# ECE 6341

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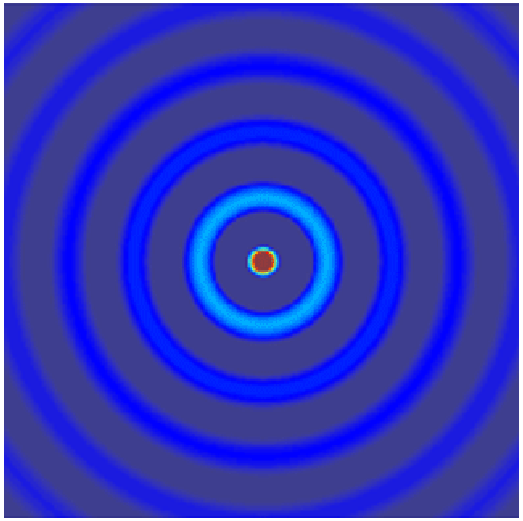

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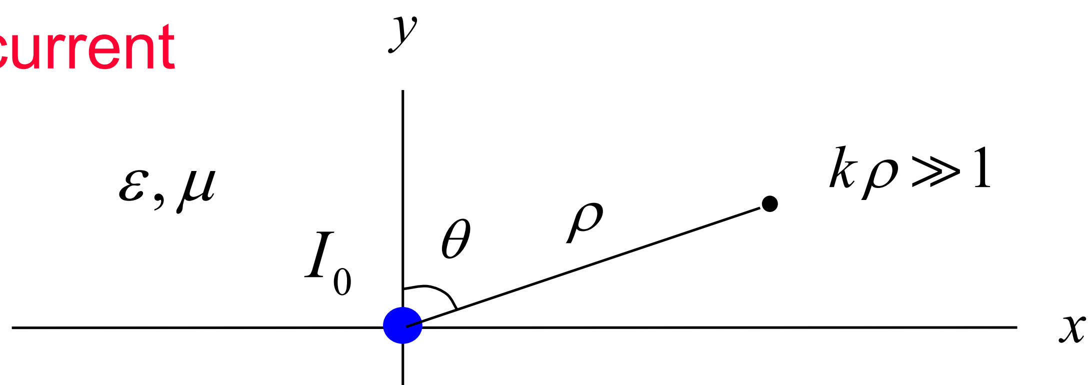
# Example

Line current




$$\theta = \frac{\pi}{2} - \phi$$

$$\underline{J} = \underline{\hat{z}}\,\delta(x)\,\delta(y)\,I_0$$

$$A_z = \frac{\mu I_0}{4j} H_0^{(2)}(k\rho)$$

By using a Fourier-transform method, the exact solution is

$$\underline{A} = \underline{\hat{z}}\,\psi(x, y)$$

where, for $y > 0$,

$$\psi = \frac{\mu I_0}{4\pi j} \int_{-\infty}^{+\infty} \frac{1}{k_y} e^{-jk_y y} e^{-jk_x x}\, dk_x$$

(Please see the appendix.)

$$k_y = \left(k^2 - k_x^{\,2}\right)^{1/2}$$

# Example (cont.)

The vertical wavenumber is $k_y = \left(k^2 - k_x^{\,2}\right)^{1/2}$

The wavenumber $k_y$ is interpreted as

$$k_y = \begin{cases} \left|k_y\right| & k_x < k \\ -j\left|k_y\right| & k_x > k \end{cases}$$

(This follows from the radiation condition at infinity.)

A convenient change of variables is the "steepest-descent transformation".

$$k_x = k \sin \zeta \qquad (\zeta = \zeta_r + j\,\zeta_i)$$

# Example (cont.)

Then

$$k_y = \left(k^2 - k^2 \sin^2 \zeta\right)^{1/2} = \pm k \cos \zeta$$

The path $C$ in the complex $\zeta$-plane is not unique until we choose either + or – here.

This is because the path is not uniquely determined by only $k_x = k \sin \zeta$

To see this in more detail, write

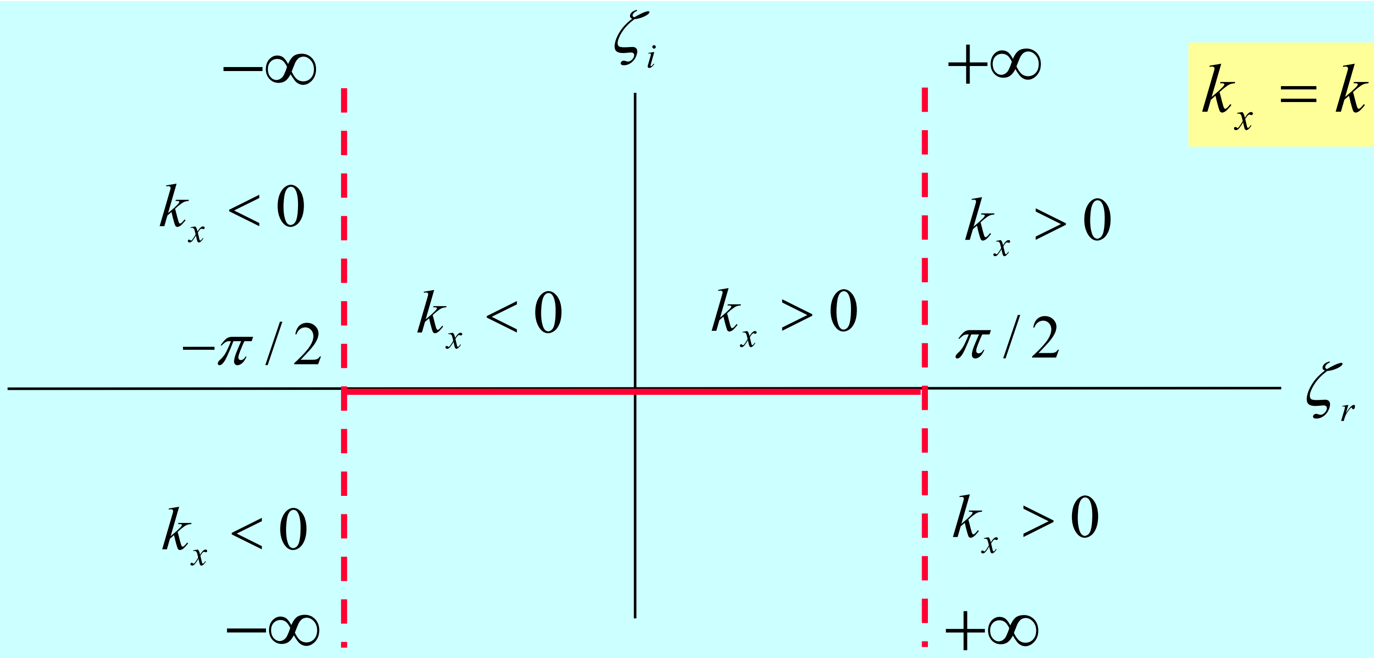
$$\begin{aligned} k_x &= k \sin\left(\zeta_r + j\zeta_i\right) \\ &= k\left[\sin \zeta_r \cosh \zeta_i + j \cos \zeta_r \sinh \zeta_i\right] \end{aligned}$$

# Example (cont.)

Because $k_x$ is real, $\cos\zeta_r \sinh\zeta_i = 0$

Hence $\zeta_i = 0$

or $\zeta_r = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$

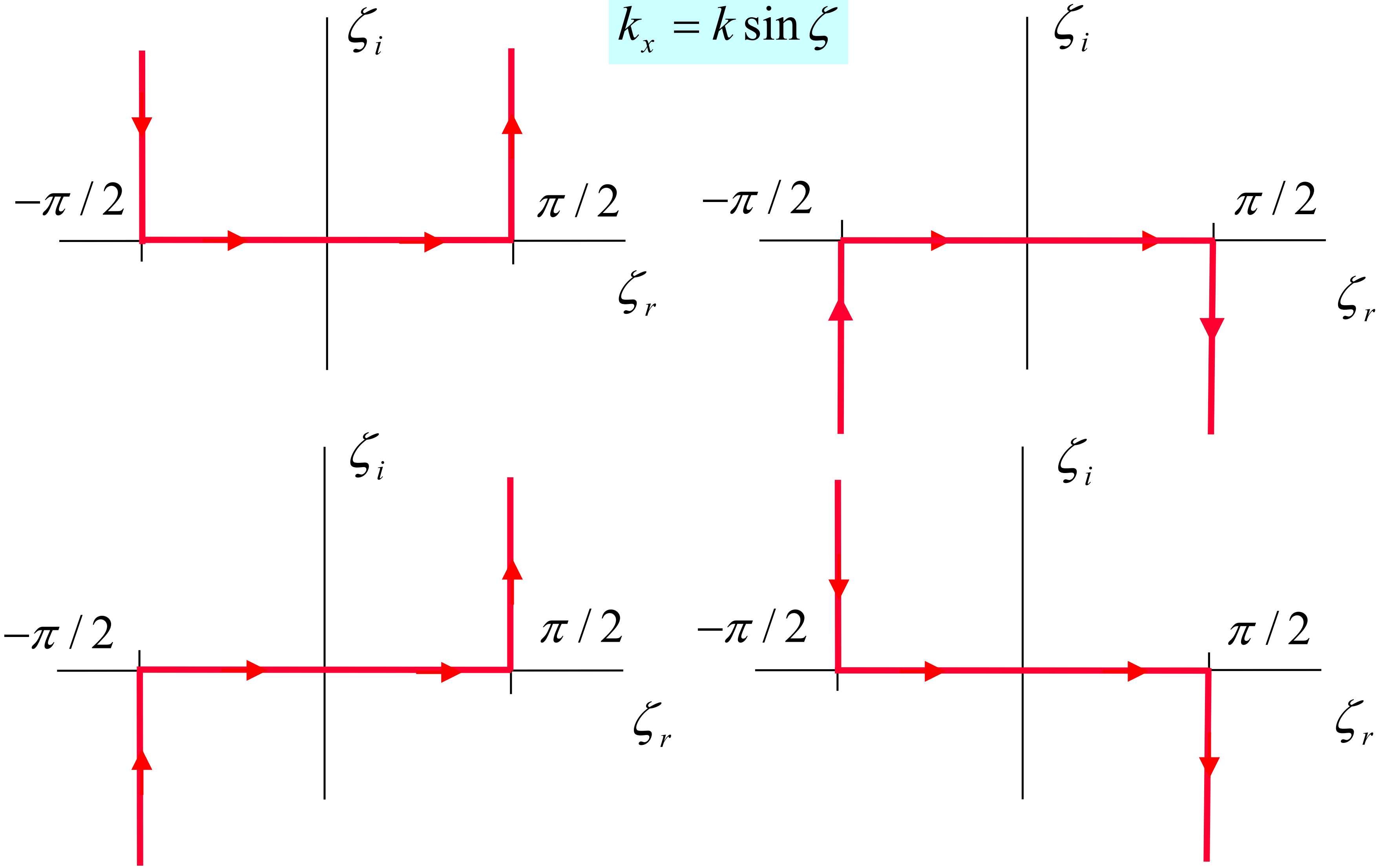



$$k_x = k \sin\zeta_r \cosh\zeta_i$$

# Example (cont.)

There are four possible paths.

$$k_x = k \sin \zeta$$



# Example (cont.)

$k_x$ will vary from -∞ to ∞ along each of these paths.

The path must be chosen so that along the path

$$\operatorname{Re}\left(k_y\right) \geq 0$$

$$\operatorname{Im}\left(k_y\right) \leq 0$$

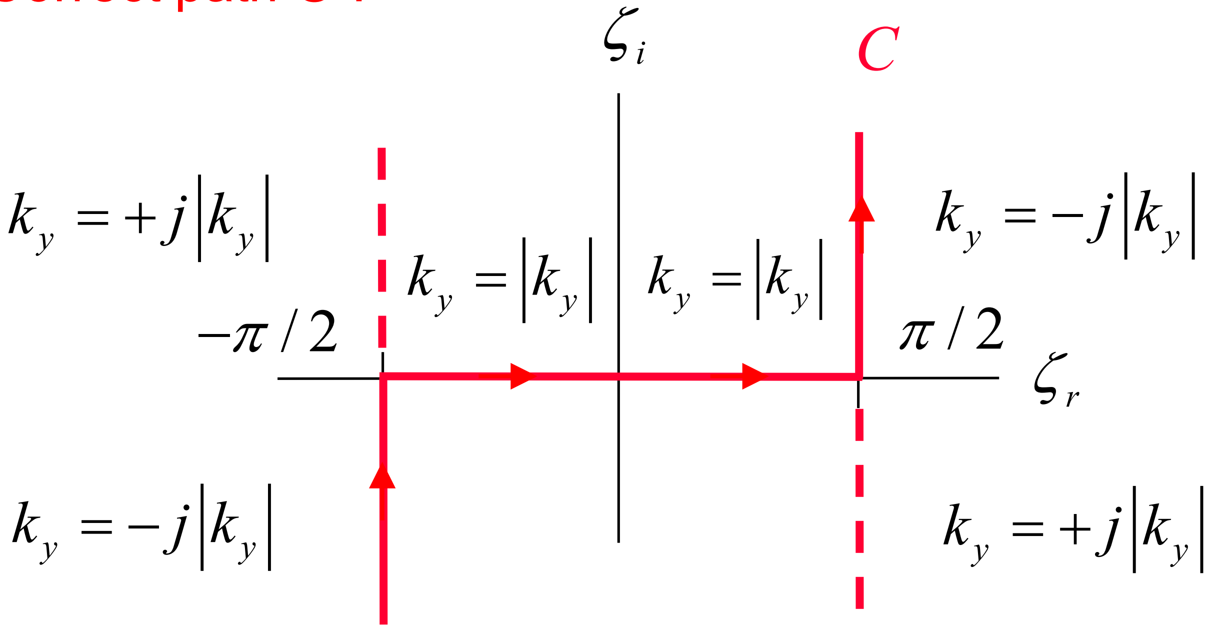
Assume we <u>choose</u> the + sign (an arbitrary choice): $k_y = +k\cos\zeta$

$$\begin{aligned} k_y &= k\cos\zeta = k\cos\left(\zeta_r + j\zeta_i\right) \\ &= k\left[\cos\zeta_r \cosh\zeta_i - j\sin\zeta_r \sinh\zeta_i\right] \end{aligned}$$

# Example (cont.)

$$k_y = k\left[\cos\zeta_r \cosh\zeta_i - j\sin\zeta_r \sinh\zeta_i\right]$$

Correct path $C$ :



# Example (cont.)

$$\psi = \frac{I_0 \mu}{4\pi j} \int_{-\infty}^{\infty} \frac{1}{k_y} e^{-jk_y y} e^{-jk_x x} dk_x$$

Now proceed with the change of variables:

$$k_x = k \sin \zeta \qquad k_y = k \cos \zeta$$

$$dk_x = k \cos \zeta \, d\zeta$$

Hence, we have

$$\psi = \frac{\mu I_0}{4\pi j} \int_C \frac{1}{k \cos \zeta} e^{-jk[x \sin \zeta + y \cos \zeta]} k \cos \zeta \, d\zeta$$

# Example (cont.)

$$\psi = \frac{\mu I_0}{4\pi j} \int_C e^{-jk(x\sin\zeta + y\cos\zeta)} d\zeta$$

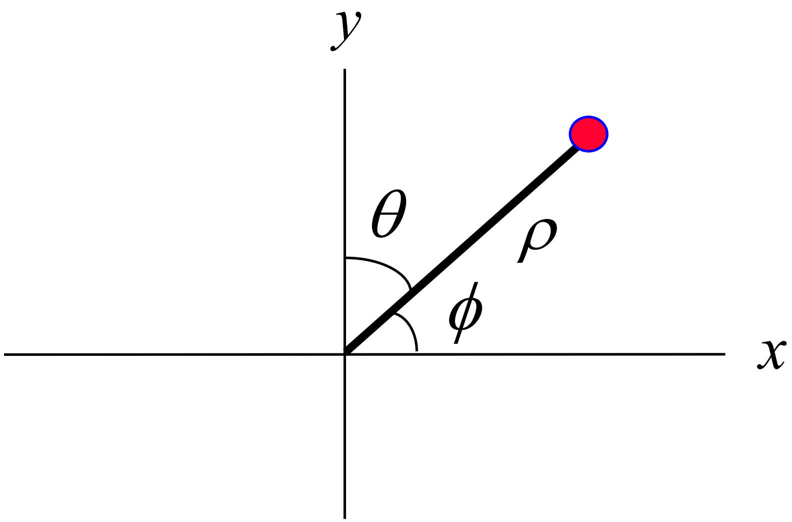
Next, let

$$x = \rho \sin\theta$$

$$y = \rho \cos\theta$$

$$\theta \equiv \frac{\pi}{2} - \phi$$

$$x\sin\zeta + y\cos\zeta = (\rho\sin\theta)\sin\zeta + (\rho\cos\theta)\cos\zeta = \rho\cos(\zeta - \theta)$$



# Example (cont.)

The integral then becomes

$$\psi = \frac{\mu I_0}{4\pi j} \int_C e^{-jk\rho\cos(\zeta-\theta)} d\zeta$$

Ignoring the constant in front, we can identify:

$$f(\zeta) = 1$$

$$\Omega = k\rho$$

$$g(\zeta) = -j\cos(\zeta - \theta)$$

$$g'(\zeta) = j\sin(\zeta - \theta)$$

$$g''(\zeta) = j\cos(\zeta - \theta)$$

Hence

$$\zeta_0 = \theta$$

$$g(\zeta_0) = -j$$

$$g''(\zeta_0) = j$$

# Example (cont.)

SDP:

$$v(\zeta) = \mathrm{Im}\left(g(\zeta)\right) = \text{constant} = \mathrm{Im}\left(g(\zeta_0)\right) = \mathrm{Im}(-j) = -1$$

$$g(\zeta) = -j\cos(\zeta - \theta)$$

$$-j\cos\left[(\zeta_r - \theta) + j\zeta_i\right]$$

$$= -j\left[\cos(\zeta_r - \theta)\cosh\zeta_i - j\sin(\zeta_r - \theta)\sinh\zeta_i\right]$$

so

$$v(\zeta) = -\left[\cos(\zeta_r - \theta)\cosh\zeta_i\right]$$

Hence

$$\cos(\zeta_r - \theta)\cosh\zeta_i = +1$$

(SDP or SAP)

# Example (cont.)

$$g(\zeta) = -j\left[\cos(\zeta_r - \theta)\cosh\zeta_i - j\sin(\zeta_r - \theta)\sinh\zeta_i\right]$$

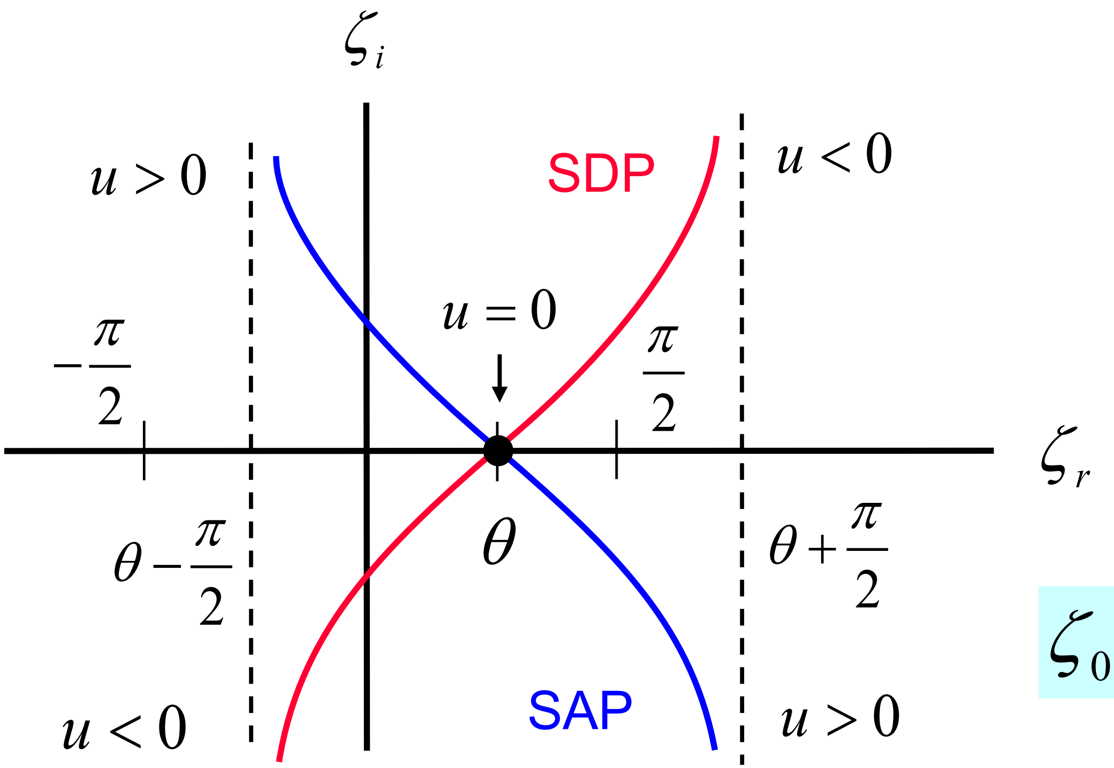
Using $u(\zeta) = \mathrm{Re}\left(g(\zeta)\right)$

we also see that

$$u(\zeta) = -\sin(\zeta_r - \theta)\sinh\zeta_i$$

This will help us determine which curve is the SDP and which is the SAP.

# Example (cont.)

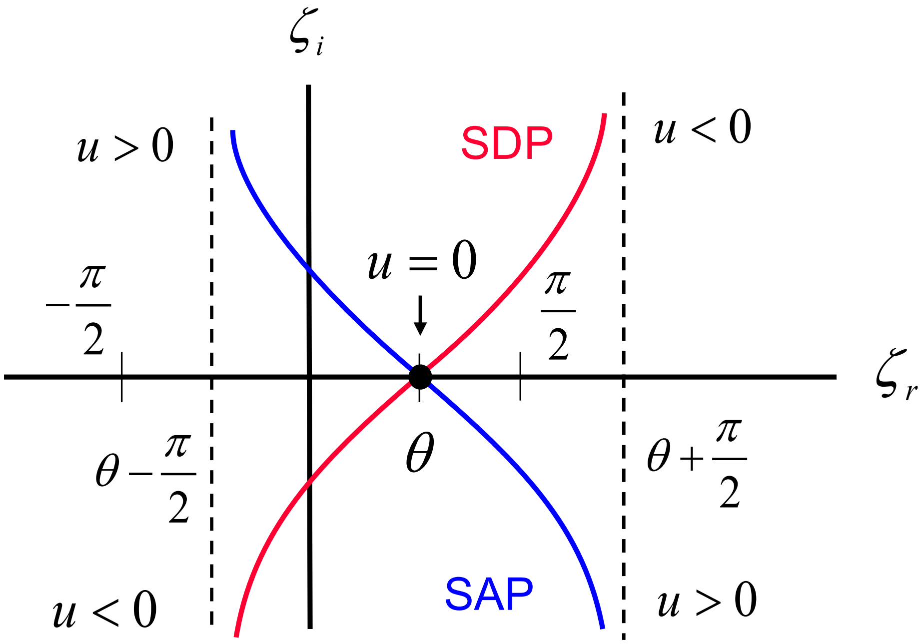



$$\zeta_0 = \theta$$

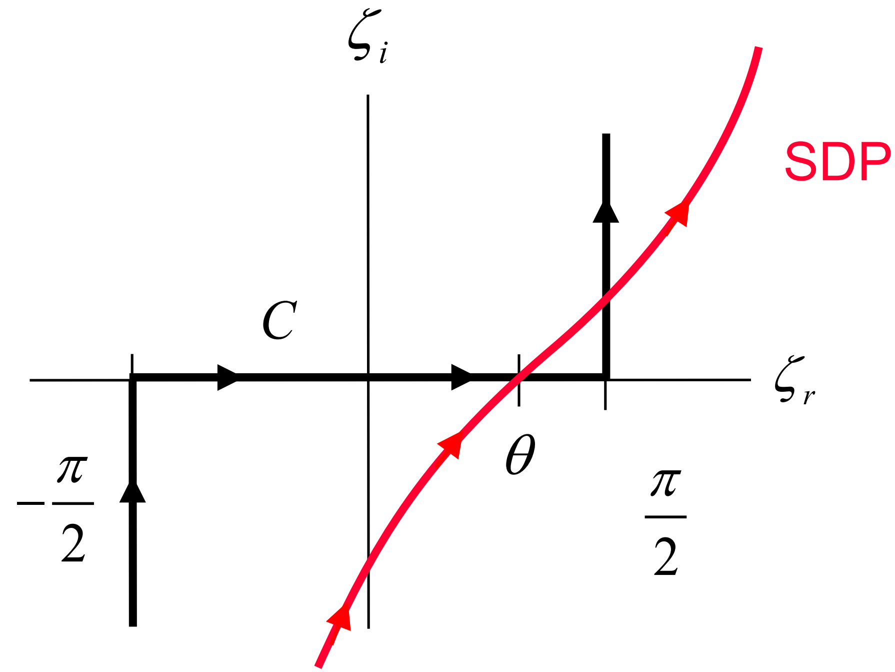
$$\cos(\zeta_r - \theta)\cosh\zeta_i = +1 \quad \text{(SDP or SAP)}$$

$$u(\zeta) = -\sin(\zeta_r - \theta)\sinh\zeta_i$$

# Example (cont.)




$$\zeta_0 = \theta$$




Examination of the original path allows us to determine the direction of integration along the SDP.

# Example (cont.)

Calculate $\theta_{SDP}$ :

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''(\zeta_0)\right) = \arg\left(j\right) = \frac{\pi}{2}$$

$$\theta_{SDP} = -\frac{\pi}{4} \pm \frac{\pi}{2}$$

so $\theta_{SDP} = +\frac{\pi}{4}$ or $\theta_{SDP} = -\frac{3\pi}{4}$

From the figure we see that the correct choice is

$$\theta_{SDP} = +\frac{\pi}{4}$$

# Example (cont.)

Method of steepest-descent recipe:

$$I(\Omega) \sim f(z_0) e^{\Omega g(z_0)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{|g''(z_0)|}} e^{j\theta_{SDP}}$$

We then have

$$\psi \sim \frac{\mu I_0}{4\pi j} \left[ e^{-jk\rho} \sqrt{\frac{\pi}{(k\rho)}} \sqrt{\frac{2}{|j|}} e^{j\frac{\pi}{4}} \right]$$

or

$$\psi \sim \frac{\mu I_0}{4\pi j} \sqrt{\frac{2\pi}{(k\rho)}} \; e^{-jk\rho} e^{j\frac{\pi}{4}}$$

$$f(\zeta) = 1$$

$$\Omega = k\rho$$

$$\zeta_0 = \theta$$

$$g(\zeta_0) = -j$$

$$g''(\zeta_0) = j$$

$$\theta_{SDP} = +\frac{\pi}{4}$$

# Example (cont.)

$$\psi \sim \frac{\mu I_0}{4\pi j}\sqrt{\frac{2\pi}{k\rho}}\; e^{-jk\rho} e^{j\frac{\pi}{4}}$$

The <u>exact</u> solution is: $\psi = \frac{\mu I_0}{4j} H_0^{(2)}\left(k\rho\right)$

It can easily be verified that the asymptotic result is correct, since

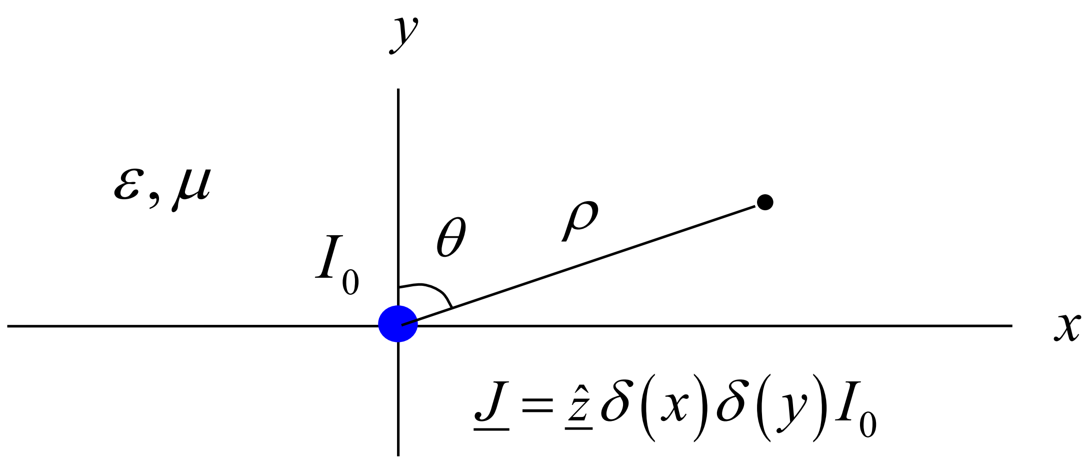
$$H_0^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-j(x-\frac{\pi}{4})}$$

so that

$$\psi \sim \frac{\mu I_0}{4j}\sqrt{\frac{2}{\pi k\rho}}\, e^{-j(k\rho-\frac{\pi}{4})}$$

# Appendix

## Derivation of formula




TM$_z$ : $\quad \psi = A_z\left(x, y\right)$

$$\nabla^2\psi + k^2\psi = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2\,\psi = 0$$

# Appendix (cont.)

Introduce the Fourier transform pair:

$$\tilde{\psi}\left(k_x, y\right) = \int_{-\infty}^{+\infty} \psi\left(x, y\right) e^{+jk_x x} dx$$

$$\psi\left(x, y\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\psi}\left(k_x, y\right) e^{-jk_x x} dk_x$$

We then have $\dfrac{\partial^2 \tilde{\psi}}{\partial y^2} + \left(k^2 - k_x^2\right)\tilde{\psi} = 0$

Define: $k_y = \left(k^2 - k_x^{\,2}\right)^{1/2}$

$$\tilde{\psi}\left(k_x, y\right) = \tilde{\psi}\left(k_x, 0\right) e^{\mp j k_y y} = \tilde{\psi}^{\pm}\left(k_x, y\right)$$

top sign, $y > 0$

bottom sign, $y < 0$

# Appendix (cont.)

$$\tilde{\psi}^{+}\left(k_x, y\right) = \tilde{\psi}\left(k_x, 0\right) e^{-jk_y y}, \quad y > 0$$

$$\tilde{\psi}^{-}\left(k_x, y\right) = \tilde{\psi}\left(k_x, 0\right) e^{+jk_y y}, \quad y < 0$$

Boundary conditions at $y = 0$:

$$E_z^{+} = E_z^{-} \qquad \text{(satisfied automatically)}$$

$$H_x^{-} - H_x^{+} = J_{sz} = I_0 \, \delta\left(x\right)$$

$$\Rightarrow \quad \tilde{H}_x^{-} - \tilde{H}_x^{+} = I_0$$

$$H_x = \frac{1}{\mu} \frac{\partial \psi}{\partial y}$$

# Appendix (cont.)

Hence

$$\tilde{H}_x^{+} = \frac{1}{\mu}\left(-jk_y\right)\tilde{\psi}\left(k_x, 0\right)$$

$$\tilde{H}_x^{-} = \frac{1}{\mu}\left(+jk_y\right)\tilde{\psi}\left(k_x, 0\right)$$

We then have

$$\frac{2j}{\mu}k_y\tilde{\psi}\left(k_x, 0\right) = I_0$$

$$\Rightarrow \quad \tilde{\psi}\left(k_x, 0\right) = \frac{\mu I_0}{2j\,k_y}$$

# Appendix (cont.)

Hence

$$\psi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\mu I_0}{2jk_y}\right)e^{-jk_x x}\,dk_x$$

And then

$$\psi(x,y) = \frac{\mu I_0}{4\pi j}\int_{-\infty}^{+\infty}\frac{1}{k_y}e^{-jk_y y}e^{-jk_x x}\,dk_x$$