\omega^2\mu\varepsilon}a_n$$

Hence we have

$$c_n = a_n\left[\frac{1}{\omega}\left(\frac{1}{n(n+1)}\right)\right]$$

EM Plane-Wave Transformation (cont.)

Hence

$$c_n = \left[(-j)^n (2n+1)\right]\left[\frac{1}{\omega}\left(\frac{1}{n(n+1)}\right)\right]$$

or

$$c_n = \frac{1}{\omega}\left[\left(\frac{(-j)^n (2n+1)}{n(n+1)}\right)\right]$$

Hence

$$A_r^i = E_0 \cos\phi \sum_{n=0}^{\infty}\left[\frac{1}{\omega}\left(\frac{(-j)^n (2n+1)}{n(n+1)}\right)\right]\hat{J}_n(kr)\, P_n^1(\cos\theta)$$

EM Plane-Wave Transformation (cont.)

Similarly, to find F_r for the incident plane wave, we use:

$$\underline{H}^i = \underline{\hat{y}} \frac{E_0}{\eta} e^{-jkz}$$

$$\underline{E}^i = \underline{\hat{x}} E_0 e^{-jkz}$$

$$\underline{H}_r^i = \frac{E_0}{\eta} \sin\theta \sin\phi \, e^{-jkz\cos\theta}$$

Note: $\sin\phi$ instead of $\cos\phi$, and $\frac{1}{\eta}$ factor included

Hence we have

$$A_r^i = E_0 \cos\phi \sum_{n=0}^{\infty} c_n \hat{J}_n(kr) P_n^1(\cos\theta)$$

$$F_r^i = \frac{E_0}{\eta} \sin\phi \sum_{n=0}^{\infty} c_n \hat{J}_n(kr) P_n^1(\cos\theta)$$