ECE 6341

Spring 2016

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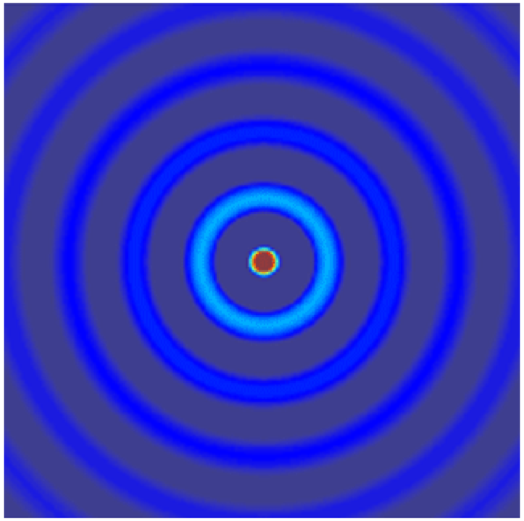
ECE Dept.



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Asymptotic Evaluation of Integral

Goal: Evaluate the following integral as Ω becomes large:

$$I(\Omega)=\int_a^b f(x)e^{j\Omega g(x)}dx$$

$g(x)$ is a real function here.

$$\Omega \to \infty$$

Case 1: $g'(x) \neq 0, \;\; x \in [a,b]$ Integration by Parts

Case 2: $g'(x_0) = 0, \;\; x_0 \in (a,b)$ Stationary-Phase Method

Integration by Parts

$$I(\Omega) = \int_a^b f(x) e^{j\Omega g(x)} dx \qquad \Omega \to \infty$$

$$g'(x) \neq 0 \qquad x \in [a, b]$$

Note that

$$\frac{d}{dx}\left[e^{j\Omega g(x)}\right] = \left(j\Omega g'(x)\right) e^{j\Omega g(x)}$$

So that we may write

$$I(\Omega) = \int_a^b f(x) \frac{1}{j\Omega g'(x)} \frac{d}{dx}\left[e^{j\Omega g(x)}\right] dx$$

Integration by Parts

$$I(\Omega)=\int_a^b \underbrace{f(x)\frac{1}{j\Omega g'(x)}}_{u}\frac{d}{dx}\Big[\underbrace{e^{j\Omega g(x)}}_{v}\Big]dx$$

$$\int_a^b u(x)\frac{d}{dx}v(x)\,dx=u(x)v(x)\Big|_a^b-\int_a^b v(x)\frac{d}{dx}u(x)\,dx$$

Hence

$$I(\Omega)=\frac{f(x)}{j\Omega g'(x)}e^{j\Omega g(x)}\Big|_a^b$$

$$-\int_a^b e^{j\Omega g(x)}\left(\frac{1}{j\Omega}\right)\frac{d}{dx}\left[\frac{f(x)}{g'(x)}\right]dx$$

Integration by Parts (cont.)

Riemann-Lebesgue Lemma:

$$\int_a^b F(x)\, e^{j\Omega g(x)}\, dx \to 0 \quad \text{as} \quad \Omega \to \infty$$

(The integrand oscillates faster, so the integral tends to zero)

Hence

$$I = \frac{f(x)}{j\Omega g'(x)} e^{j\Omega g(x)} \Big|_a^b + o(1/\Omega)$$

or

$$I \sim \frac{f(x)}{j\Omega g'(x)} e^{j\Omega g(x)} \Big|_a^b$$

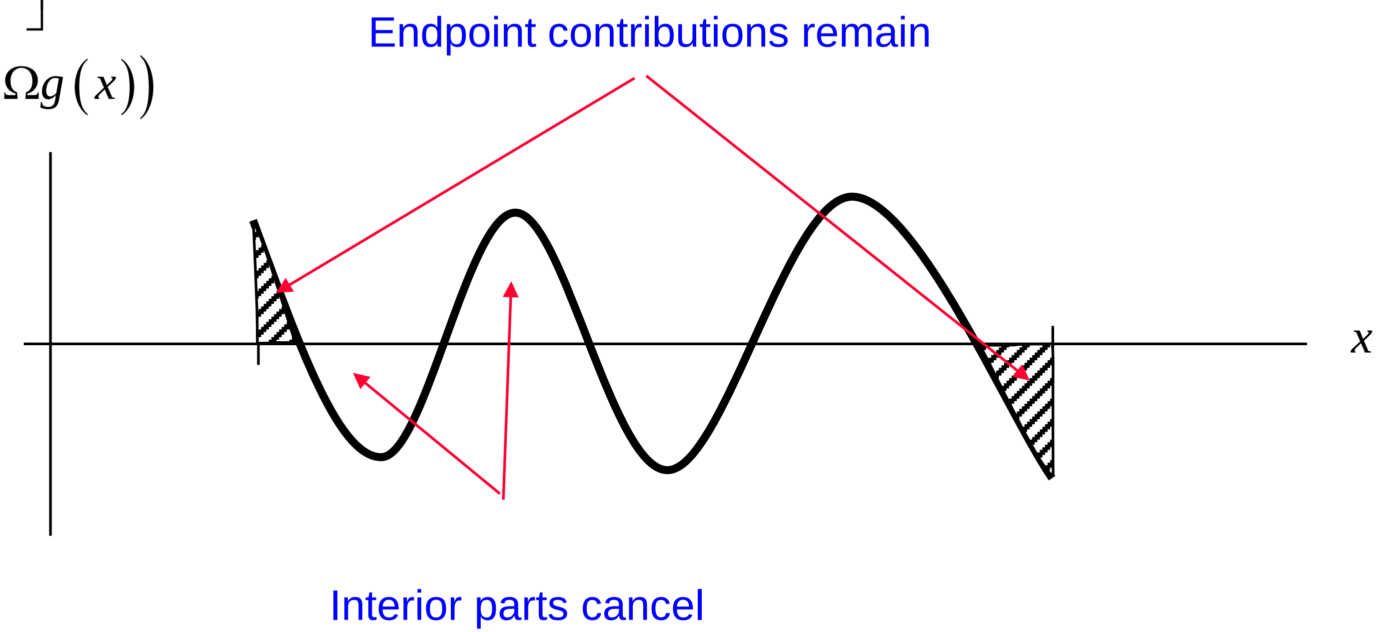
"Small o" notation:
"It decays faster than this as Ω gets large."

Integration by Parts (cont.)

Physical interpretation

$$I(\Omega) = \int_a^b f(x) e^{j\Omega g(x)} dx \qquad I \sim \frac{f(x)}{j\Omega g'(x)} e^{j\Omega g(x)} \Big|_a^b$$

$$\mathrm{Re}\left[f(x) e^{j\Omega g(x)} \right] = f(x)\cos\left(\Omega g(x)\right)$$



Stationary-Phase Method

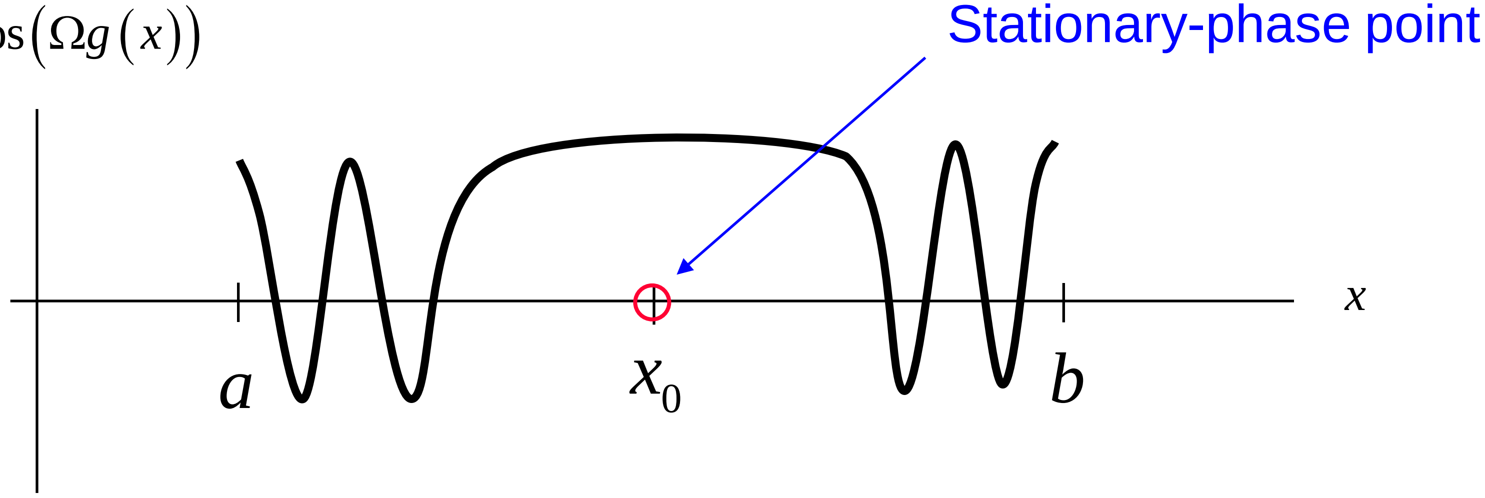
$$I(\Omega) = \int_a^b f(x) e^{j\Omega g(x)} dx$$

$$g'(x_0) = 0 \qquad x_0 \in (a, b)$$

Assume
$$f(x_0) \neq 0$$
$$g''(x_0) \neq 0$$

$$g(x) \approx g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2$$

$$\mathrm{Re}\left[f(x) e^{j\Omega g(x)} \right] = f(x) \cos\left(\Omega g(x)\right)$$



Stationary-Phase Method (cont.)

Conjecture:

$$I \sim \int_{x_0-\Delta}^{x_0+\Delta} f(x)\, e^{j\Omega g(x)} dx$$

If $\Delta \to 0$ (slow enough)

To ensure that the stationary phase point region dominates over the outside regions, we define:

$$I_{SPP}(\Omega) = \int_{x_0-\Delta}^{x_0+\Delta} f(x)\, e^{j\Omega g(x)} dx = \int_{x_0-\Delta}^{x_0+\Delta} f(x) \frac{1}{j\Omega g'(x)} \frac{d}{dx}\left[e^{j\Omega g(x)}\right] dx$$

$$I_{left}(\Omega) = \int_{a}^{x_0-\Delta} f(x)\, e^{j\Omega g(x)} dx = \int_{a}^{x_0-\Delta} f(x) \frac{1}{j\Omega g'(x)} \frac{d}{dx}\left[e^{j\Omega g(x)}\right] dx$$

$$I_{right}(\Omega) = \int_{x_0+\Delta}^{b} f(x)\, e^{j\Omega g(x)} dx = \int_{x_0+\Delta}^{b} f(x) \frac{1}{j\Omega g'(x)} \frac{d}{dx}\left[e^{j\Omega g(x)}\right] dx$$

Stationary-Phase Method (cont.)

We have that

$$\left|I_{left}\right| \le \frac{A_1}{\Omega\left|g'\left(x_0-\Delta\right)\right|} \qquad \left|I_{right}\right| \le \frac{A_2}{\Omega\left|g'\left(x_0+\Delta\right)\right|}$$

For some constants A_1 and A_2.

We will show later that

$$\left|I_{SPP}\right| = O\left(\frac{1}{\sqrt{\Omega}}\right)$$

"Big O" notation:
"It behaves like this as Ω gets large."

Stationary-Phase Method (cont.)

Hence, we require that

$$\left|I_{left}\right| \ll \frac{1}{\sqrt{\Omega}} \qquad \left|I_{right}\right| \ll \frac{1}{\sqrt{\Omega}}$$

Approximating the first derivative of *g* by a Taylor series,

$$g'(x_0 - \Delta) \approx g'(x_0) + (-\Delta) g''(x_0)$$

We require:

$$\frac{1}{\Omega \left| g''(x_0) \Delta \right|} \ll \frac{1}{\sqrt{\Omega}}$$

Hence $\Delta\sqrt{\Omega} \gg 1$ (This needs to be one of our assumptions.)

Stationary-Phase Method (cont.)

Hence, we assume that:

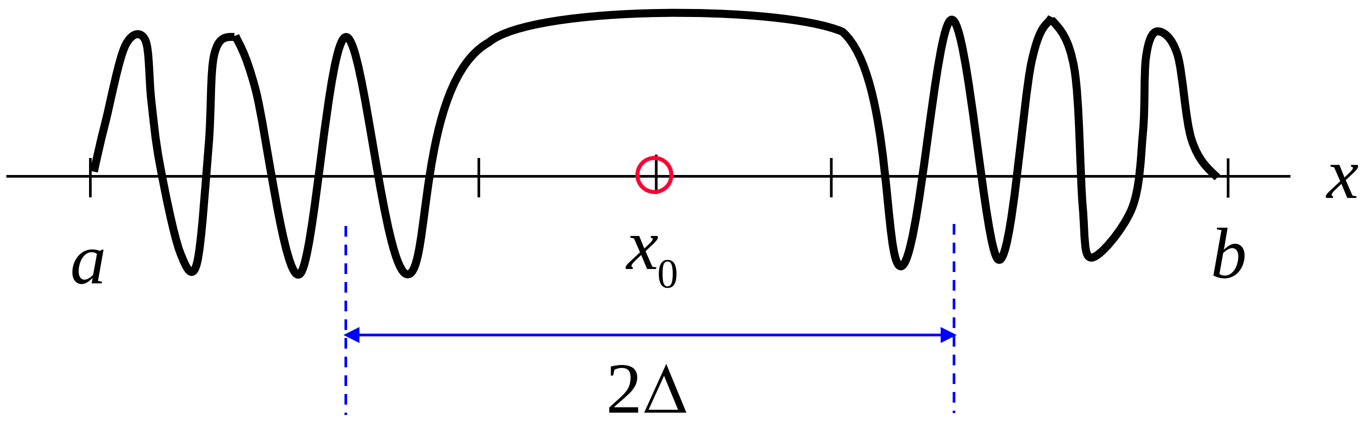
$$\Omega \to \infty$$

$$\Delta \to 0$$

$$\Delta\sqrt{\Omega} \to \infty$$

This ensures that

$$I \sim \int_{x_0-\Delta}^{x_0+\Delta} f(x)\, e^{j\Omega g(x)}\, dx$$



Stationary-Phase Method (cont.)

Since $\Delta \rightarrow 0$

Assume $f(x) \approx f(x_0)$

Hence $I \sim f(x_0) \int_{x_0 - \Delta}^{x_0 + \Delta} e^{j\Omega g(x)} dx$

Stationary-Phase Method (cont.)

$$g(x) \approx g(x_0) + \cancel{g'(x_0)(x - x_0)} + \frac{1}{2} g''(x_0)(x - x_0)^2$$

$$I(\Omega) \approx f(x_0) e^{j\Omega g(x_0)} \int_{x_0-\Delta}^{x_0+\Delta} e^{j\frac{1}{2} g''(x_0)\Omega(x-x_0)^2} dx$$

or

$$I(\Omega) \approx f(x_0) e^{j\Omega g(x_0)} \int_{x_0-\Delta}^{x_0+\Delta} e^{\pm j\frac{1}{2} |g''(x_0)|\Omega(x-x_0)^2} dx$$

where $g''(x_0) = \pm |g''(x_0)|$

Stationary-Phase Method (cont.)

Let

$$s = (x - x_0)\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}} \qquad ds = dx\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}}$$

Then

$$I(\Omega) \approx f(x_0)\, e^{j\Omega g(x_0)} \sqrt{\frac{2}{\Omega\left|g''(x_0)\right|}} \int_{-S_L}^{+S_L} e^{\pm js^2}\, ds$$

where

$$S_L = \Delta\sqrt{\frac{\Omega\left|g''(x_0)\right|}{2}}$$

As $\Omega \to \infty, \quad S_L \to \infty$ since $\Delta\sqrt{\Omega} \to \infty$

Stationary-Phase Method (cont.)

Therefore

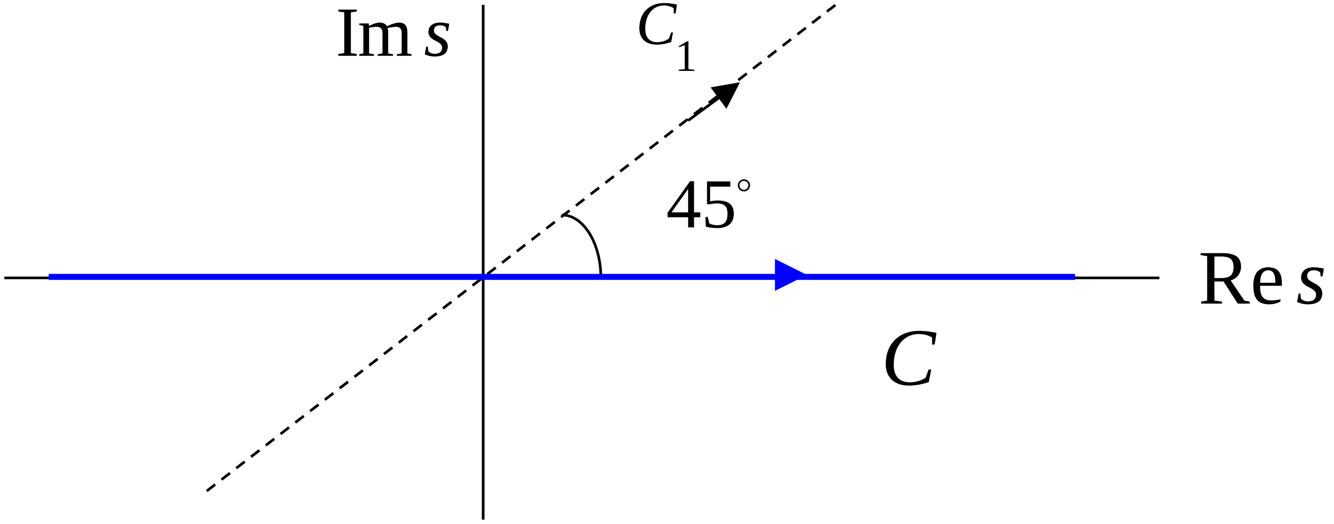
$$I_L \equiv \int_{-S_L}^{+S_L} e^{\pm js^2} ds \to \int_{-\infty}^{+\infty} e^{\pm js^2} ds \qquad \Delta\sqrt{\Omega} \to \infty$$

Let

$$I_{L1} = \int_{-\infty}^{+\infty} e^{+js^2} ds = \int_C e^{+js^2} ds$$

Deform the path:

$$\int_C \to \int_{C_1}$$



Stationary-Phase Method (cont.)

Let

$$s = te^{j\frac{\pi}{4}} \Rightarrow \begin{cases} s^2 = t^2 e^{j\frac{\pi}{2}} = jt^2 \\ ds = dt\, e^{j\frac{\pi}{4}} \end{cases}$$

Hence

$$I_{L1} = e^{j\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{j\left(jt^2\right)} dt$$

$$= e^{j\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{-t^2} dt$$

$$= e^{j\frac{\pi}{4}} \sqrt{\pi}$$

Stationary-Phase Method (cont.)

Similarly,

$$I_{L2} \equiv \int_{-\infty}^{+\infty} e^{-js^2} ds = e^{-j\frac{\pi}{4}} \sqrt{\pi}$$

Then

$$I \approx f(x_0) e^{j\Omega g(x_0)} \sqrt{\frac{2}{\Omega \left| g''(x_0) \right|}} e^{\pm j\frac{\pi}{4}} \sqrt{\pi}$$

Note: $I = O\left(\frac{1}{\sqrt{\Omega}} \right)$

Stationary-Phase Method (cont.)

Hence

$$I \approx f(x_0)\, e^{j\Omega g(x_0)} \sqrt{\frac{2\pi}{\Omega \left| g''(x_0) \right|}}\, e^{\pm j\frac{\pi}{4}}$$

$$\begin{array}{ll} +, & g''(x_0) > 0 \\ -, & g''(x_0) < 0 \end{array}$$

Example

$$J_0(\Omega) = \frac{1}{\pi}\int_0^{\pi} \cos(\Omega \sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\,\mathrm{Re}\int_0^{\pi} e^{j\Omega\sin\theta}\, d\theta$$

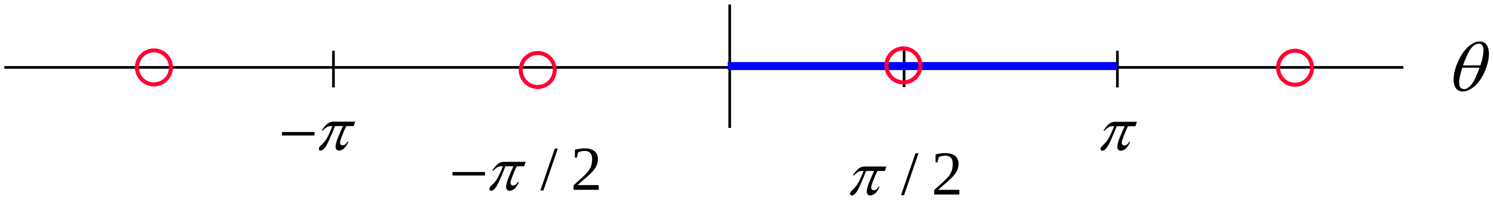
$$f(\theta) = (1)$$

$$g(\theta) = \sin\theta$$

$$g'(\theta_0) = \cos\theta_0 = 0$$

$$\Rightarrow \quad \theta_0 = \frac{\pi}{2} + n\pi$$

Example (cont.)



$$g(\theta) = \sin\theta$$

$$g'(\theta) = \cos\theta$$

$$g''(\theta) = -\sin\theta$$

$$g''(\theta_0) = -\sin\theta_0 = -1 < 0$$

Hence

$$J_0(\Omega) \sim \frac{1}{\pi}\mathrm{Re}\left\{ e^{j\Omega(1)} \sqrt{\frac{2\pi}{\Omega|-1|}} e^{-j\frac{\pi}{4}} \right\}$$

Example (cont.)

$$J_0(\Omega) \sim \frac{1}{\pi} \operatorname{Re}\left\{ e^{j\Omega(1)} \sqrt{\frac{2\pi}{\Omega|-1|}} e^{-j\frac{\pi}{4}} \right\}$$

Hence

$$J_0(\Omega) \sim \sqrt{\frac{2}{\pi\Omega}} \cos\left(\Omega - \frac{\pi}{4}\right)$$

as $\Omega \to \infty$