ECE 6341

Spring 2016

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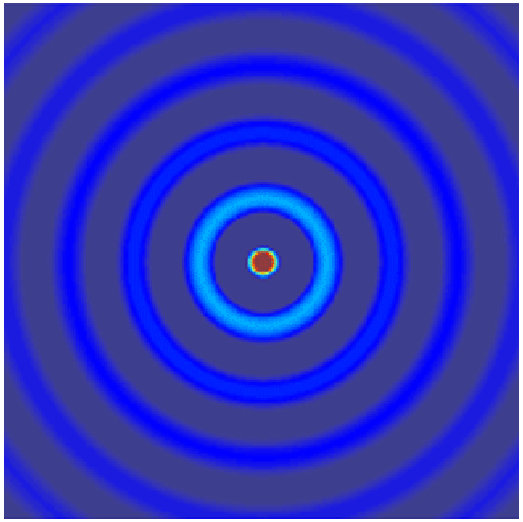
ECE Dept.



Notes 33

Steepest-Descent Method

Complex Integral:

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

This is an extension of Laplace's method to treat integrals in the *complex plane*.

The method was published by Peter Debye in 1909. Debye noted in his work that the method was developed in a unpublished note by Bernhard Riemann (1863).

Peter Joseph William Debye (March 24, 1884 – November 2, 1966) was a Dutch physicist and physical chemist, and Nobel laureate in Chemistry.

Georg Friedrich Bernhard Riemann (September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis and differential geometry, some of them enabling the later development of general relativity.

http://en.wikipedia.org/wiki/Peter_Debye

http://en.wikipedia.org/wiki/Bernhard_Riemann

Steepest-Descent Method (cont.)

Complex Integral:

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

The functions $f(z)$ and $g(z)$ are analytic (except for poles or branch points), so that the path C may be deformed if necessary (possibly adding residue contributions or branch-cut integrals).

Saddle Point (SP): $g'(z_0) = 0$

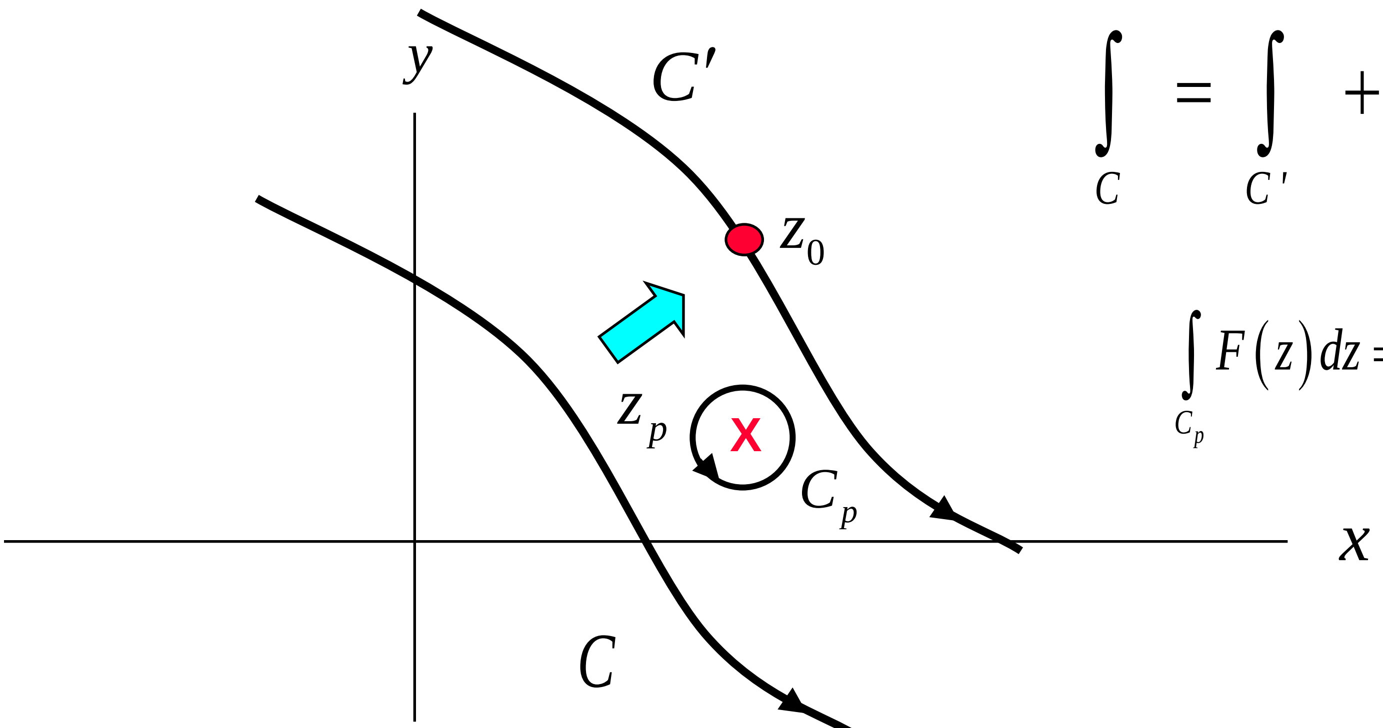
$$\Rightarrow \quad \frac{\partial g}{\partial x} = 0 \qquad \frac{\partial g}{\partial y} = 0$$

Steepest-Descent Method (cont.)

Path deformation:

If the path does not go through a saddle point, we assume that it can be deformed to do so.

If any singularities are encountered during the path deformation, they must be accounted for (e.g., residue of captured poles).



$$\int_C = \int_{C'} + \int_{C_p}$$

$$\int_{C_p} F(z)\,dz = 2\pi j \,\mathrm{Res}\, F(z_p)$$

Steepest-Descent Method (cont.)

Denote $$g(z) = u(z) + j\,v(z)$$

Cauchy Reimann eqs.:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$= -\frac{\partial^2 u}{\partial y^2}$$

Steepest-Descent Method (cont.)

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

⇒ If $u_{xx} < 0,$ then $u_{yy} > 0$

Near the saddle point:

$$u(x,y) \approx u(x_0, y_0) + \frac{1}{2} u_{xx} (x - x_0)^2$$

$$+ \frac{1}{2} u_{yy} (y - y_0)^2 + u_{xy} (x - x_0)(y - y_0)$$

Ignore (rotate coordinates to eliminate).

Steepest-Descent Method (cont.)

$$u(x,y) \approx u(x_0,y_0) + \frac{1}{2}u_{xx}(x-x_0)^2 + \frac{1}{2}u_{yy}(y-y_0)^2 + u_{xy}(x-x_0)(y-y_0)$$

In the rotated coordinate system:

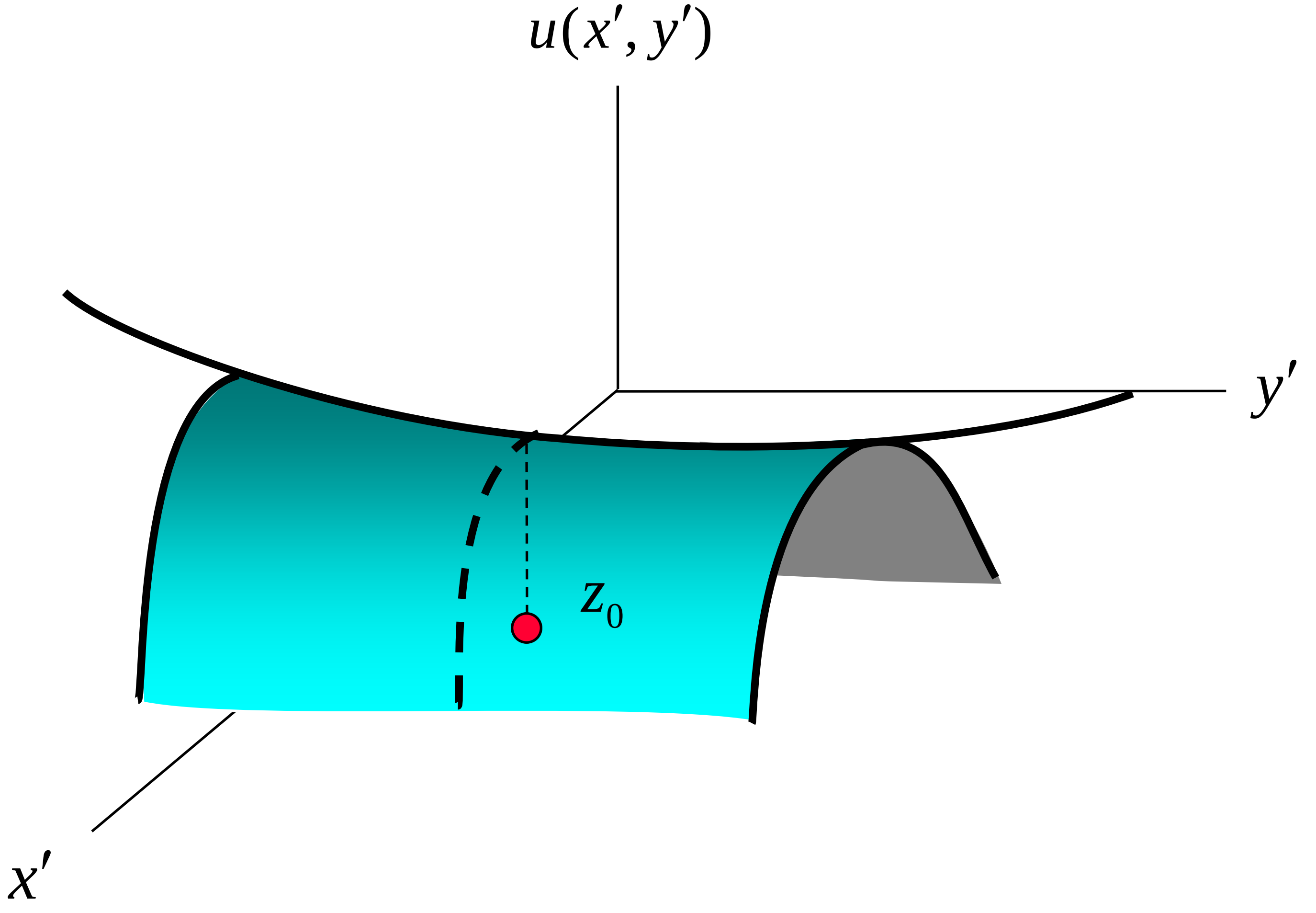
$$u(x',y') \approx u(x'_0,y'_0) + \frac{1}{2}u_{x'x'}(x'-x'_0)^2 + \frac{1}{2}u_{y'y'}(y'-y'_0)^2$$

Assume that the coordinate system is rotated so that

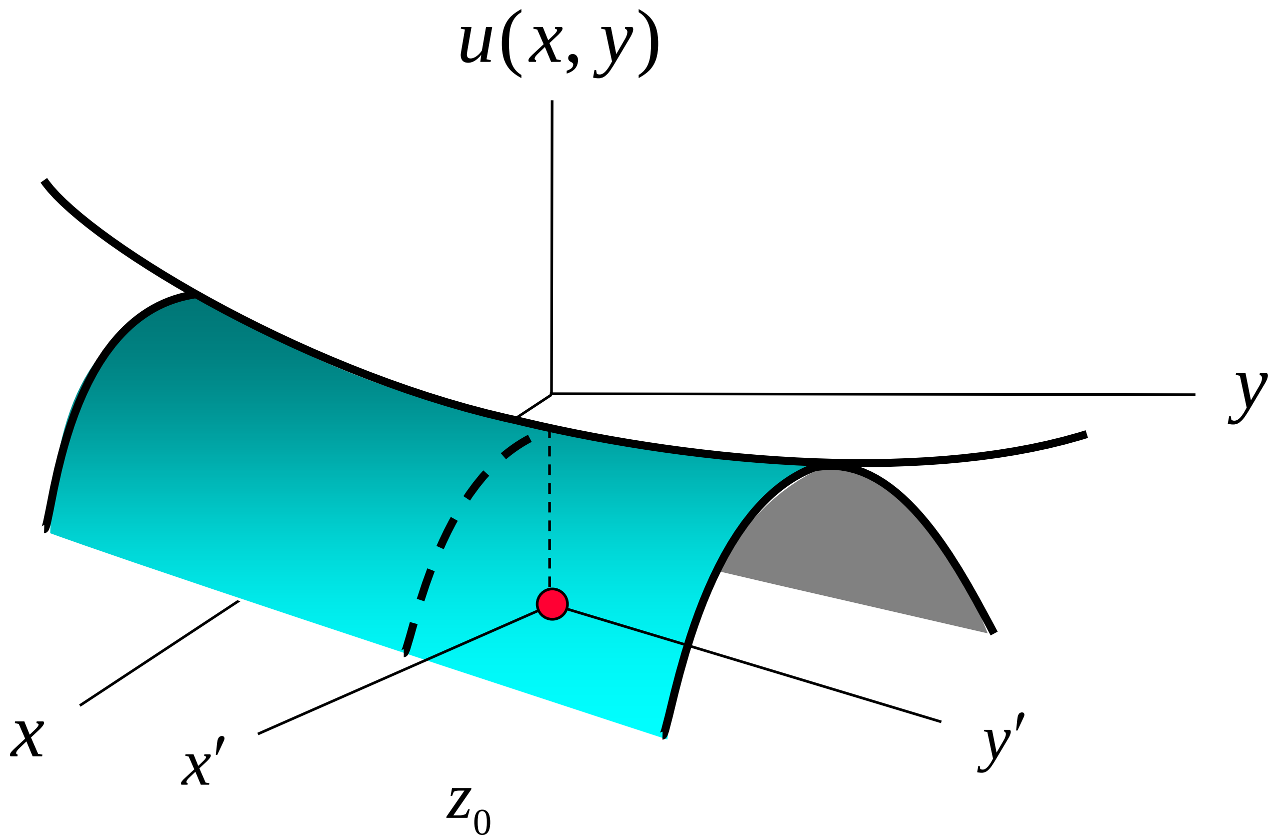
$$u_{x'x'} < 0 \qquad u_{y'y'} > 0$$

Steepest-Descent Method (cont.)

The $u(x', y')$ function has a "saddle" shape near the saddle point:



Steepest-Descent Method (cont.)



Note: The saddle does not necessarily open along one of the principal axes (only when $u_{xy}(x_0, y_0) = 0$).

Steepest-Descent Method (cont.)

Along any descending path (where the u function decreases):

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \int_C e^{\Omega[g(z) - g(z_0)]} dz$$

or

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \int_C e^{j\Omega[v(z) - v(z_0)]} e^{\Omega[u(z) - u(z_0)]} dz$$

Behaves like a delta function

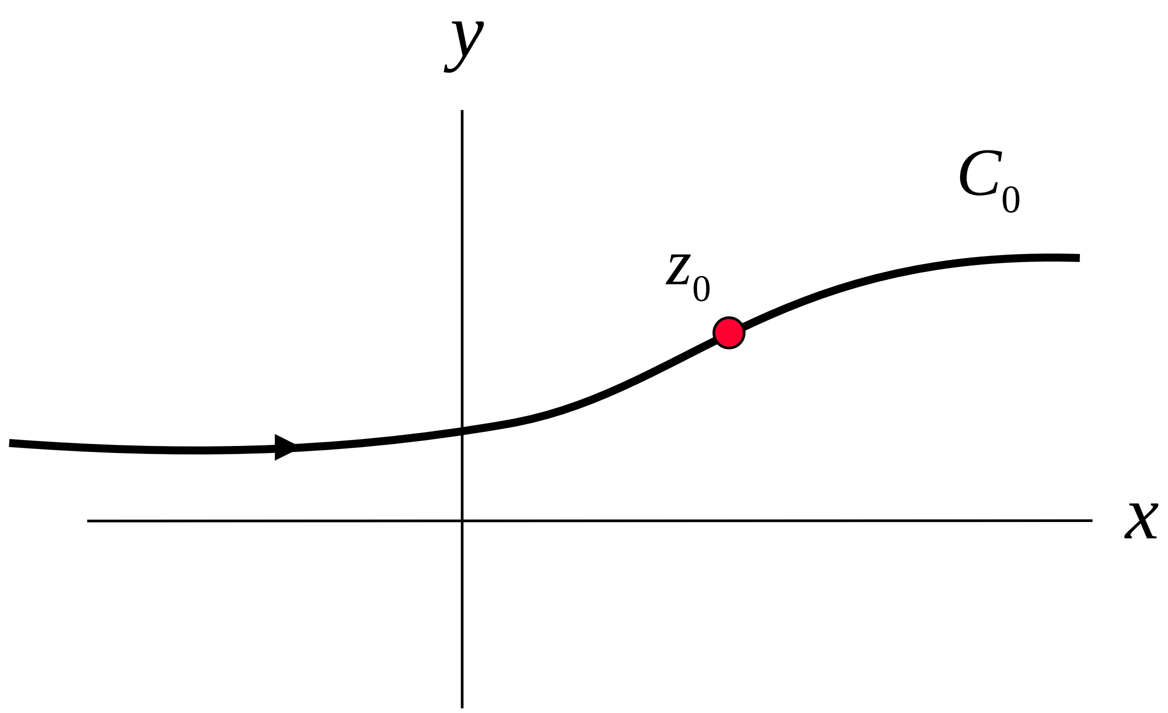
Both the phase and amplitude change along an arbitrary descending path C.

If we can find a path along which the phase does not change (v is constant), then the integral will look like that in Laplace's method.

Steepest-Descent Method (cont.)

Choose path of constant phase:

$$C_0: \;\; v(z) = v(z_0) = \text{constant}$$



Steepest-Descent Method (cont.)

Gradient Property (proof on next slide):

$$\nabla u\left(x, y\right) \text{ is parallel to } C_0$$

Hence C_0 is either a "path of steepest descent" (SDP)
or a "path of steepest ascent" (SAP).

(Of course, we want to choose the SDP.)

SDP: $u(x,y)$ decreases as fast as possible along the path away from the saddle point.

SAP: $u(x,y)$ increases as fast as possible along the path away from the saddle point.

Steepest-Descent Method (cont.)

Proof of gradient property

$$g(z) = u(z) + j\,v(z)$$

$$\nabla u = \underline{\hat{x}}\frac{\partial u}{\partial x} + \underline{\hat{y}}\frac{\partial u}{\partial y}$$

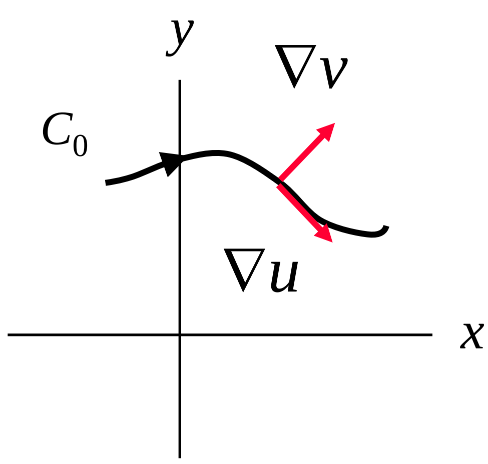
$$\nabla v = \underline{\hat{x}}\frac{\partial v}{\partial x} + \underline{\hat{y}}\frac{\partial v}{\partial y}$$

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$= \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= 0$$

Hence, $\nabla u \perp \nabla v$



Also, $\nabla v \perp C_0$ (v is constant on C_0)

Hence $\nabla u \,\|\, C_0$

Steepest-Descent Method (cont.)

Recall:

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \int_C e^{j\Omega[v(z)-v(z_0)]} e^{\Omega[u(z)-u(z_0)]} dz$$

Because the v function is constant along the SDP, we have

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \int_{SDP} e^{\Omega[u(z)-u(z_0)]} dz$$

or

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \int_{SDP} e^{\Omega \underline{[g(z)-g(z_0)]}} dz$$

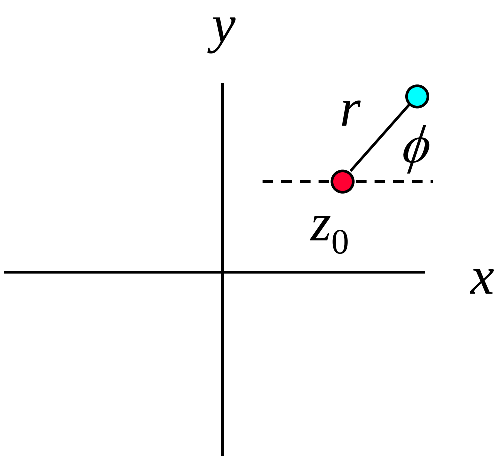
This is real on the SDP.

Steepest-Descent Method (cont.)

Local behavior near the saddle point

$$g(z) \approx g(z_0) + g'(z_0)(z - z_0) + \frac{1}{2} g''(z_0)(z - z_0)^2$$

so $$g(z) - g(z_0) \approx \frac{1}{2} g''(z_0)(z - z_0)^2$$



Denote $$g''(z_0) = R\, e^{j\alpha}$$

$$z - z_0 = r e^{j\phi}$$

Then we have

$$g(z) - g(z_0) \approx \frac{1}{2} R\, r^2\, e^{j(\alpha + 2\phi)}$$

Steepest-Descent Method (cont.)

$$g(z)-g(z_0)\approx\frac{1}{2}\left(Rr^2\right)e^{j(\alpha+2\phi)} \Rightarrow u(z)-u(z_0)\approx\frac{1}{2}\left(Rr^2\right)\cos(\alpha+2\phi)$$

SAP:

$$\alpha+2\phi=0+2\pi n$$

$$\phi=-\frac{\alpha}{2},-\frac{\alpha}{2}+\pi$$

$$u(z)-u(z_0)=\frac{1}{2}Rr^2$$

$$v(z)-v(z_0)=0$$

SDP:

$$\alpha+2\phi=\pi+2\pi n$$

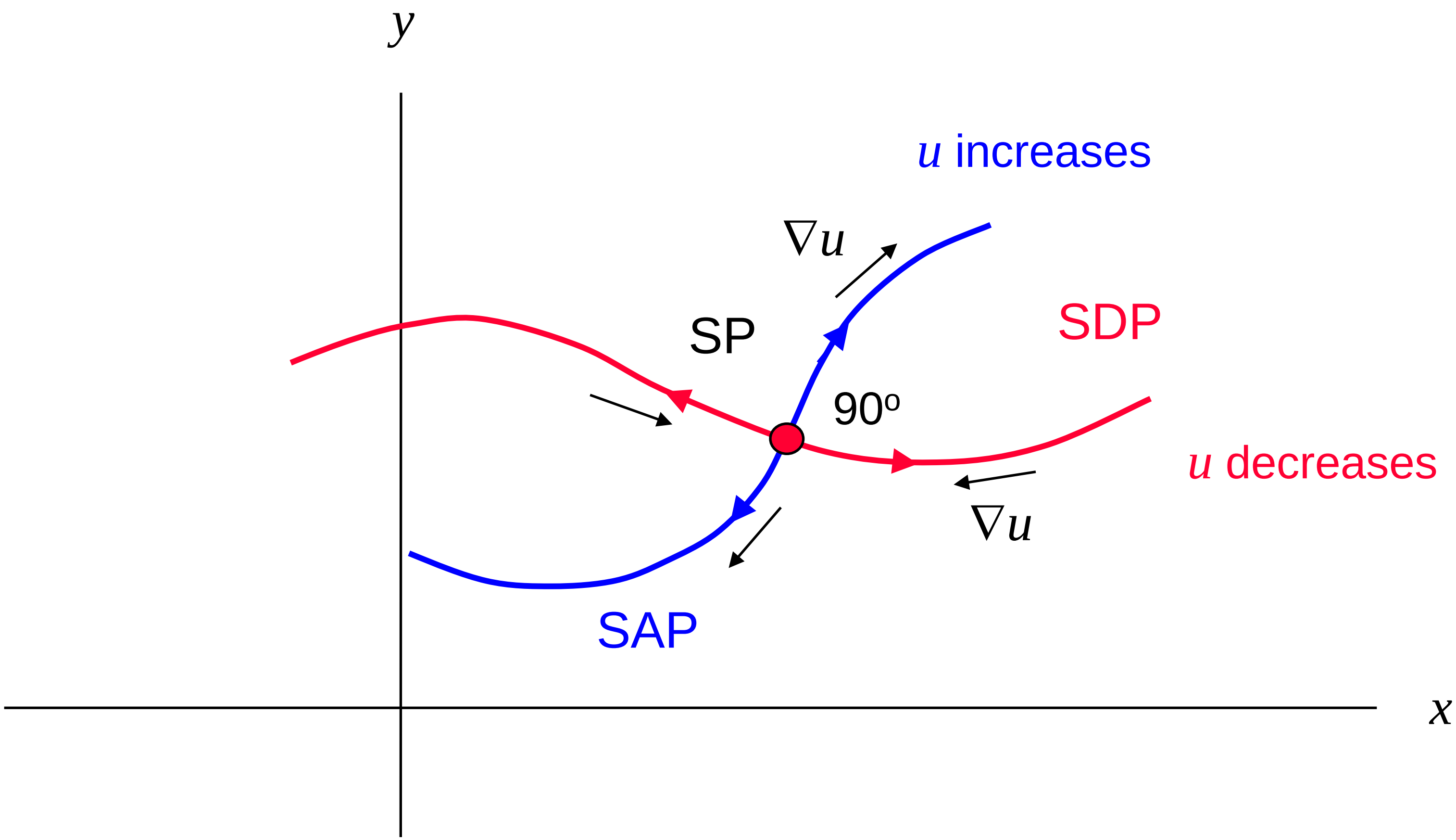
$$\phi=-\frac{\alpha}{2}-\frac{\pi}{2},-\frac{\alpha}{2}-\frac{\pi}{2}+\pi$$

$$u(z)-u(z_0)=-\frac{1}{2}Rr^2$$

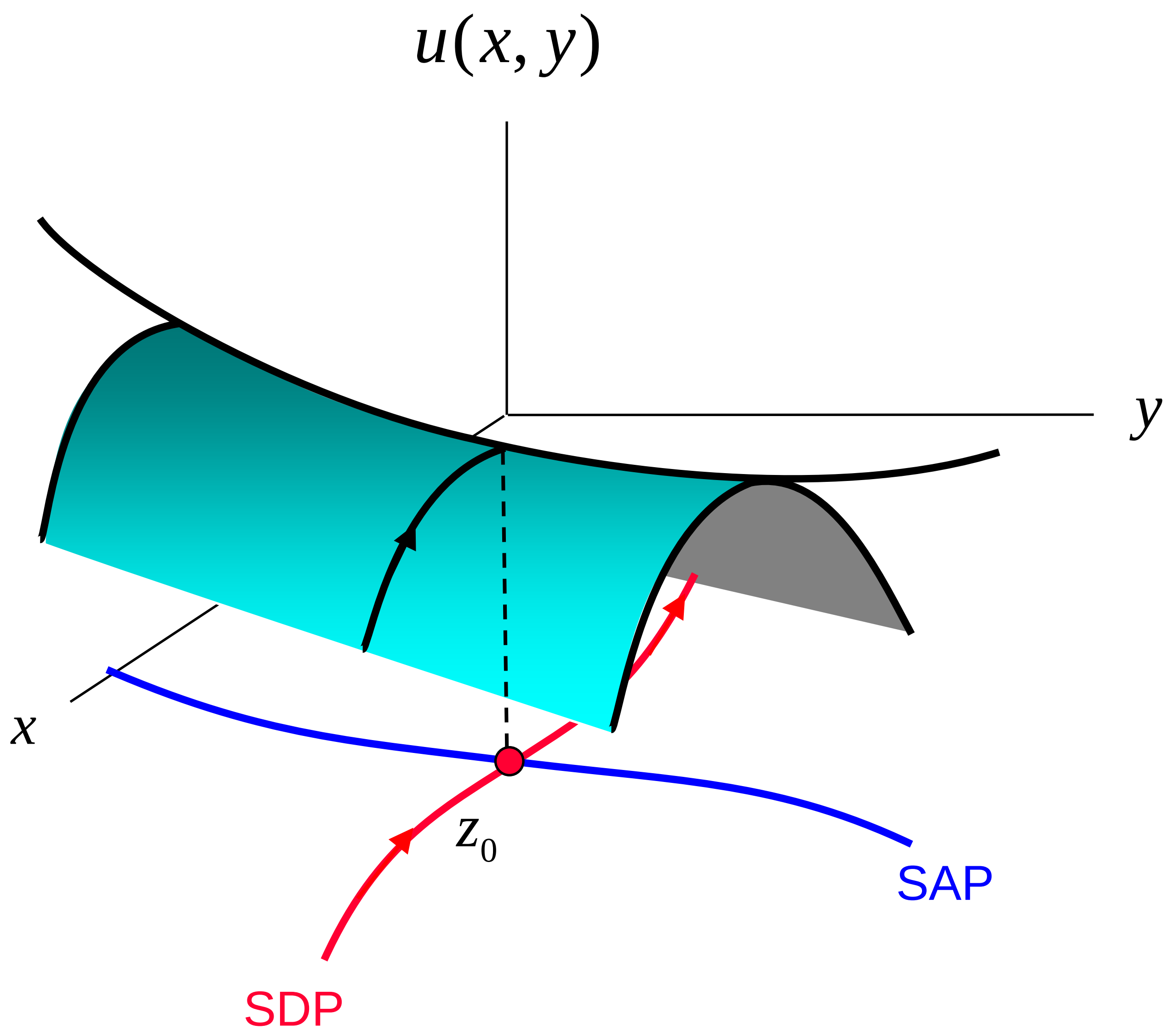
$$v(z)-v(z_0)=0$$

Note: The two paths are 90° apart at the saddle point.

Steepest-Descent Method (cont.)



Steepest-Descent Method (cont.)



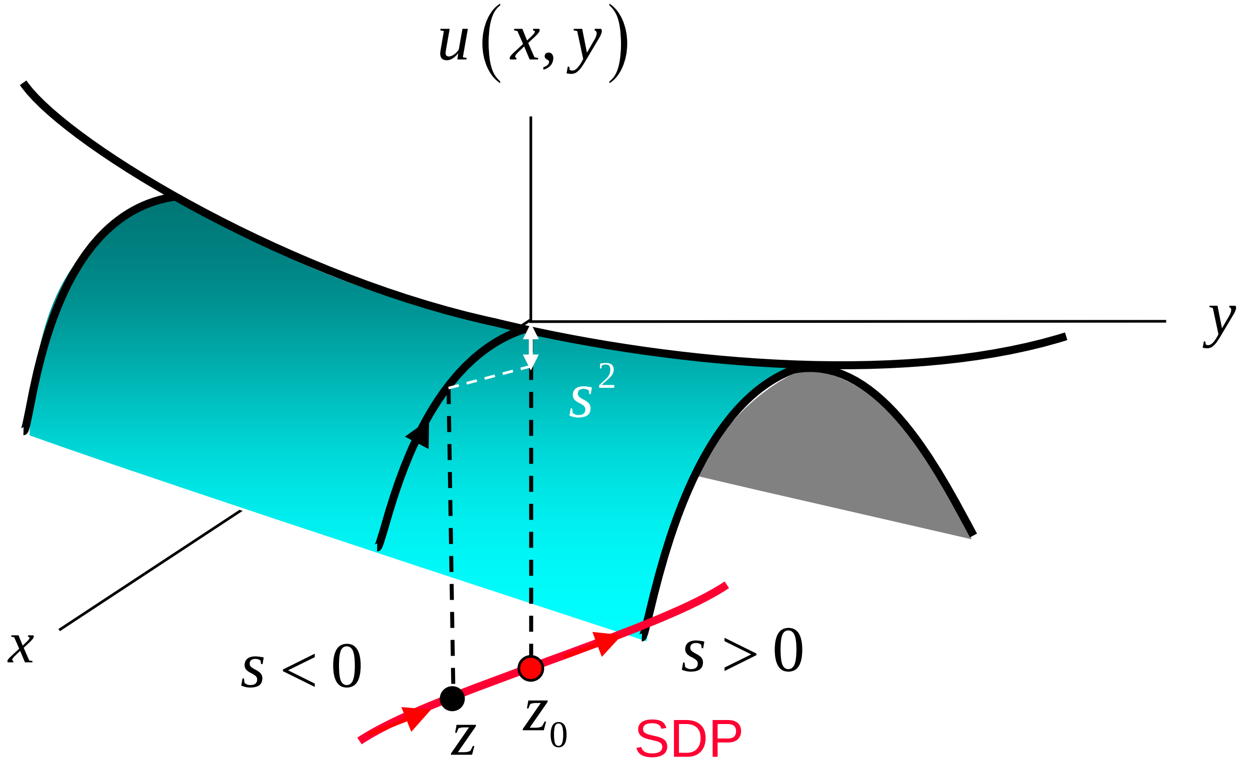
Steepest-Descent Method (cont.)

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \int_{SDP} e^{\Omega\left[u(z)-u(z_0)\right]} dz$$

Define $s^2 \equiv u(z_0) - u(z)$

This defines

$$z = z(s)$$



Steepest-Descent Method (cont.)

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \int_{-\infty}^{+\infty} e^{-\Omega s^2} \left(\frac{dz}{ds} \right) ds$$

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{dz}{ds} \right)_{s=0} \int_{-\infty}^{+\infty} e^{-\Omega s^2} ds$$

(This gives the leading term of the asymptotic expansion.)

Hence

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{dz}{ds} \right)_{s=0} \sqrt{\frac{\pi}{\Omega}}$$

Steepest-Descent Method (cont.)

To evaluate the derivative:

$$
\begin{aligned}
-s^2 &= u(z) - u(z_0) \\
&= g(z) - g(z_0)
\end{aligned}
$$

(Recall: v is constant along SDP.)

$$-2s\left(\frac{ds}{dz}\right) = g'(z)$$

At the saddle point this gives $0 = 0$.

Take one more derivative:

$$-2\left(\frac{ds}{dz}\right)\left(\frac{ds}{dz}\right) - 2s\frac{d^2 s}{dz^2} = g''(z)$$

Steepest-Descent Method (cont.)

At $s = 0$:

$$\left(\frac{ds}{dz}\right)_{z_0} = \left(\frac{g''(z_0)}{-2}\right)^{1/2}$$

so

$$\left(\frac{dz}{ds}\right)_{s=0} = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

Hence, we have

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \left(\frac{-2}{g''(z_0)}\right)^{1/2} \sqrt{\frac{\pi}{\Omega}}$$

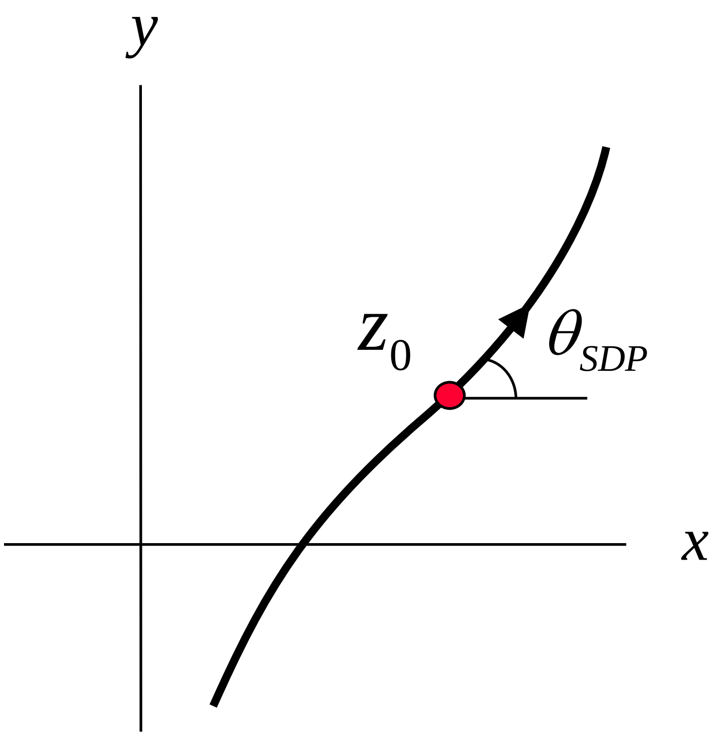
Steepest-Descent Method (cont.)

Note: There is an ambiguity in sign for the square root:

$$\left(\frac{dz}{ds}\right)_{s=0} = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

To avoid this ambiguity, define

$$\arg\left(\frac{dz}{ds}\right)_{s=0} = \theta_{SDP}$$



Steepest-Descent Method (cont.)

The derivative term is therefore

$$\left(\frac{dz}{ds}\right)_{s=0} = \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{j\theta_{SDP}}$$

Hence

$$I(\Omega) \sim f(z_0)\, e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{j\theta_{SDP}}$$

Steepest-Descent Method (cont.)

To find θ_{SDP} :

Denote:

$$g''(z_0) = R\, e^{j\alpha} \qquad \alpha = \arg\left(g''(z_0)\right)$$

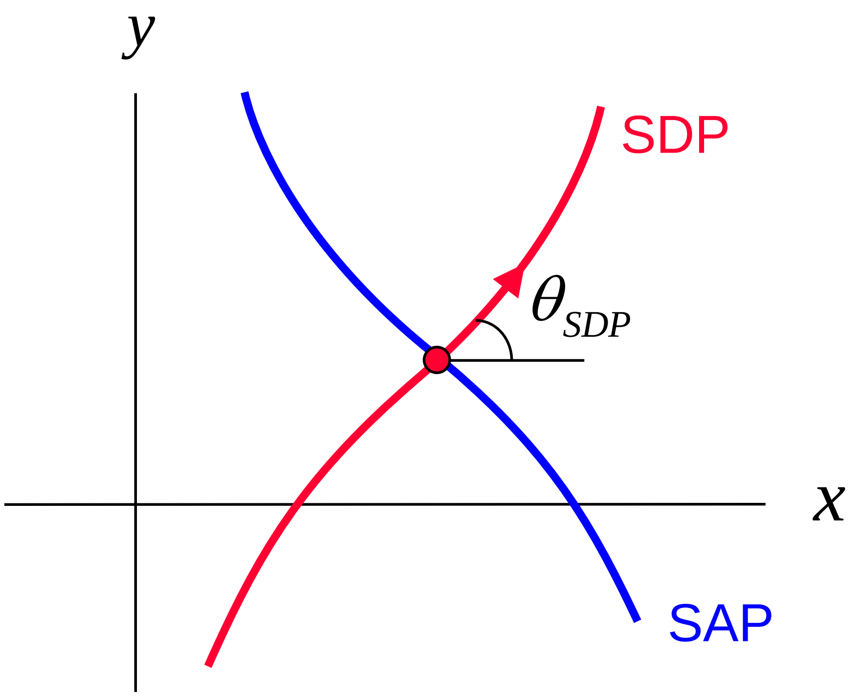
$$z - z_0 = r\, e^{j\theta_{SDP}}$$

$$g(z) - g(z_0) \approx \frac{1}{2} g''(z_0)(z - z_0)^2 = \frac{1}{2}\left(Rr^2\right) e^{j(\alpha + 2\theta_{SDP})}$$

$$\alpha + 2\theta_{SDP} = \pm\pi$$

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

Steepest-Descent Method (cont.)



$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''\left(z_0\right)\right)$$

Note: The direction of integration determines The sign.

The “user” must determine this.

Steepest-Descent Method (cont.)

Summary

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''(z_0)\right|}} e^{j\theta_{SDP}}$$

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''(z_0)\right)$$

Example

$$J_0(\Omega) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(\Omega\cos z)\,dz$$

$$= \frac{1}{\pi}\operatorname{Re} I(\Omega)$$

where

$$I(\Omega) = \int_{-\pi/2}^{\pi/2} e^{\Omega(j\cos z)}\,dz$$

Hence, we identify:

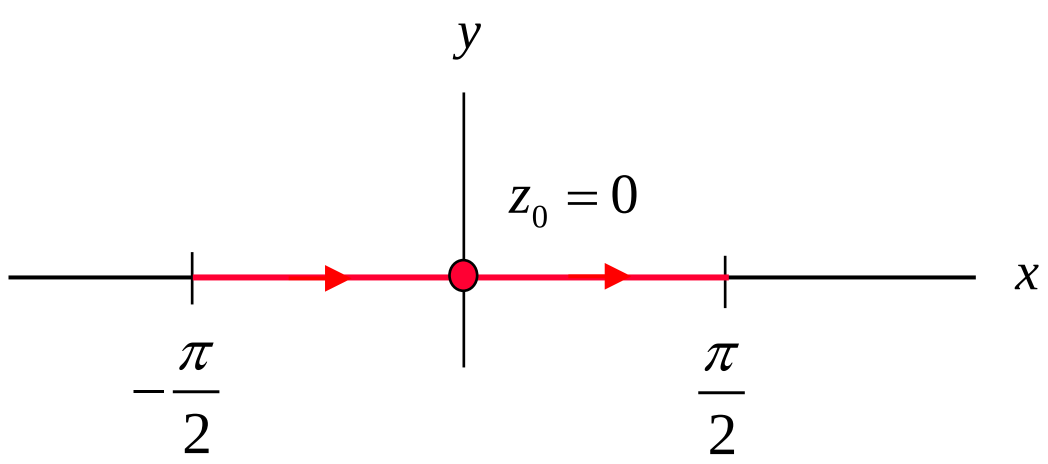
$$g'(z_0) = -j\sin z_0 = 0$$

$$f(z) = 1$$

$$g(z) = j\cos z$$

$$\Rightarrow \quad z_0 = 0 \pm n\pi$$

Example (cont.)



$$g(z) = j\cos z$$

$$g''(z_0) = -j\cos z_0 = -j$$

$$\alpha = \arg g''(z_0) = -\frac{\pi}{2}$$

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2} = \frac{\pi}{4} \pm \frac{\pi}{2}$$

$$\theta_{SDP} = -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Example (cont.)

Identify the SDP and SAP:

$$g(z) = j\cos(x + jy)$$
$$= j\left[\cos x \cosh y - j\sin x \sinh y\right]$$

$$u(x, y) = \sin x \sinh y$$
$$v(x, y) = \cos x \cosh y$$

SDP and SAP: $v(z) = v(z_0) = \text{constant}$

$$\cos x \cosh y = \text{constant} = \cos(0)\cosh(0) = 1$$

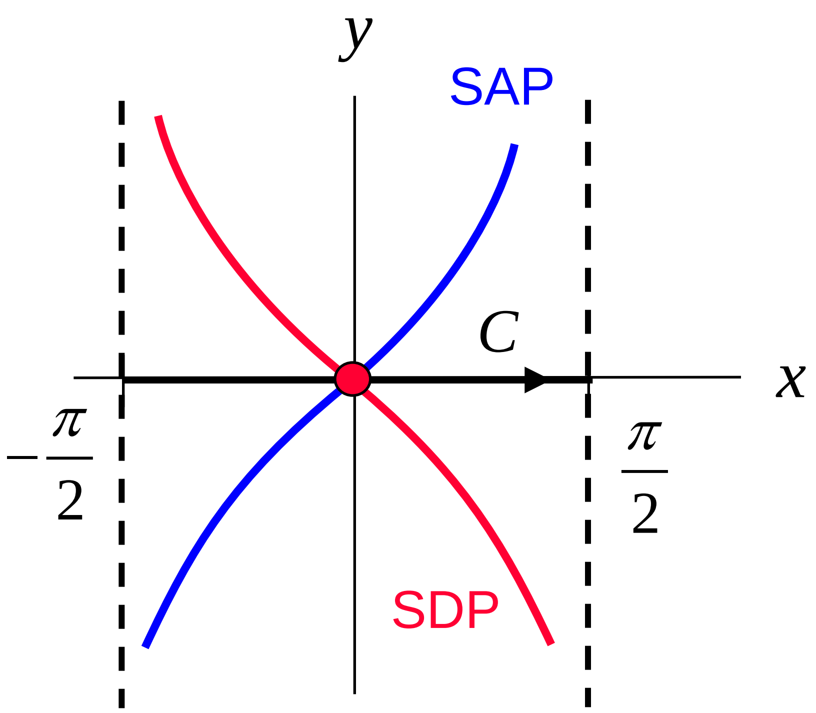
Example (cont.)

SDP and SAP:

$$\cos x \cosh y = 1$$

Examination of the u function reveals which of the two paths is the SDP.

$$u(x, y) = \sin x \,\sinh y$$



Example (cont.)

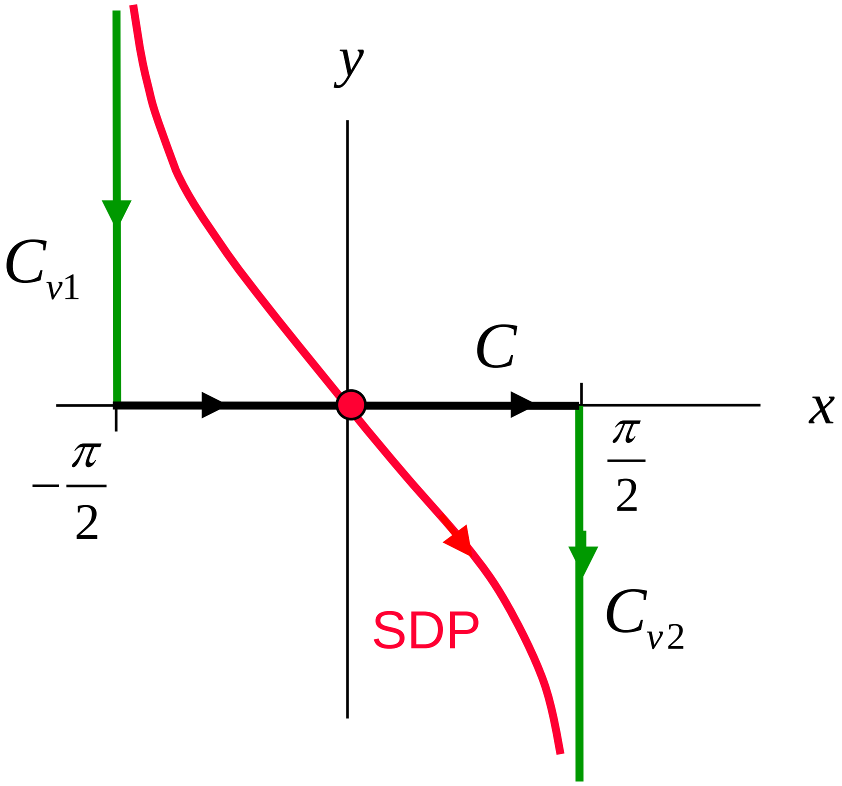
Vertical paths are added so that the path now has limits at infinity.

$$\text{SDP} = C + C_{v1} + C_{v2}$$

It is now clear which choice is correct for the departure angle:

$$\theta_{SDP} = -\frac{\pi}{4}$$

$$I(\Omega) = I_{SDP} - I_{v1} - I_{v2}$$



Example (cont.)

$$I_{SDP} \sim f\left(z_0\right) e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''\left(z_0\right)\right|}} e^{j\theta_{SDP}}$$

This is the answer for $I(\Omega)$ if we ignore the contributions from the vertical paths.

Hence,

$$I\left(\Omega\right) \sim \left(1\right) e^{\Omega j \cos(0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|-j\right|}} e^{-j\frac{\pi}{4}}$$

or

$$I\left(\Omega\right) \sim \sqrt{\frac{2\pi}{\Omega}} e^{j\left(\Omega - \frac{\pi}{4}\right)}$$

so

$$J_0\left(\Omega\right) \sim \frac{1}{\pi} \operatorname{Re}\left\{ \sqrt{\frac{2\pi}{\Omega}} e^{j\left(\Omega - \frac{\pi}{4}\right)} \right\}$$

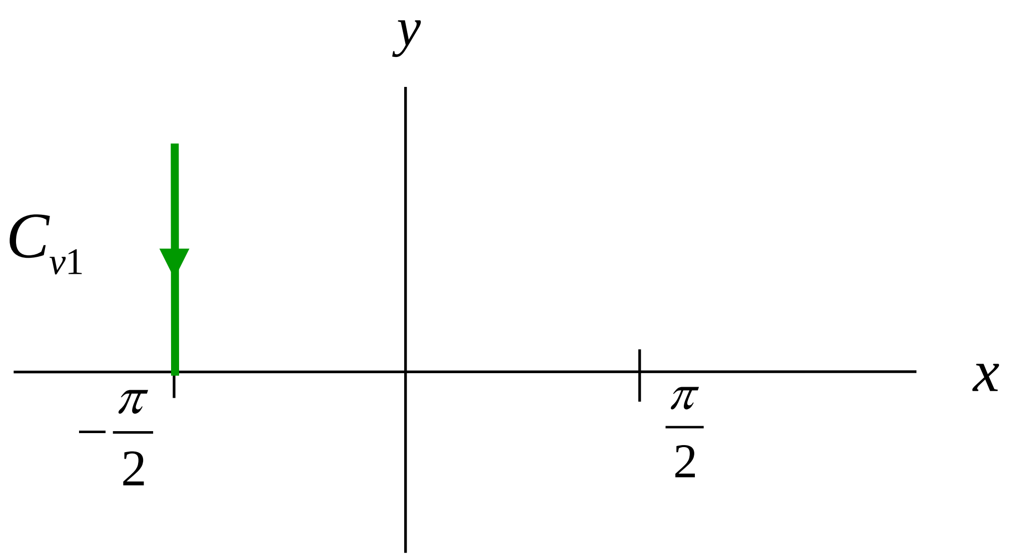
Example (cont.)

Hence

$$J_0(\Omega) \sim \sqrt{\frac{2}{\pi\Omega}}\cos\left(\Omega - \frac{\pi}{4}\right)$$

Example (cont.)

Examine the path C_{v1} (the path C_{v2} is similar).



$$I_{v1} = \int_{-\pi/2+j\infty}^{-\pi/2} e^{\Omega(j\cos z)}\, dz$$

Let $z = -\frac{\pi}{2} + jy$

Example (cont.)

$$I_{v1} = j\int_{\infty}^{0} e^{j\Omega\cos\left(-\frac{\pi}{2}+jy\right)}dy = j\int_{\infty}^{0} e^{-\Omega\sinh y}dy = -j\int_{0}^{\infty} e^{-\Omega\sinh y}dy$$

since $\cos\left(-\dfrac{\pi}{2}+jy\right) = \sin(jy) = j\sinh y$

Use integration by parts (we can also use Watson's Lemma):

$$I_{v1} = -j\int_{0}^{\infty}\left(\frac{-1}{\Omega\cosh y}\right)\frac{d}{dy}\left(e^{-\Omega\sinh y}\right)dy$$

$$\sim -j\left(\frac{-1}{\Omega\cosh y}\right)\left(e^{-\Omega\sinh y}\right)\Big|_{0}^{\infty}$$

$$I_{v1} \sim -j\left(\frac{1}{\Omega}\right)$$

Example (cont.)

Hence,

$$I_{v1} = O\left(\frac{1}{\Omega}\right)$$

Note: I_{v1} is negligible compared to the saddle-point contribution as $\Omega \to \infty$.

However, if we want an asymptotic expansion that is accurate to order 1/ Ω, then the vertical paths must be considered.

Example (cont.)

Alternative evaluation of I_{v1} using Watson's Lemma (alternative form):

$$I_{v1} = -j\int_0^\infty e^{-\Omega \sinh y}\,dy$$

Use

$$s = \sinh(y)$$

$$\frac{ds}{dy} = \cosh(y)$$

$$I_{v1} = -j\int_0^\infty e^{-\Omega s}\left(\frac{dy}{ds}\right)ds$$

$$= -j\int_0^\infty e^{-\Omega s}\,\frac{1}{\cosh(y)}\,ds$$

$$= -j\int_0^\infty e^{-\Omega s}\,\frac{1}{\sqrt{1+\sinh^2(y)}}\,ds$$

$$= -j\int_0^\infty e^{-\Omega s}\,\frac{1}{\sqrt{1+s^2}}\,ds$$

Applying Watson's Lemma:

$$\frac{1}{\sqrt{1+s^2}} \sim 1-\frac{s^2}{2}+\ldots$$

so

$$I_{v1} \sim -j\int_0^\infty e^{-\Omega s}(1)\,ds - j\int_0^\infty e^{-\Omega s}\left(-\frac{s^2}{2}\right)ds + \ldots$$

Example (cont.)

$$I_{v1} \sim -j\int_0^{\infty} e^{-\Omega s}(1)\,ds - j\int_0^{\infty} e^{-\Omega s}\left(-\frac{s^2}{2}\right)ds + \ldots$$

so

$$I_{v1} \sim -j\int_0^{\infty} e^{-\Omega s}\,ds + \frac{j}{2}\int_0^{\infty} e^{-\Omega s}\,s^2\,ds + \ldots$$

Recall:

$$\int_0^{\infty} s^{\alpha_n} e^{-\Omega s} ds = \frac{1}{\Omega^{\alpha_n+1}}\Gamma\left(\alpha_n+1\right)$$

$$\Gamma\left(x\right) = \left(x-1\right)! \equiv \int_0^{\infty} t^{x-1} e^{-t} dt$$

Hence,

$$I_{v1} \sim -j\left(\frac{1}{\Omega}\right) + \frac{j}{2}\left(\frac{\Gamma(3)}{\Omega^3}\right) + \ldots$$

Complete Asymptotic Expansion

By using Watson's lemma, we can obtain the complete asymptotic expansion of the integral in the steepest-descent method, exactly as we did in Laplace's method.

$$I(\Omega) = \int_a^b f(z) e^{\Omega g(z)} dz$$

Define:

$$s^2 \equiv \left[g(z_0) - g(z) \right]_{SDP} \qquad h(s) \equiv f(z(s)) \left(\frac{dz(s)}{ds} \right)$$

Assume: $h(s) \sim \sum_{n=0,1,2\ldots} a_n s^n$ as $s \to 0$

Then we have

$$I(\Omega) \sim e^{\Omega g(z_0)} \sum_{n=0,2,4\ldots} \frac{a_n}{\Omega^{\frac{(n+1)}{2}}} \Gamma\left(\frac{n+1}{2} \right)$$