ECE 6341

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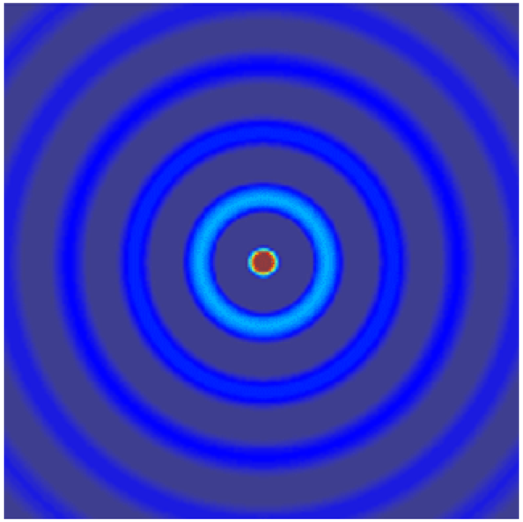

Notes 30

2-D Stationary Phase Method

$$I(\Omega) = \iint_S f(x,y) e^{j\Omega g(x,y)} dx\, dy$$

2D stationary phase point:

$$g_x(x_0, y_0) = 0$$
$$g_y(x_0, y_0) = 0$$

Assume $(x,y) \approx (x_0, y_0)$

$$\begin{aligned} g(x,y) \approx g(x_0,y_0) &+ \cancel{g_x}(x-x_0) + \cancel{g_y}(y-y_0) \\ &+ \frac{1}{2} g_{xx}(x-x_0)^2 + \frac{1}{2} g_{yy}(y-y_0)^2 \\ &+ g_{xy}(x-x_0)(y-y_0) \end{aligned}$$

2-D Stationary Phase (cont.)

Denote

$$\alpha = \frac{1}{2} g_{xx}\left(x_0, y_0\right)$$

$$\beta = \frac{1}{2} g_{yy}\left(x_0, y_0\right)$$

$$\gamma = g_{xy}\left(x_0, y_0\right)$$

Then

$$I(\Omega) \sim f\left(x_0, y_0\right) e^{j\Omega g\left(x_0, y_0\right)} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j\Omega\left[\alpha\left(x-x_0\right)^2+\beta\left(y-y_0\right)^2+\gamma\left(x-x_0\right)\left(y-y_0\right)\right]} \, dx\, dy$$

2-D Stationary Phase (cont.)

Let
$$\bar{x} = x - x_0$$
$$\bar{y} = y - y_0$$

and
$$\alpha = |\alpha| \, (\pm)_x$$
$$\beta = |\beta| \, (\pm)_y$$

where
$$(\pm)_x = \begin{cases} +1, & g_{xx} > 0 \\ -1, & g_{xx} < 0 \end{cases}$$
$$(\pm)_y = \begin{cases} +1, & g_{yy} > 0 \\ -1, & g_{yy} < 0 \end{cases}$$

We then have

$$I(\Omega) \sim f(x_0, y_0) e^{j\Omega g(x_0, y_0)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_x |\alpha| \bar{x}^2 + (\pm)_y |\beta| \bar{y}^2 + \gamma \bar{x}\bar{y}\right]} \, d\bar{x} \, d\bar{y}$$

2-D Stationary Phase (cont.)

$$I(\Omega) \sim f(x_0, y_0) e^{j\Omega g(x_0, y_0)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_x |\alpha| \overline{x}^2 + (\pm)_y |\beta| \overline{y}^2 + \gamma \overline{x}\overline{y}\right]} d\overline{x}\, d\overline{y}$$

Let

$$s = \sqrt{|\alpha|}\, \overline{x} \qquad t = \sqrt{|\beta|}\, \overline{y}$$

We then have

$$I \sim f(x_0, y_0) e^{j\Omega g(x_0, y_0)} \left(\frac{1}{\sqrt{|\alpha\beta|}} \right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_x s^2 + (\pm)_y t^2 + \frac{\gamma(st)}{\sqrt{|\alpha\beta|}}\right]} ds\, dt$$

2-D Stationary Phase (cont.)

$$I \sim f\left(x_0, y_0\right) e^{j\Omega g\left(x_0, y_0\right)} \left(\frac{1}{\sqrt{|\alpha\beta|}}\right) \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_x s^2 + (\pm)_y t^2 + \frac{\gamma(st)}{\sqrt{|\alpha\beta|}}\right]} ds\, dt}_{I_2}$$

Complete the square:

$$(\pm)_x s^2 + (\pm)_y t^2 + \frac{\gamma(st)}{\sqrt{|\alpha\beta|}} = (\pm)_x \left[s + \frac{(\pm)_x \gamma t}{2\sqrt{|\alpha\beta|}}\right]^2 - (\pm)_x \frac{\gamma^2 t^2}{4|\alpha\beta|} + (\pm)_y t^2$$

$$= (\pm)_x \left[s + \frac{(\pm)_x \gamma t}{2\sqrt{|\alpha\beta|}}\right]^2 + (\pm)_y \left[t^2 - (\pm)_x (\pm)_y \frac{\gamma^2 t^2}{4|\alpha\beta|}\right]$$

2-D Stationary Phase (cont.)

The integral I_2 is then

$$I_2 = \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_y\left(t^2-(\pm)_x(\pm)_y\frac{\gamma^2 t^2}{4|\alpha\beta|}\right)\right]}\left\{\int_{-\infty}^{+\infty} e^{j\Omega(\pm)_x\left[s+(\pm)_x\frac{\gamma t}{2\sqrt{|\alpha\beta|}}\right]^2} ds\right\} dt$$

Now use

$$\bar{s} = s + (\pm)_x \frac{\gamma t}{2\sqrt{|\alpha\beta|}} \qquad d\bar{s} = ds$$

so

$$I_2 = \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_y\left(t^2-(\pm)_x(\pm)_y\frac{\gamma^2 t^2}{4|\alpha\beta|}\right)\right]}\left\{\int_{-\infty}^{+\infty} e^{j\Omega(\pm)_x \bar{s}^2} d\bar{s}\right\} dt$$

2-D Stationary Phase (cont.)

The integral I_2 is then in the form of the product of two 1-D integrals:

$$I_2 = \left\{ \int_{-\infty}^{+\infty} e^{j\Omega\left[(\pm)_y\left(t^2-(\pm)_x(\pm)_y\frac{\gamma^2t^2}{4|\alpha\beta|}\right)\right]}dt \right\} \left\{ \int_{-\infty}^{+\infty} e^{j\Omega(\pm)_x\overline{s}^2}d\overline{s} \right\}$$

2-D Stationary Phase (cont.)

This has the form $I_2 = I_t \, I_{\bar{s}}$

Integral in $\bar{s}$:

$$I_{\bar{s}} = \int_{-\infty}^{+\infty} e^{j\Omega(\pm)_x \bar{s}^2} d\bar{s}$$

$$= \sqrt{\frac{1}{\Omega}} \int_{-\infty}^{+\infty} e^{j\,(\pm)_x u^2} du$$

$$= \sqrt{\frac{\pi}{\Omega}} \, e^{j(\pm)_x \frac{\pi}{4}}$$

Use

$$u = \bar{s}\sqrt{\Omega}$$

Recall that

$$\int_{-\infty}^{+\infty} e^{j\,(\pm) x^2} dx = \sqrt{\pi}\, e^{\pm j\pi/4}$$

2-D Stationary Phase (cont.)

Integral in *t*:

$$I_t = \int_{-\infty}^{+\infty} e^{j\Omega t^2\left[(\pm)_y\left(1-(\pm)_x(\pm)_y\frac{\gamma^2}{4|\alpha\beta|}\right)\right]}dt$$

Define:

$$\Delta \equiv 1-(\pm)_x(\pm)_y\frac{\gamma^2}{4|\alpha\beta|}$$

And then let

$$\Delta = (\pm)_\Delta|\Delta|$$

2-D Stationary Phase (cont.)

Then we have

$$I_t = \int_{-\infty}^{+\infty} e^{j\Omega t^2 (\pm)_y (\pm)_\Delta |\Delta|} dt$$

Use

$$u = t\sqrt{\Omega|\Delta|}$$

$$I_t(\Omega) = \sqrt{\frac{\pi}{\Omega}}\left(\frac{1}{\sqrt{|\Delta|}}\right) e^{j\frac{\pi}{4}(\pm)_y(\pm)_\Delta}$$

Hence

$$I(\Omega) \sim f(x_0, y_0)\, e^{j\Omega g(x_0, y_0)} \left(\frac{1}{\sqrt{|\alpha\beta|}}\right) \underbrace{\left[\sqrt{\frac{\pi}{\Omega}} e^{j\frac{\pi}{4}(\pm)_x}\right]}_{I_s} \underbrace{\left[\sqrt{\frac{\pi}{\Omega}}\left(\frac{1}{\sqrt{|\Delta|}}\right) e^{j\frac{\pi}{4}(\pm)_y(\pm)_\Delta}\right]}_{I_t}$$

2-D Stationary Phase (cont.)

We then have

$$I(\Omega) \sim \frac{\pi}{\Omega} f(x_0, y_0) e^{j\Omega g(x_0, y_0)}$$

$$\cdot e^{j\frac{\pi}{4}\left[(\pm)_x + (\pm)_y (\pm)_\Delta\right]}$$

$$\cdot \left(\frac{1}{\sqrt{\left| 1 - (\pm)_x (\pm)_y \dfrac{\gamma^2}{4|\alpha\beta|} \right|}} \right) \left(\frac{1}{\sqrt{|\alpha\beta|}} \right)$$

2-D Stationary Phase (cont.)

or

$$I(\Omega) \sim \frac{\pi}{\Omega} f(x_0, y_0) e^{j\Omega g(x_0, y_0)} e^{j\frac{\pi}{4}\left[(\pm)_x + (\pm)_y (\pm)_\Delta\right]}$$

$$\cdot \left(\frac{1}{\sqrt{\left| |\alpha\beta| - (\pm)_x (\pm)_y \left(\frac{\gamma^2}{4}\right) \right|}} \right)$$

2-D Stationary Phase (cont.)

Important special case (often met in practice):

α, β both ≥ 0 or ≤ 0

and $\alpha\beta - \dfrac{\gamma^2}{4} > 0$

In this case: $\Delta \equiv 1 - (\pm)_x (\pm)_y \dfrac{\gamma^2}{4|\alpha\beta|} = 1 - \dfrac{\gamma^2}{4\alpha\beta} > 0$

so $(\pm)_\Delta = 1$

2-D Stationary Phase (cont.)

We then have:

$$I(\Omega) \sim \frac{\pi}{\Omega} f\left(x_0, y_0\right) e^{j\Omega g\left(x_0, y_0\right)} \left(e^{\pm j\pi/2}\right) \left(\frac{1}{\sqrt{\alpha\beta - \frac{\gamma^2}{4}}} \right)$$

where

$$+, \quad \alpha \text{ and } \beta > 0$$
$$-, \quad \alpha \text{ and } \beta < 0$$

2-D Stationary Phase (cont.)

Hence, the final result is

$$I(\Omega) \sim \frac{\pi}{\Omega} f(x_0, y_0) e^{j\Omega g(x_0, y_0)} (\pm j) \left(\frac{1}{\sqrt{\alpha\beta - \frac{\gamma^2}{4}}} \right)$$

where

$$\begin{array}{ll} +, & \alpha \text{ and } \beta > 0 \\ -, & \alpha \text{ and } \beta < 0 \end{array}$$