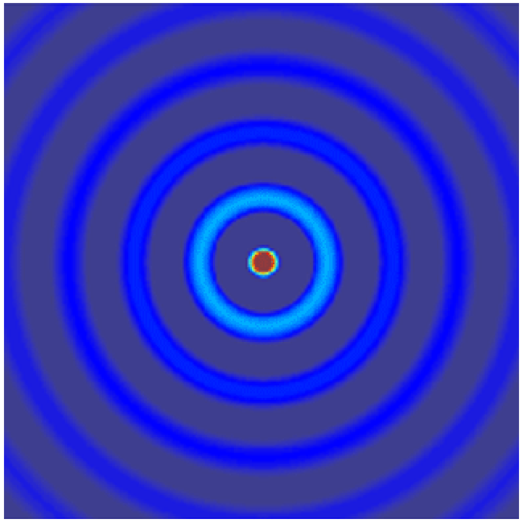
ECE 6341

Spring 2016

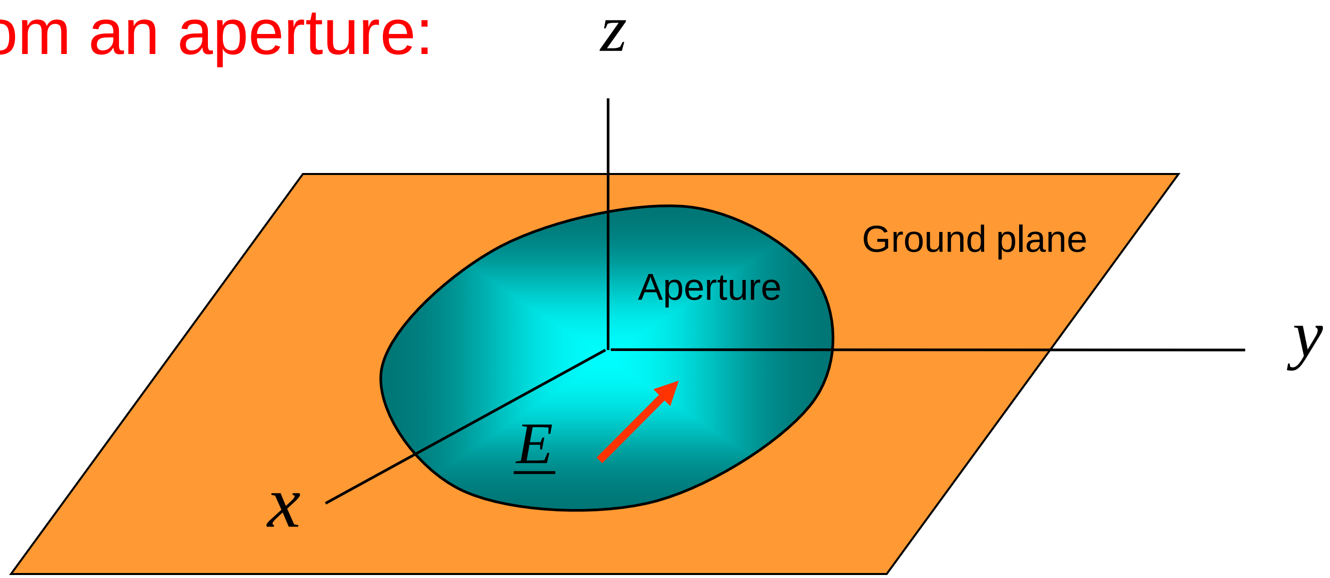
Prof. David R. Jackson
ECE Dept.



Notes 31

Example

Far field from an aperture:



Given: A known field on the aperture $\underline{E}(x, y, 0)$

From previous Fourier transform analysis, we know that above the aperture,

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underline{\tilde{E}}\left(k_x,k_y,0\right)e^{-jk_z z}e^{-j\left(k_x x+k_y y\right)}dk_x\,dk_y$$

Example (cont.)

Goal: Approximate this result for $k_0 r \gg 1$

Use spherical coordinates:

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

and define

$$\bar{k}_x = \frac{k_x}{k_0} \quad \bar{k}_y = \frac{k_y}{k_0} \quad \bar{k}_z = \frac{k_z}{k_0}$$

$$k_x x = \bar{k}_x (k_0 r) \sin\theta \cos\phi$$
$$k_y y = \bar{k}_y (k_0 r) \sin\theta \sin\phi$$
$$k_z z = \bar{k}_z (k_0 r) \cos\theta$$

Example (cont.)

We then have

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}k_0^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underline{\tilde{E}}^N\left(\bar{k}_x,\bar{k}_y,0\right)$$

$$\cdot\, e^{-j(k_0 r)\left[\bar{k}_z\cos\theta+\bar{k}_x\sin\theta\cos\phi+\bar{k}_y\sin\theta\sin\phi\right]}\,d\bar{k}_x\,d\bar{k}_y$$

where

$$\underline{\tilde{E}}\left(k_x,k_y,z\right)=\underline{\tilde{E}}\left(k_0\bar{k}_x,k_0\bar{k}_y,z\right)\equiv\underline{\tilde{E}}^N\left(\bar{k}_x,\bar{k}_y,z\right)$$

$$\bar{k}_z=\frac{k_z}{k_0}=\frac{\left(k_0^2-k_x^2-k_y^2\right)^{1/2}}{k_0}=\left(1-\bar{k}_x^2-\bar{k}_y^2\right)^{1/2}$$

Example (cont.)

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}k_0^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underline{\tilde{E}}^N\left(\bar{k}_x,\bar{k}_y,0\right)\cdot e^{-j(k_0 r)\left[\bar{k}_z\cos\theta+\bar{k}_x\sin\theta\cos\phi+\bar{k}_y\sin\theta\sin\phi\right]}\,d\bar{k}_x\,d\bar{k}_y$$

Compare with: $I(\Omega)=\iint_S f\left(k_x,k_y\right)e^{j\Omega g\left(k_x,k_y\right)}dk_x\,dk_y$

We identify:

$$\Omega=k_0 r$$

$$f\left(\bar{k}_x,\bar{k}_y\right)=\left(\frac{k_0}{2\pi}\right)^2\underline{\tilde{E}}^N\left(\bar{k}_x,\bar{k}_y,0\right)$$

$$g\left(\bar{k}_x,\bar{k}_y\right)=-\left[\left(1-\bar{k}_x^2-\bar{k}_y^2\right)^{1/2}\cos\theta+\bar{k}_x\sin\theta\cos\phi+\bar{k}_y\sin\theta\sin\phi\right]$$

Example (cont.)

2-D stationary-phase point:

$$\frac{\partial g}{\partial \bar{k}_x} = 0$$

$$\frac{\partial g}{\partial \bar{k}_y} = 0$$

Hence

$$\frac{\bar{k}_{x0}}{\left(1 - \bar{k}_{x0}^2 - \bar{k}_{y0}^2\right)^{1/2}} \cos\theta = \sin\theta \cos\phi$$

$$\frac{\bar{k}_{y0}}{\left(1 - \bar{k}_{x0}^2 - \bar{k}_{y0}^2\right)^{1/2}} \cos\theta = \sin\theta \sin\phi$$

Example (cont.)

Squaring both sides, we have:

$$\bar{k}_{x0}^2 \cos^2\theta = \sin^2\theta \, \cos^2\phi \left(1-\bar{k}_{x0}^2-\bar{k}_{y0}^2\right)$$

$$\bar{k}_{y0}^2 \cos^2\theta = \sin^2\theta \, \sin^2\phi \left(1-\bar{k}_{x0}^2-\bar{k}_{y0}^2\right)$$

Add these two equations:

$$\left(\bar{k}_{x0}^2+\bar{k}_{y0}^2\right)\cos^2\theta = \sin^2\theta \left(1-\bar{k}_{x0}^2-\bar{k}_{y0}^2\right)$$

or

$$\bar{k}_{x0}^2+\bar{k}_{y0}^2 = \sin^2\theta$$

Example (cont.)

Hence, from the first equation on the previous slide,

$$\bar{k}_{x0}^2 \cos^2\theta = \sin^2\theta \, \cos^2\phi \left(1 - \bar{k}_{x0}^2 - \bar{k}_{y0}^2\right)$$

$$\Downarrow$$

$$\bar{k}_{x0}^2 \cos^2\theta = \sin^2\theta \, \cos^2\phi \, \left(1 - \sin^2\theta\right)$$

$$= \sin^2\theta \, \cos^2\phi \, \cos^2\theta$$

Therefore,

$$\bar{k}_{x0} = \sin\theta \cos\phi$$

Similarly,

$$\bar{k}_{y0} = \sin\theta \sin\phi$$

Also,

$$\bar{k}_{z0} = \left(1 - \bar{k}_{x0}^2 - \bar{k}_{y0}^2\right)^{1/2} = \cos\theta$$

Example (cont.)

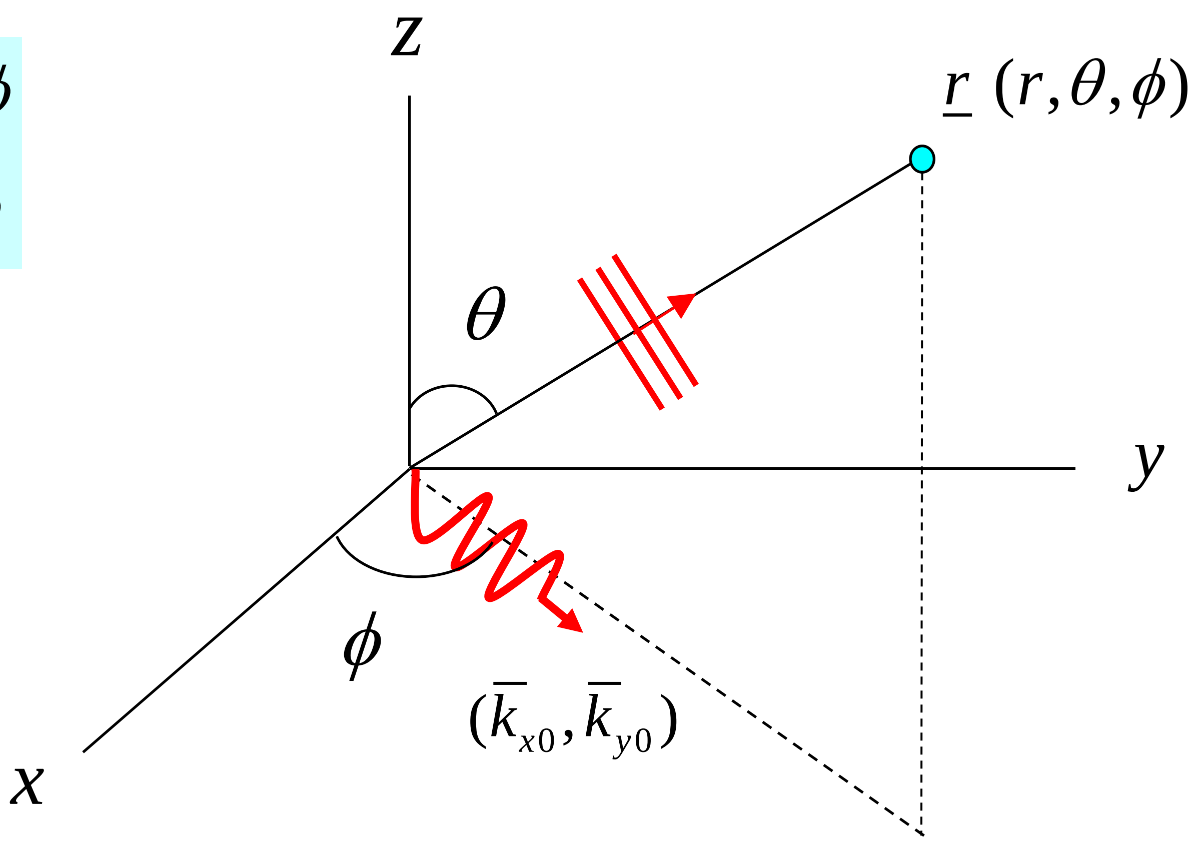
Physical interpretation:

The plane wave that propagates at angles (θ,ϕ).

$$\bar{k}_{x0} = \sin\theta\cos\phi$$

$$\bar{k}_{y0} = \sin\theta\sin\phi$$

$$\bar{k}_{z0} = \cos\theta$$



Example (cont.)

$$\alpha = \frac{1}{2}\sec^2\theta\left[\cos^2\theta + \sin^2\theta\cos^2\phi\right]$$

$$\beta = \frac{1}{2}\sec^2\theta\left[\cos^2\theta + \sin^2\theta\sin^2\phi\right]$$

$$\gamma = \sec^2\theta\sin^2\theta\ \sin\phi\cos\phi$$

(The details of these calculations have been omitted.)

Example (cont.)

Hence

$$\alpha > 0$$
$$\beta > 0$$

Also,

$$\alpha\beta - \frac{\gamma^2}{4} = \frac{1}{4}\sec^4\theta\left[\cos^2\theta + \sin^2\theta\cos^2\phi\right]\left[\cos^2\theta + \sin^2\theta\sin^2\phi\right]$$
$$-\frac{1}{4}\sec^4\theta\sin^4\theta\sin^2\phi\cos^2\phi$$
$$= \frac{1}{4}\sec^4\theta\left[\begin{array}{l}\cos^4\theta + \cos^2\theta\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right) \\ +\sin^4\theta\cos^2\phi\sin^2\phi - \sin^4\theta\sin^2\phi\cos^2\phi\end{array}\right]$$

Example (cont.)

$$= \frac{1}{4}\sec^4\theta\left[\cos^4\theta + \cos^2\theta\sin^2\theta\right]$$

$$= \frac{1}{4}\sec^2\theta\left[\cos^2\theta + \sin^2\theta\right]$$

$$= \frac{1}{4}\sec^2\theta$$

Hence

$$\alpha\beta - \frac{\gamma^2}{4} = \frac{1}{4}\sec^2\theta > 0$$

This problem thus fits into the “important special case” mentioned in Notes 30.

Example (cont.)

Recall that for this "important special case,"

$$I(\Omega) \sim \frac{\pi}{\Omega} f(x_0, y_0) e^{j\Omega g(x_0, y_0)} (\pm j) \left(\frac{1}{\sqrt{\alpha\beta - \frac{\gamma^2}{4}}} \right)$$

Choose + sign, since $\alpha > 0$ and $\beta > 0$.

Also, recall that $f(\bar{k}_x, \bar{k}_y) = \left(\frac{k_0}{2\pi} \right)^2 \underline{\tilde{E}}^N (\bar{k}_x, \bar{k}_x, 0)$

Example (cont.)

Therefore we have

$$\underline{E}(x,y,z) \sim \frac{\pi}{\Omega}\left(\frac{k_0}{2\pi}\right)^2 \underline{\tilde{E}}^N\left(\bar{k}_{x0},\bar{k}_{y0},0\right) e^{j\Omega g\left(\bar{k}_{x0},\bar{k}_{y0}\right)}$$

$$(+j)\left(\frac{1}{\sqrt{\alpha\beta - \frac{\gamma^2}{4}}}\right)$$

Example (cont.)

At the stationary phase point we have

$$g\left(\bar{k}_{x0},\bar{k}_{y0}\right)=-\left[\left(1-\bar{k}_{x0}^{2}-\bar{k}_{y0}^{2}\right)^{1/2}\cos\theta+\bar{k}_{x0}\sin\theta\cos\phi+\bar{k}_{y0}\sin\theta\sin\phi\right]$$

$$-\left[\left(1-\sin^{2}\theta\right)^{1/2}\cos\theta+\left(\sin\theta\cos\phi\right)^{2}+\left(\sin\theta\sin\phi\right)^{2}\right]$$

$$=-\left[\cos^{2}\theta+\sin^{2}\theta\right]$$

$$=-1$$

Hence

$$e^{j\Omega g\left(\bar{k}_{x0},\bar{k}_{y0}\right)}=e^{-jk_{0}r}$$

Example (cont.)

Also,

$$\frac{1}{\sqrt{\alpha\beta - \frac{\gamma^2}{4}}} = \frac{1}{\sqrt{\frac{1}{4}\sec^2\theta}} = 2\cos\theta$$

Example (cont.)

Hence

$$\underline{E}(x,y,z) \sim j\left(\frac{\pi}{k_0 r}\right)\left(\frac{k_0}{2\pi}\right)^2 \underline{\tilde{E}}^N\left(\bar{k}_{x0},\bar{k}_{y0},0\right)e^{-jk_0 r}\left(2\cos\theta\right)$$

Note: $$\left(\frac{\pi}{k_0}\right)\left(\frac{k_0}{2\pi}\right)^2(2) = \frac{k_0}{2\pi} = \frac{1}{\lambda_0}$$

Hence

$$\underline{E}(x,y,z) \sim j\left(\frac{1}{\lambda_0}\right)\cos\theta\left(\frac{e^{-jk_0 r}}{r}\right)\underline{\tilde{E}}\left(k_{x0},k_{y0},0\right)$$

(We are returning to unnormalized variable notation.)

Example (cont.)

Assume that only the tangential aperture field is known:

$$\underline{E}_t = \underline{\hat{x}}\, E_x + \underline{\hat{y}}\, E_y \qquad @\, z = 0$$

The uniqueness principle indicates that this should be sufficient to determine the far field.

$$E_{x,y}^{FF} \sim j\left(\frac{1}{\lambda_0}\right)\cos\theta\left(\frac{e^{-jk_0 r}}{r}\right)\tilde{E}_{x,y}\left(k_{x0}, k_{y0}, 0\right)$$

It should be possible to find the far-field components E_θ and E_ϕ in spherical coordinates from these.

Example (cont.)

To find E_z, use

$$\nabla \cdot \underline{E} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Assuming a plane-wave field in the far field,

$$-jk_{x0}E_x^{FF} - jk_{y0}E_y^{FF} - jk_{z0}E_z^{FF} = 0$$

Example (cont.)

Hence

$$E_z^{FF} = -\frac{1}{k_{z0}}\left(k_{x0}\, E_x^{FF} + k_{y0}\, E_y^{FF}\right)$$

For the plane wave that corresponds to the stationary-phase point, we have

$$E_z^{FF} = -\frac{1}{\cos\theta}\left(\sin\theta\cos\phi\, E_x^{FF} + \sin\theta\sin\phi\, E_y^{FF}\right)$$

Example (cont.)

In the far-field,

$$E_\phi^{FF} = E_x^{FF}\left(-\sin\phi\right) + E_y^{FF}\left(\cos\phi\right)$$

$$E_\theta^{FF} = E_x^{FF}\left(\cos\theta\cos\phi\right) + E_y^{FF}\left(\cos\theta\sin\phi\right) + E_z^{FF}\left(-\sin\theta\right)$$

To eliminate E_z in the second equation above, we use

$$E_z^{FF}\left(-\sin\theta\right) = \left[-\frac{1}{\cos\theta}\left(E_x^{FF}\sin\theta\cos\phi + E_y^{FF}\sin\theta\sin\phi\right)\right]\left(-\sin\theta\right)$$

Hence

$$E_\theta^{FF} = E_x^{FF}\left[\cos\theta\cos\phi + \frac{\sin^2\theta}{\cos\theta}\cos\phi\right] + E_y^{FF}\left[\cos\theta\sin\phi + \frac{\sin^2\theta}{\cos\theta}\sin\phi\right]$$

$$= \left(E_x^{FF}\cos\phi + E_y^{FF}\sin\phi\right)\sec\theta$$

Example (cont.)

The far-field spherical components are thus expressed in terms of the far-field rectangular components (x,y) as

$$E_{\phi}^{FF} = E_x^{FF}\left(-\sin\phi\right) + E_y^{FF}\left(\cos\phi\right)$$

$$E_{\theta}^{FF} = \left(E_x^{FF}\cos\phi + E_y^{FF}\sin\phi\right)\sec\theta$$

We now use the result from the 2-D stationary-phase method, which is

$$E_{x,y}^{FF} \sim j\left(\frac{1}{\lambda_0}\right)\cos\theta\left(\frac{e^{-jk_0 r}}{r}\right)\tilde{E}_{x,y}\left(k_{x0}, k_{y0}, 0\right)$$

Example (cont.)

Final result:

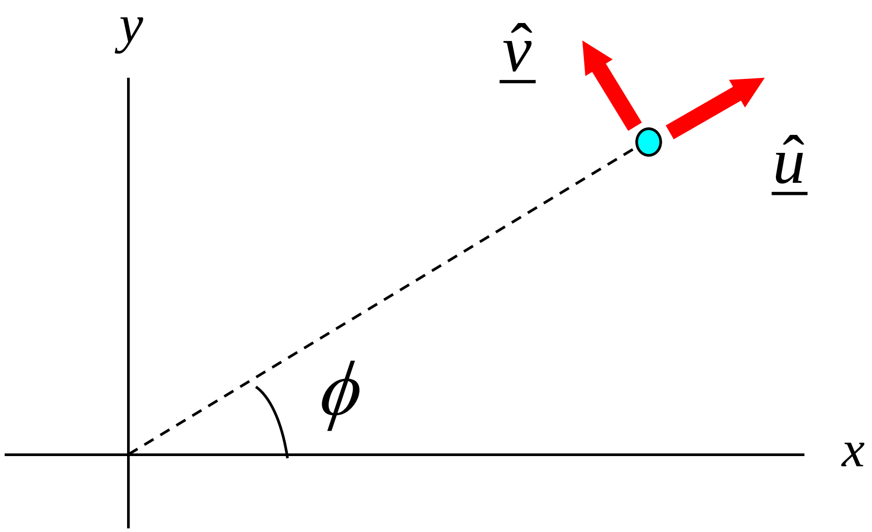
$$E_\phi^{FF} = j\left(\frac{1}{\lambda_0}\right)\cos\theta\left(\frac{e^{-jk_0 r}}{r}\right)\left[\tilde{E}_x\left(-\sin\phi\right)+\tilde{E}_y\left(\cos\phi\right)\right]$$

$$E_\theta^{FF} = j\left(\frac{1}{\lambda_0}\right)\left(\frac{e^{-jk_0 r}}{r}\right)\left[\tilde{E}_x\cos\phi+\tilde{E}_y\sin\phi\right]$$

Introduce new unit vectors:

$$\underline{\hat{u}} = \underline{\hat{x}}\cos\phi + \underline{\hat{y}}\sin\phi$$

$$\underline{\hat{v}} = \underline{\hat{x}}\left(-\sin\phi\right) + \underline{\hat{y}}\cos\phi$$



Example (cont.)

The final simplified form is

$$E_{\phi}^{FF} \sim j\left(\frac{1}{\lambda_0}\right)\left(\frac{e^{-jk_0 r}}{r}\right)\left[\underline{\widetilde{E}}_t \cdot \underline{\hat{v}}\right]$$

$$E_{\theta}^{FF} \sim j\left(\frac{1}{\lambda_0}\right)\cos\theta\left(\frac{e^{-jk_0 r}}{r}\right)\left[\underline{\widetilde{E}}_t \cdot \underline{\hat{u}}\right]$$

$$\underline{\hat{u}} = \underline{\hat{x}}\cos\phi + \underline{\hat{y}}\sin\phi$$

$$\underline{\hat{v}} = \underline{\hat{x}}\left(-\sin\phi\right) + \underline{\hat{y}}\cos\phi$$