ECE 6341

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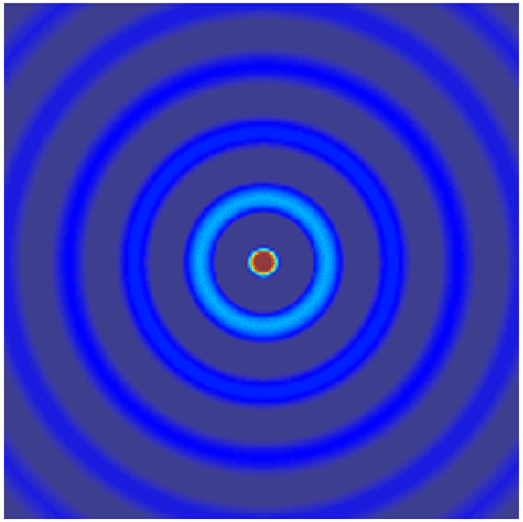

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Laplace's Method

Consider

$$I(\Omega) = \int_a^b f(x) e^{\Omega g(x)} dx$$

$$\Omega \to \infty$$

Assumptions:

$$g'(x_0) = 0, \quad x_0 \in (a, b)$$ ("saddle point")

$$g(x_0) > g(x), \quad x \in [a, b]$$

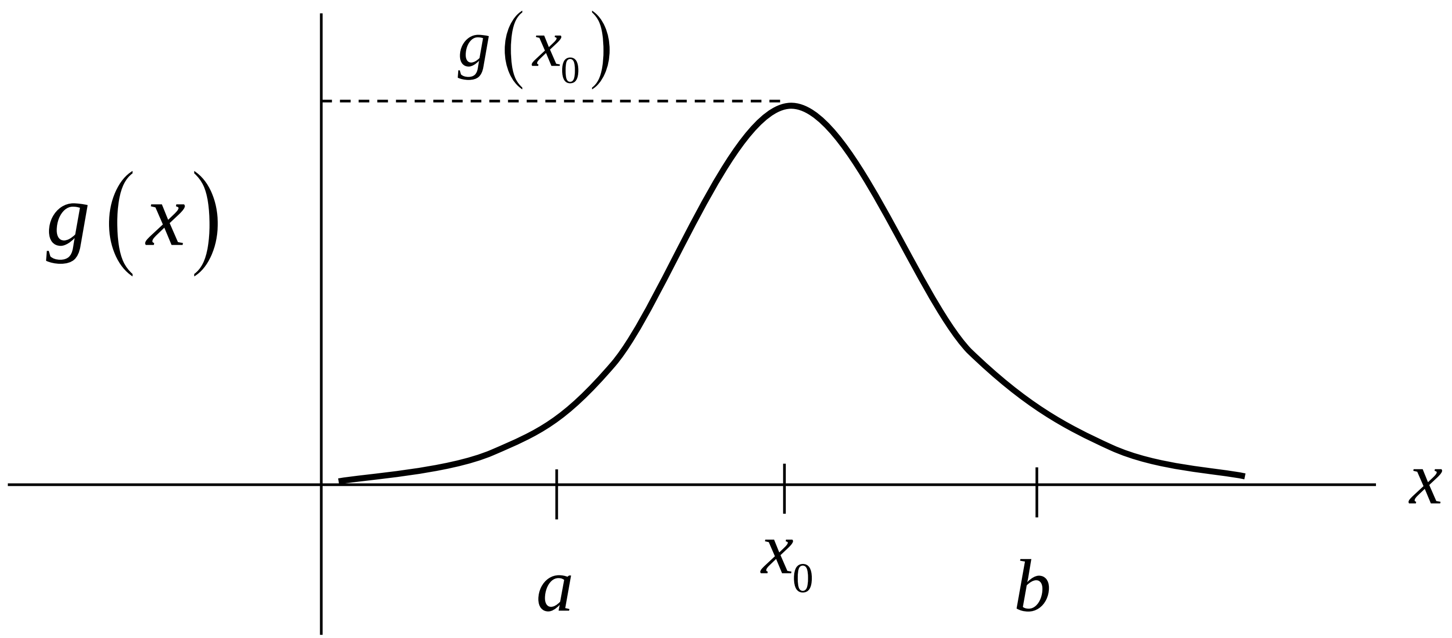
Note: If there is no point where the derivative of the g function vanishes, then integration by parts may be used, in exactly the same manner as was done for the case when there was a j in the exponent (Notes 28).

Laplace's Method (cont.)

$$I(\Omega) = \int_a^b f(x) e^{\Omega g(x)} dx$$

Note:

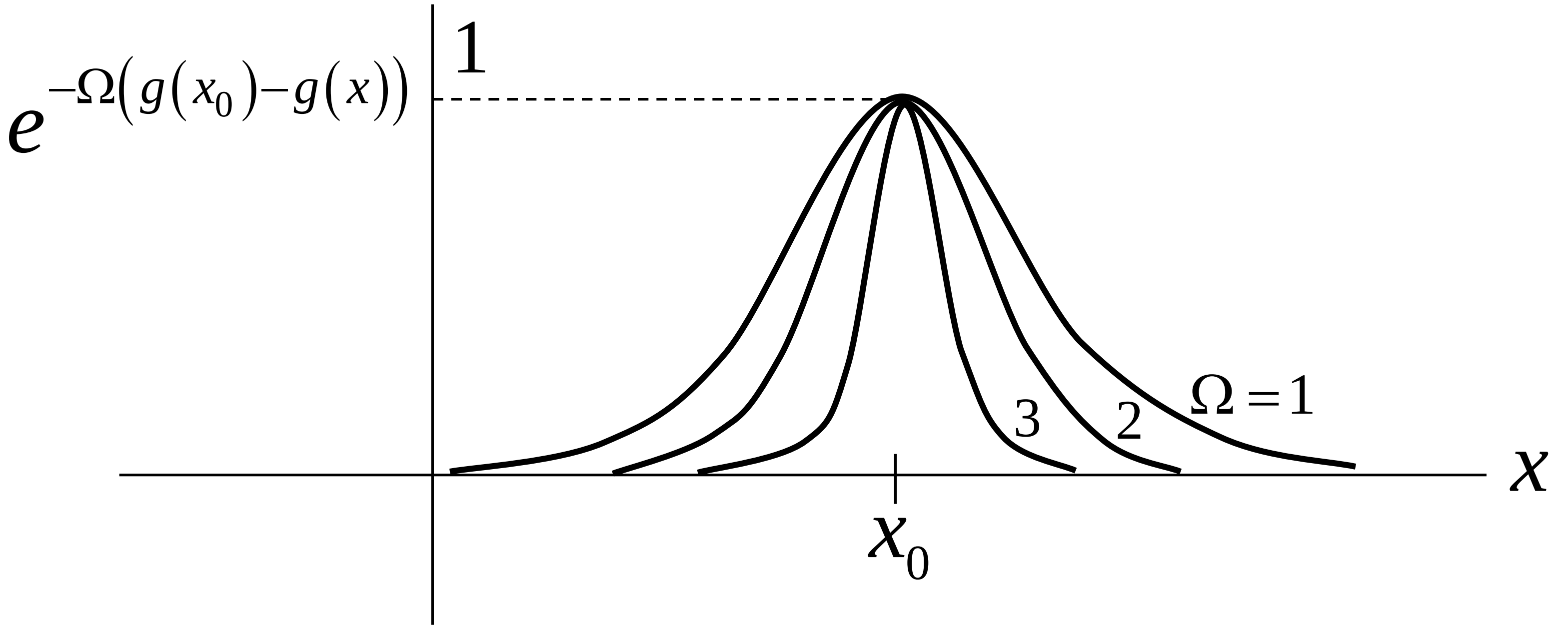
$$g''(x_0) < 0$$



Laplace's Method (cont.)

$$I(\Omega) = e^{\Omega g(x_0)} \int_a^b f(x) e^{-\Omega(g(x_0) - g(x))} dx$$

The exponential function behaves similar to a δ function as $\Omega \to \infty$.



Laplace's Method (cont.)

$$I(\Omega) \sim f(x_0)\, e^{\Omega g(x_0)} \int_a^b e^{-\Omega\left(g(x_0) - g(x)\right)} dx$$

$$g(x) \approx g(x_0) + \cancel{g'(x_0)(x - x_0)} + \frac{1}{2} g''(x_0)(x - x_0)^2$$

Hence

$$g(x_0) - g(x) \approx -\frac{1}{2} g''(x_0)(x - x_0)^2$$

$$= \frac{1}{2} \left| g''(x_0) \right| (x - x_0)^2$$

Laplace's Method (cont.)

Hence

$$I(\Omega) \sim f(x_0)\, e^{\Omega g(x_0)} \int_a^b e^{-\frac{1}{2}\Omega |g''(x_0)| (x-x_0)^2} dx$$

Since only the local neighborhood of x_0 is important, we can write

$$I(\Omega) \sim f(x_0)\, e^{\Omega g(x_0)} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\Omega |g''(x_0)| (x-x_0)^2} dx$$

Next, let

$$s = (x - x_0)\sqrt{\frac{\Omega |g''(x_0)|}{2}} \qquad ds = dx\sqrt{\frac{\Omega |g''(x_0)|}{2}}$$

Laplace's Method (cont.)

Then

$$I(\Omega) \sim f(x_0)\, e^{\Omega g(x_0)} \sqrt{\frac{2}{\Omega \left| g''(x_0) \right|}} \int_{-\infty}^{+\infty} e^{-s^2} ds$$

Next use $\int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi}$

We then have

$$I(\Omega) \sim f(x_0)\, e^{\Omega g(x_0)} \sqrt{\frac{2\pi}{\Omega \left| g''(x_0) \right|}}$$

Laplace's Method (cont.)

Summary

$$I(\Omega)=\int_a^b f(x)e^{\Omega g(x)}dx$$

$$g'(x_0)=0,\quad x_0\in(a,b)$$

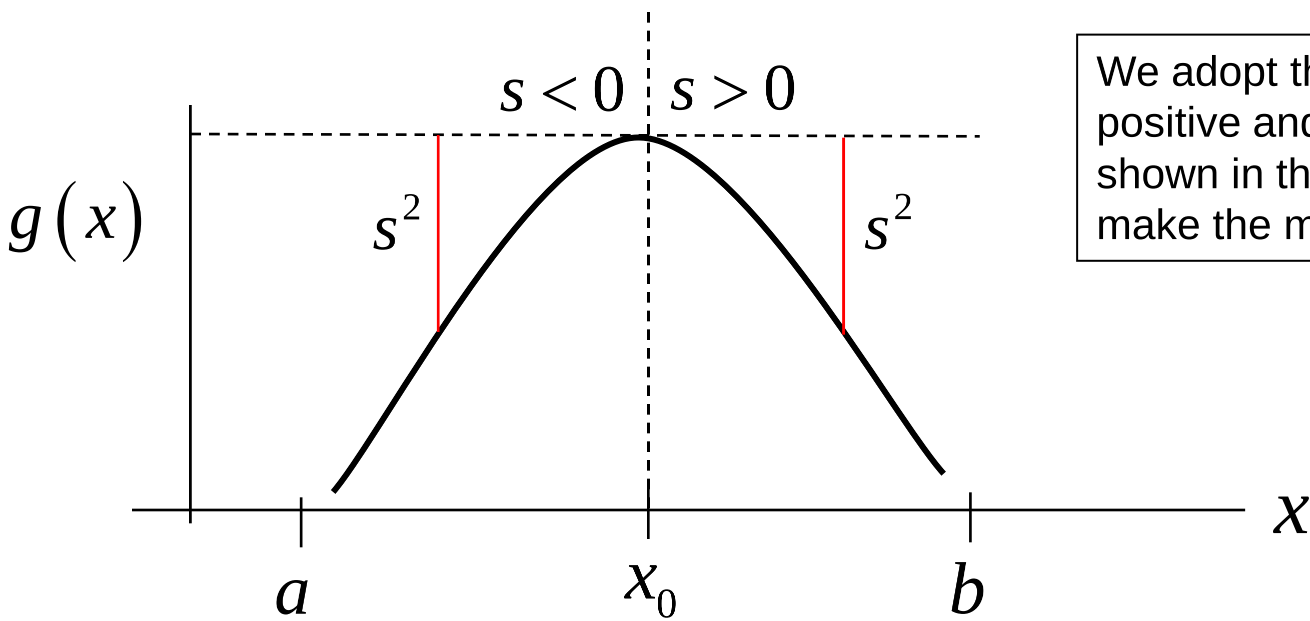
$$g(x_0)>g(x),\quad x\in[a,b]$$

$$I(\Omega)\sim f(x_0)e^{\Omega g(x_0)}\sqrt{\frac{-2\pi}{\Omega g''(x_0)}}$$

Complete Asymptotic Expansion (Using Watson's Lemma)

$$I(\Omega) = e^{\Omega g(x_0)} \int_a^b f(x) e^{-\Omega(g(x_0) - g(x))} dx$$

Let $s^2 \equiv g(x_0) - g(x)$



We adopt the convention of positive and negative s as shown in the figure, in order to make the mapping $x(s)$ unique.

Complete Asymptotic Expansion (cont.)

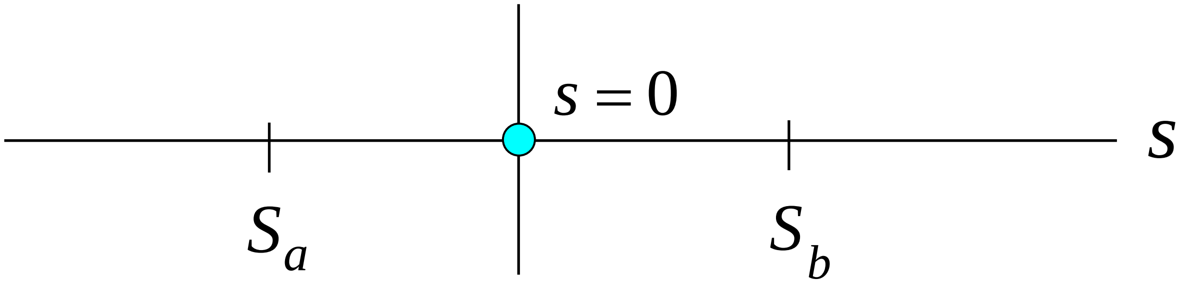
$$I(\Omega) = e^{\Omega g(x_0)} \int_{S_a}^{S_b} f(x(s)) e^{-\Omega s^2} \left(\frac{dx}{ds} \right) ds$$

Define

$$h(s) \equiv f(x(s)) \left(\frac{dx(s)}{ds} \right)$$

We then have

$$I(\Omega) = e^{\Omega g(x_0)} \int_{S_a}^{S_b} h(s) e^{-\Omega s^2} ds$$



Complete Asymptotic Expansion (cont.)

$$I(\Omega) = e^{\Omega g(x_0)} \int_{S_a}^{S_b} h(s) e^{-\Omega s^2} ds$$

Assume $h(s) \sim \sum_{n=0,1,2\ldots} a_n s^n$ as $s \to 0$

Then $I(\Omega) \sim e^{\Omega g(x_0)} \sum_{n=0,1,2\ldots} a_n \int_{S_a}^{S_b} s^n e^{-\Omega s^2} ds$

(This is Watson's Lemma.)

For a proof of Watson's Lemma, please see: Norman Bleistein and Richard A. Handelsman, *Asymptotic Expansion of Integrals*, Holt, Rinehart, and Winston, 1975.

Complete Asymptotic Expansion (cont.)

Since only the local neighborhood of $s = 0$ is important,

$$I(\Omega) \sim e^{\Omega g(x_0)} \sum_{n=0,1,2\ldots} a_n \int_{-\infty}^{+\infty} s^n e^{-\Omega s^2} ds$$

(The error made in extending the limits is exponentially small.)

Because of symmetry,

$$I(\Omega) \sim 2e^{\Omega g(x_0)} \sum_{n=0,2,4\ldots} a_n \int_{0}^{\infty} s^n e^{-\Omega s^2} ds$$

$n =$ even

Complete Asymptotic Expansion (cont.)

Denote $$I_n \equiv \int_0^\infty s^n e^{-\Omega s^2} ds$$

Use $$t = \Omega s^2$$

$$dt = 2\Omega s\, ds$$

Then we have

$$I_n = \int_0^\infty \left(\frac{t}{\Omega}\right)^{\frac{n}{2}} e^{-t} \frac{dt}{2\Omega\sqrt{\frac{t}{\Omega}}} = \frac{1}{2\Omega^{\frac{(n+1)}{2}}} \int_0^\infty t^{\frac{(n-1)}{2}} e^{-t} dt$$

Complete Asymptotic Expansion (cont.)

$$I_n = \frac{1}{2\Omega^{\frac{(n+1)}{2}}} \int_0^\infty t^{\frac{(n-1)}{2}} e^{-t} dt$$

Now use $$\Gamma(x) = (x-1)! \equiv \int_0^\infty t^{x-1} e^{-t} dt$$

Hence

$$I_n = \frac{1}{2\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Complete Asymptotic Expansion (cont.)

Recall that

$$I(\Omega) \sim 2e^{\Omega g(x_0)} \sum_{n=0,2,4\ldots} a_n \int_0^\infty s^n e^{-\Omega s^2} ds$$

$$I_n = \int_0^\infty s^n e^{-\Omega s^2} ds = \frac{1}{2\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Hence

$$I(\Omega) \sim e^{\Omega g(x_0)} \sum_{n=0,2\ldots} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

Complete Asymptotic Expansion (cont.)

Summary

$$I(\Omega) = \int_a^b f(x) e^{\Omega g(x)} dx$$

$$g'(x_0) = 0, \quad x_0 \in (a,b)$$
$$g(x_0) > g(x), \quad x \in [a,b]$$

$$s^2 \equiv g(x_0) - g(x) \qquad h(s) \equiv f(x(s)) \left(\frac{dx(s)}{ds} \right)$$

Assume

$$h(s) \sim \sum_{n=0,2,4\ldots} a_n s^n \quad \text{as} \quad s \to 0$$

Note: Integer powers are assumed.

Then

$$I(\Omega) \sim e^{\Omega g(x_0)} \sum_{n=0,2,4\ldots} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2} \right)$$

Note: The hard part is determining the a_n coefficients!

Complete Asymptotic Expansion (cont.)

Calculation of <u>Leading term</u>:

$$I(\Omega) \sim e^{\Omega g(x_0)} \left[\frac{a_0}{\Omega^{1/2}} \Gamma\left(\frac{1}{2}\right) \right]$$

Note:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$h(s) \sim a_0 = h(0)$$

so that

$$I(\Omega) \sim a_0 \sqrt{\frac{\pi}{\Omega}} e^{\Omega g(x_0)}$$

Complete Asymptotic Expansion (cont.)

Recall that $a_0 = h(0)$

and that $h(s) \equiv f(x(s))\dfrac{dx}{ds}$

$$\Rightarrow \quad h(0) = f(x_0)\left.\frac{dx}{ds}\right|_{s=0}$$

To find $h(0)$, we must evaluate the derivative term. To do this, we take the derivative with respect to x:

$$s^2 = g(x_0) - g(x)$$

$$2s\frac{ds}{dx} = -g'(x)$$

Note:
At $s = 0$ ($x = x_0$), this yields $0 = 0$ (not useful).

Complete Asymptotic Expansion (cont.)

$$2\,s\frac{ds}{dx} = -g'(x)$$

Take one more derivative:

$$2\left(\frac{ds}{dx}\right)^2 + 2s\frac{d^2s}{dx^2} = -g''(x)$$

At $x = x_0, \quad (s=0)$

$$2\left(\frac{ds}{dx}\right)^2 = -g''(x_0)$$

Complete Asymptotic Expansion (cont.)

Hence

$$\frac{dx}{ds} = \sqrt{\frac{-2}{g''(x_0)}}$$

We then have

$$h(0) = f(x_0)\left.\frac{dx}{ds}\right|_{s=0} = f(x_0)\sqrt{\frac{-2}{g''(x_0)}}$$

Therefore

$$I(\Omega) \sim \left[f(x_0)\sqrt{\frac{-2}{g''(x_0)}}\right]\sqrt{\frac{\pi}{\Omega}}e^{\Omega g(x_0)}$$

Complete Asymptotic Expansion (cont.)

or

$$I(\Omega) \sim f(x_0)\sqrt{\frac{-2\pi}{\Omega\, g''(x_0)}}\, e^{\Omega\, g(x_0)}$$

This agrees with the result from Laplace’s method.

Watson's Lemma (Alternative Form)

$$I(\Omega) = \int_0^\infty h(s) e^{-\Omega s} ds$$

Assume

$$h(s) \sim \sum_n a_n s^{\alpha_n} \quad \text{as} \quad s \to 0$$

Here we do not necessarily assume integer powers in the expansion of the function, and we also start the integral at $s = 0$.

Note: This one-sided form occurs when integrating along branch cuts in the complex plane (discussed later).

Watson's Lemma (Alternative Form) (cont.)

$$I(\Omega) = \int_0^\infty h(s) e^{-\Omega s} ds$$

Assume

$$h(s) \sim \sum_n a_n s^{\alpha_n} \quad \text{as} \;\; s \to 0$$

Then

$$I(\Omega) \sim \sum_n a_n \int_0^\infty s^{\alpha_n} e^{-\Omega s} ds$$

(This is another form of Watson's Lemma.)

For a proof of Watson's Lemma, please see: Norman Bleistein and Richard A. Handelsman, *Asymptotic Expansion of Integrals*, Holt, Rinehart, and Winston, 1975.

Watson's Lemma (Alternative Form) (cont.)

$$I_n \equiv \int_0^\infty s^{\alpha_n} e^{-\Omega s} ds$$

$$= \int_0^\infty \left(\frac{t}{\Omega}\right)^{\alpha_n} e^{-t} \frac{dt}{\Omega}$$

$$= \frac{1}{\Omega^{\alpha_n+1}} \int_0^\infty t^{\alpha_n} e^{-t} dt$$

$$= \frac{1}{\Omega^{\alpha_n+1}} \Gamma(\alpha_n + 1)$$

Let

$$t = \Omega s$$

$$dt = \Omega \, ds$$

Hence

$$I(\Omega) \sim \sum_n a_n \frac{1}{\Omega^{\alpha_n+1}} \Gamma(\alpha_n + 1)$$

Watson's Lemma (Alternative Form) (cont.)

Summary

$$I(\Omega) = \int_0^\infty h(s)\, e^{-\Omega s}\, ds$$

Assume

$$h(s) \sim \sum_n a_n s^{\alpha_n} \quad \text{as} \;\; s \to 0$$

Then

$$I(\Omega) \sim \sum_n a_n \frac{1}{\Omega^{\alpha_n + 1}} \Gamma(\alpha_n + 1)$$