ECE 6341

Spring 2016

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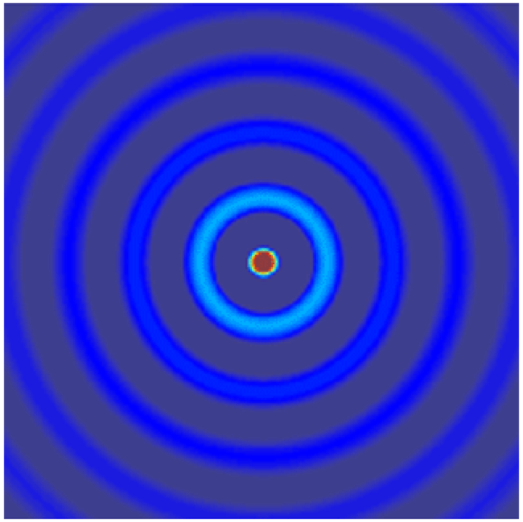


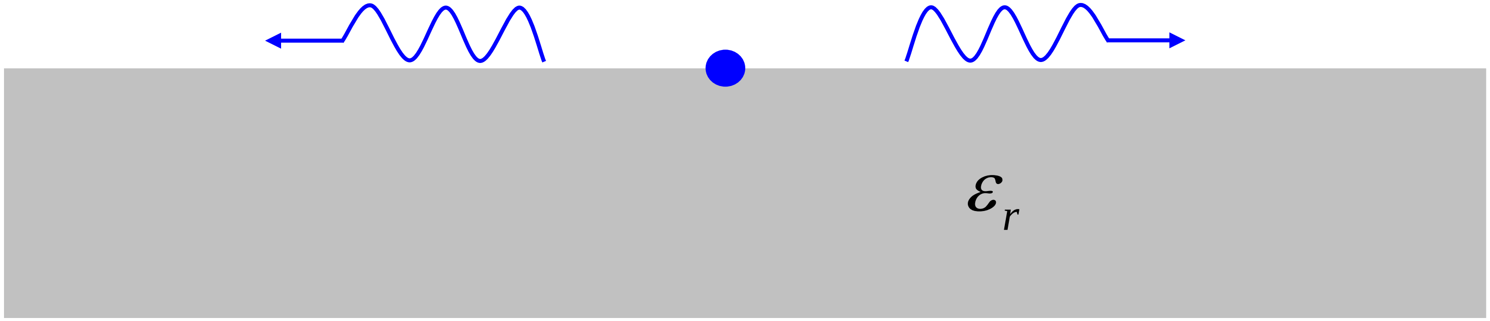
Notes 35

Higher-order Steepest-Descent Method

$$I(\Omega) = \int_C f(z) e^{\Omega g(z)} dz$$

Assume $f(z_0) = 0$

This important special case arises when asymptotically evaluating the field along an interface (discussed later in the notes).



Higher-order SDM (cont.)

$$I(\Omega) = \int_C f(z)\, e^{\Omega g(z)} dz$$

$$I(\Omega) = e^{\Omega g(z_0)} \int_{SDP} f(z)\, e^{-\Omega[g(z_0)-g(z)]}\, dz$$

Along SDP: $s^2 \equiv g(z_0) - g(z)$ (real)

Note: The variable s is taken as positive after we leave the saddle point, and negative before we enter the saddle point.

Hence $I(\Omega) = e^{\Omega g(z_0)} \int_{-\infty}^{+\infty} f(z(s)) \left(\dfrac{dz}{ds} \right) e^{-\Omega s^2} ds$

Higher-order SDM (cont.)

$$I(\Omega) = e^{\Omega g(z_0)} \int_{-\infty}^{+\infty} f\left(z(s)\right)\left(\frac{dz}{ds}\right) e^{-\Omega s^2} ds$$

Define $\quad h(s) \equiv f\left(z(s)\right)\frac{dz}{ds}$

Then $\quad I(\Omega) = e^{\Omega g(z_0)} \int_{-\infty}^{+\infty} h(s) e^{-\Omega s^2} ds$

$$f(z_0) = 0$$

Assume $\quad h(s) \sim \cancel{h(0)} + h'(0)s + \frac{1}{2}h''(0)s^2$

Higher-order SDM (cont.)

Note: $$\int_{-\infty}^{+\infty} s\,e^{-\Omega s^2}\,ds = 0$$ (s is odd)

Hence $$I(\Omega) \sim \frac{1}{2} h''(0)\, e^{\Omega g(z_0)}\, I_s$$

where $$I_s \equiv \int_{-\infty}^{+\infty} s^2\, e^{-\Omega s^2} ds = 2\int_0^{\infty} s^2 e^{-\Omega s^2} ds$$

Use $$t = \Omega s^2$$

$$dt = 2\,\Omega\, s\, ds$$

Higher-order SDM (cont.)

$$I_s = 2\int_0^\infty s^2 e^{-\Omega s^2}\, ds$$

$$t = \Omega s^2$$

$$dt = 2\,\Omega\, s\, ds$$

$$I_s = 2\int_0^\infty \left(\frac{t}{\Omega}\right) e^{-t} \left(\frac{1}{2\Omega\sqrt{\frac{t}{\Omega}}}\right) dt$$

$$= \frac{1}{\Omega^{3/2}} \int_0^\infty t^{1/2} e^{-t}\, dt$$

$$= \frac{1}{\Omega^{3/2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\Omega^{3/2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\Omega^{3/2}} \left(\frac{\sqrt{\pi}}{2}\right)$$

Recall:

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$$

$$\Gamma(x) = (x-1)! = (x-1)(x-2)!$$

$$= (x-1)\Gamma(x-1)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Higher-order SDM (cont.)

Hence

$$I(\Omega) \sim \frac{1}{4} h''(0) e^{\Omega g(z_0)} \sqrt{\pi} \left(\frac{1}{\Omega^{3/2}} \right)$$

Note: The leading term of the expansion now behaves as 1 / $\Omega^{3/2}$.

We now we need to calculate $h''(0)$

Higher-order SDM (cont.)

$$h(s) = f\left(z(s)\right) z'(s)$$

$$h'(s) = f'\left(z(s)\right)\left(z'(s)\right)^2 + f\left(z(s)\right) z''(s)$$

$$\begin{aligned} h''(s) = {} & f''\left(z(s)\right)\left(z'(s)\right)^3 + f'\left(z(s)\right)(2)\left(z'(s)\right) z''(s) \\ & + f'\left(z(s)\right) z'(s) z''(s) + f\left(z(s)\right) z'''(s) \end{aligned}$$

$$h''(0) = f''\left(z_0\right)\left(z'(0)\right)^3 + 3 f'\left(z_0\right)\left(z'(0)\right) z''(0) + \cancel{f\left(z_0\right)} z'''(0)$$

Higher-order SDM (cont.)

Hence $h''(0) = f''(z_0)\left(z'(0)\right)^3 + 3f'(z_0)z'(0)z''(0)$

We next take two derivatives with respect to s
in order to calculate $z'(0),\ z''(0)$

$$s^2 \equiv g(z_0) - g(z)$$

(1st) $2s = -g'(z)z'(s)$

(2nd) $2 = -g''(z)\left(z'(s)\right)^2 - g'(z)z''(s)$

Higher-order SDM (cont.)

From (2nd): $2 = -g''(z_0)\left(z'(0)\right)^2 - \cancel{g'(z_0)}\, z''(0)$

We then have

$$z'(0) = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

or

$$\boxed{z'(0) = \sqrt{\frac{2}{\left|g''(z_0)\right|}}\, e^{j\theta_{SDP}}}$$

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''(z_0)\right)$$

Higher-order SDM (cont.)

(2nd) $$2 = -g''(z)\left(z'(s)\right)^2 - g'(z)z''(s)$$

Take one more derivative to calculate z'':

$$\begin{aligned} 0 &= -g'''(z)\left(z'(s)\right)^3 - 2g''(z)\left(z'(s)\right)z''(s) - g''(z)z'(s)z''(s) - g'(z)z'''(s) \\ &= -g'''(z)\left(z'(s)\right)^3 - 3g''(z)\left(z'(s)\right)z''(s) - g'(z)z'''(s) \end{aligned}$$ (3rd)

$$0 = -g'''(z_0)\left(z'(0)\right)^3 - 3g''(z_0)\left(z'(0)\right)z''(0) - \cancel{g'(z_0)}z'''(0)$$

or

$$0 = -g'''(z_0)\left(z'(0)\right)^2 - 3g''(z_0)z''(0)$$

Higher-order SDM (cont.)

Hence,

$$z''(0) = -\frac{1}{3}\frac{g'''(z_0)\left(z'(0)\right)^2}{g''(z_0)}$$

where

$$z'(0) = \left(\frac{-2}{g''(z_0)}\right)^{1/2}$$

Therefore,

$$\boxed{z''(0) = \frac{2}{3}\frac{g'''(z_0)}{\left(g''(z_0)\right)^2}}$$

Higher-order SDM (cont.)

We then have

$$h''(0) = f''(z_0)\left(\frac{2}{\left|g''(z_0)\right|}\right)^{3/2} e^{j3\theta_{SDP}}$$

$$+3f'(z_0)\sqrt{\frac{2}{\left|g''(z_0)\right|}}e^{j\theta_{SDP}}\left(\frac{2}{3}\frac{g'''(z_0)}{\left(g''(z_0)\right)^2}\right)$$

Higher-order SDM (cont.)

Summary

$$I(\Omega) = \int\limits_C f(z) e^{\Omega g(z)} dz \qquad f(z_0) = 0$$

$$I(\Omega) \sim \frac{1}{4} h''(0) e^{\Omega g(z_0)} \sqrt{\pi} \left(\frac{1}{\Omega^{3/2}} \right)$$

$$h''(0) = f''(z_0) \left(\frac{2}{\left| g''(z_0) \right|} \right)^{3/2} e^{j3\theta_{SDP}}$$

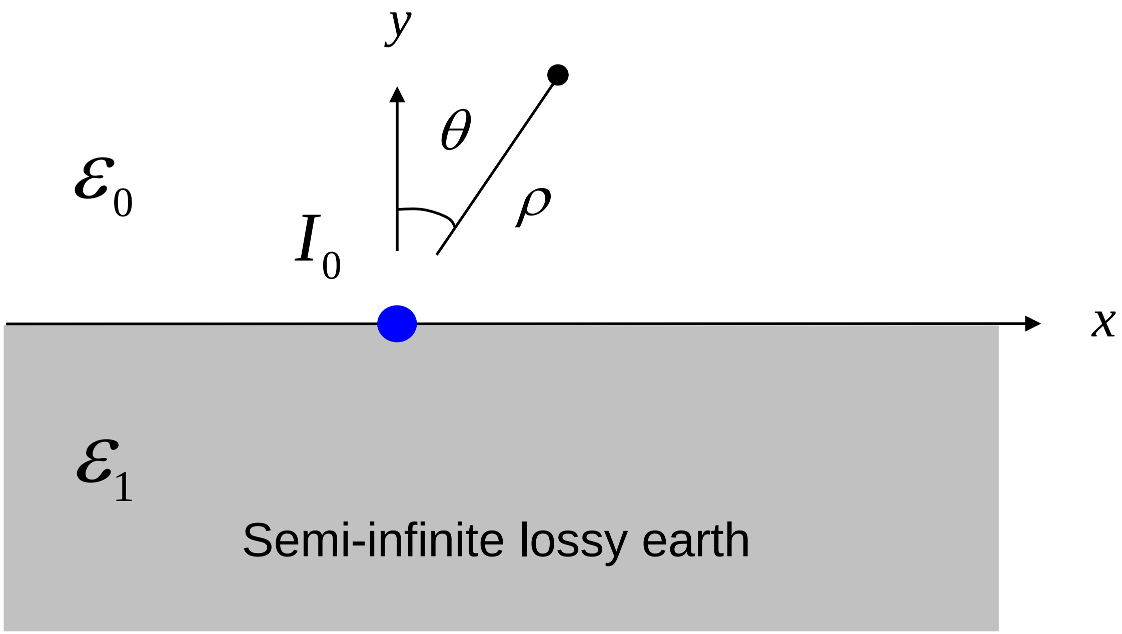
$$+3 f'(z_0) \sqrt{\frac{2}{\left| g''(z_0) \right|}} e^{j\theta_{SDP}} \left(\frac{2}{3} \frac{g'''(z_0)}{\left(g''(z_0) \right)^2} \right)$$

$$\theta_{SDP} = -\frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$\alpha = \arg\left(g''(z_0) \right)$$

Example

A line source is located on the interface.



Example

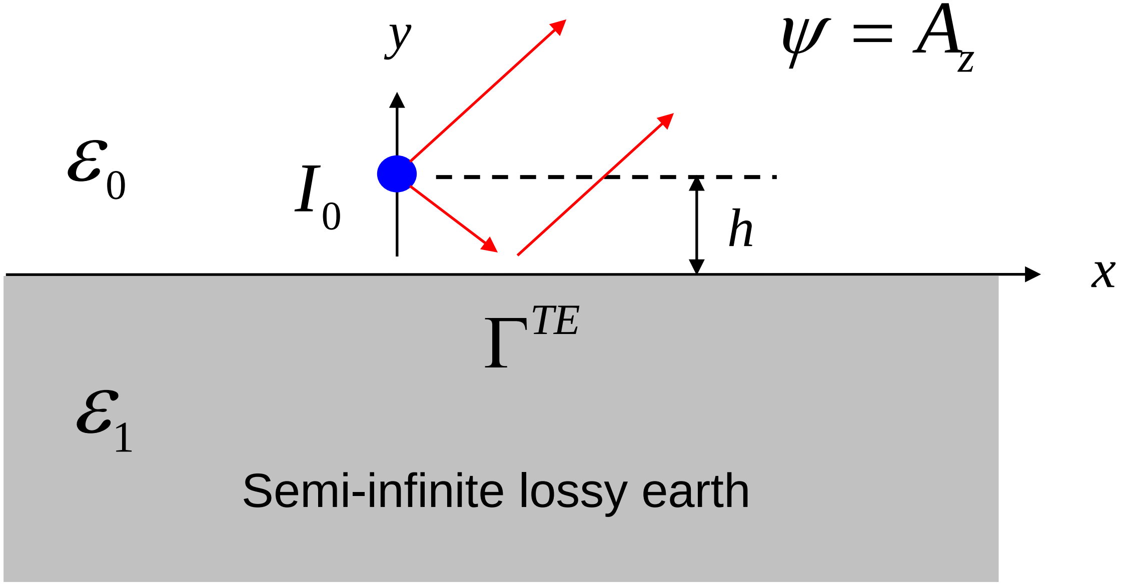
Consider first the line source at a height h above the interface.

The field is TM$_z$ and also TE$_y$.

$$\underline{E} = \underline{\hat{z}}E_z$$

$$\underline{E} = -j\omega\underline{A} - \cancel{\nabla\Phi}$$

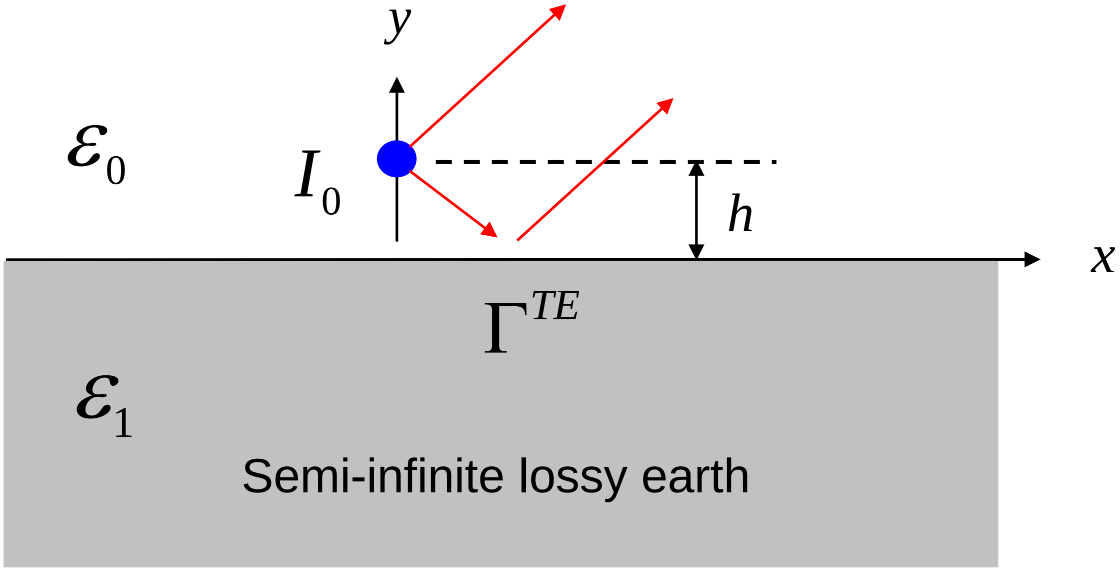
$$\psi = A_z$$



Example

$$\psi = \frac{\mu_0 I_0}{4\pi j} \int_{-\infty}^{+\infty} \left[\frac{1}{k_{y0}} e^{-jk_{y0}|y-h|} + \frac{1}{k_{y0}} e^{-jk_{y0}h}\, \Gamma^{TE} e^{-jk_{y0}y} \right] e^{-jk_x x} dk_x$$

$$\Gamma^{TE} = \frac{Z_1^{TE} - Z_0^{TE}}{Z_1^{TE} + Z_0^{TE}}$$



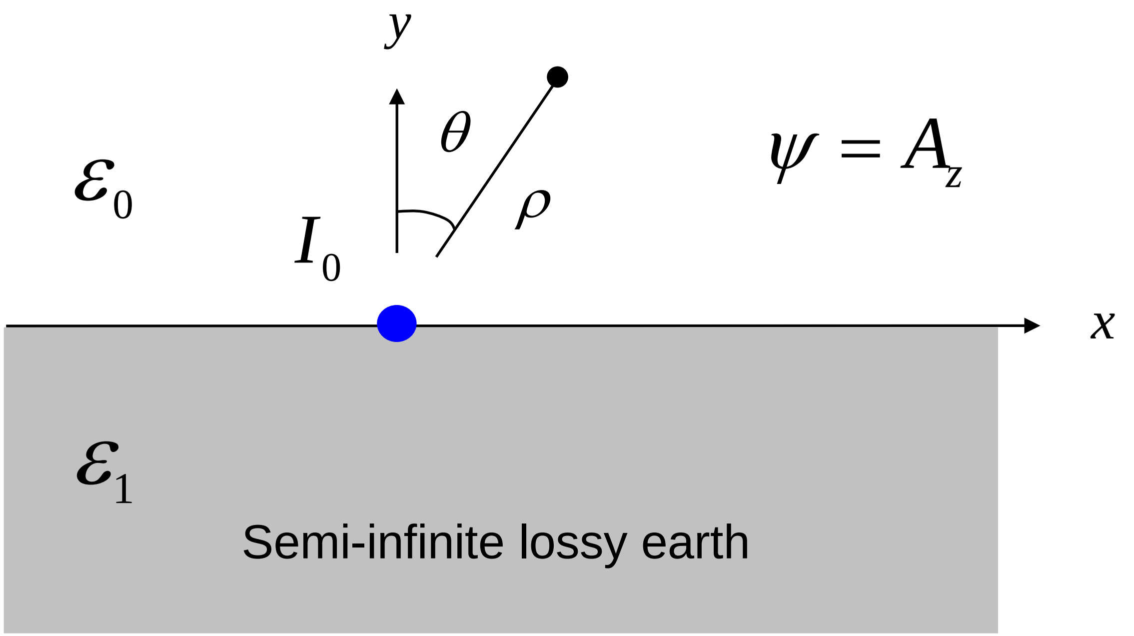
TE$_y$:

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{y0}}$$

$$Z_1^{TE} = \frac{\omega\mu_0}{k_{y1}}$$

Example

The line source is now at the interface ($h = 0$).



$$\psi = A_z$$

$$\psi = \frac{\mu_0 I_0}{4\pi j} \int_{-\infty}^{+\infty} \frac{1}{k_{y0}} \left[1 + \left(\frac{Z_1^{TE} - Z_0^{TE}}{Z_1^{TE} + Z_0^{TE}} \right) \right] e^{-jk_{y0}y} e^{-jk_x x} dk_x$$

$$k_{y0} = \left(k_0^2 - k_x^2 \right)^{1/2}$$

Example (cont.)

Simplifying,

$$1+\frac{Z_1^{TE}-Z_0^{TE}}{Z_1^{TE}+Z_0^{TE}}=\frac{2Z_1^{TE}}{Z_0^{TE}+Z_1^{TE}}$$

$$Z_0^{TE}=\frac{\omega\mu_0}{k_{y0}} \qquad k_{y0}=\left(k_0^2-k_x^2\right)^{1/2}$$

$$Z_1^{TE}=\frac{\omega\mu_0}{k_{y1}} \qquad k_{y1}=\left(k_1^2-k_x^2\right)^{1/2}$$

Example (cont.)

Hence, we have

$$\frac{2Z_1^{TE}}{Z_0^{TE}+Z_1^{TE}}=\frac{2\left(\dfrac{1}{k_{y1}}\right)}{\dfrac{1}{k_{y0}}+\dfrac{1}{k_{y1}}}=\frac{2k_{y0}}{k_{y0}+k_{y1}}$$

Therefore,

$$\psi=\frac{\mu_0 I_0}{2\pi j}\int_{-\infty}^{+\infty}\left(\frac{1}{k_{y0}+k_{y1}}\right)e^{-jk_{y0}y}e^{-jk_x x}dk_x$$

Now use:

$$k_x=k_0\sin\zeta$$
$$dk_x=k_0\cos\zeta\,d\zeta$$
$$k_{y0}=k_0\cos\zeta$$

$$x=\rho\sin\theta$$
$$y=\rho\cos\theta$$

Example (cont.)

$$\psi = \frac{\mu_0 I_0}{2\pi j} \int_C \left(\frac{k_0 \cos\zeta}{k_0 \cos\zeta + \left(k_1^2 - k_0^2 \sin^2\zeta\right)^{1/2}} \right) e^{-j(k_0\rho)\cos(\zeta-\theta)} d\zeta$$

or

$$\psi = \frac{\mu_0 I_0}{2\pi j} \int_C \left(\frac{\cos\zeta}{\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}} \right) e^{-j(k_0\rho)\cos(\zeta-\theta)} d\zeta$$

$$n_1 \equiv k_1 / k_0 = \sqrt{\varepsilon_r \mu_r}$$

Note:

For a lossless earth

$$k_{y1} = \left(k_1^2 - k_0^2 \sin^2\zeta\right)^{1/2} = \begin{cases} k_0\sqrt{k_1^2 - k_0^2 \sin^2\zeta}, & k_0 \sin\zeta < k_1 \\ -jk_0\sqrt{k_0^2 \sin^2\zeta - k_1^2}, & k_0 \sin\zeta > k_1 \end{cases}$$

Example (cont.)

We then identify (ignoring the constant in front of the integral):

$$\Omega = k\rho$$

$$g(\zeta) = -j\cos(\zeta - \theta)$$

$$f(\zeta) = \frac{\cos\zeta}{\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}}$$

Saddle point: $\zeta_0 = \theta$

Notes:

There are branch points in the complex ζ plane arising from k_{y1}, but we are ignoring these for a lossy earth (n_1 is complex).

There are no branch points in the ζ plane arising from k_{y0}, since the steepest-descent transformation has removed them.

$$k_{y0} = \left(k^2 - k_x^2\right)^{1/2} = k_0\cos\zeta$$

Example (cont.)

$$f(\zeta)=\frac{\cos\zeta}{\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}}$$

Fields on the interface:

As $\theta \to 90^\circ, \quad \zeta_0 \to 90^\circ$

Hence $\boxed{f(\zeta_0)=0}$ (unless $n_1 = 1$)

We need the higher-order steepest-descent method when we evaluate the fields along the interface.

Example (cont.)

Far-field (antenna) radiation pattern: $0 \le \theta < \pi / 2$

$$\Omega = k\rho$$

$$f\left(\zeta_0\right) = \frac{\cos\theta}{\cos\theta + \left(n_1^2 - \sin^2\theta\right)^{1/2}}$$

$$\zeta_0 = \theta$$

$$g\left(\zeta_0\right) = -j \qquad g''\left(\zeta_0\right) = j \qquad f\left(\zeta_0\right) = f\left(\theta\right) \neq 0$$

The usual steepest-descent method applies:

$$I\left(\Omega\right) \sim f\left(\zeta_0\right) e^{\Omega g\left(\zeta_0\right)} \sqrt{\frac{\pi}{\Omega}} \sqrt{\frac{2}{\left|g''\left(\zeta_0\right)\right|}} e^{j\theta_{SDP}}$$

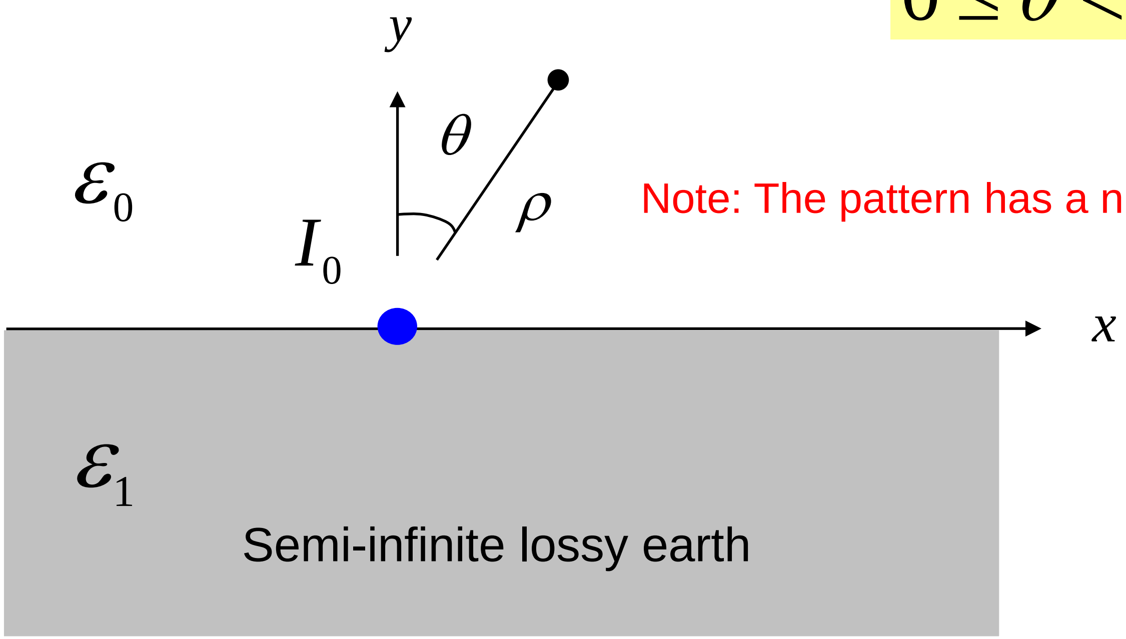
$$\psi \sim \frac{\mu_0 I_0}{2\pi j} \left(\frac{\cos\theta}{\cos\theta + \left(n_1^2 - \sin^2\theta\right)^{1/2}} \right) e^{-j\left(k_0\rho\right)} \sqrt{\frac{\pi}{k_0\rho}} \sqrt{\frac{2}{\left|j\right|}} e^{j\pi/4}$$

Example (cont.)

Final far-field radiation pattern:

$$E_z \sim -j\omega\left(\frac{\mu_0 I_0}{2\pi j}\right)\left(\frac{\cos\theta}{\cos\theta+\left(n_1^2-\sin^2\theta\right)^{1/2}}\right)e^{-j(k_0\rho)}\sqrt{\frac{2\pi}{k_0\rho}}\,e^{j\pi/4}$$

$$0 \le \theta < \pi/2$$



Note: The pattern has a null as we approach the horizon.

Example (cont.)

Interface field: $\theta = \pi / 2$

$$\psi = \frac{\mu_0 I_0}{2\pi j} \int_C \left(\frac{\cos\zeta}{\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}} \right) e^{-j(k_0\rho)\cos(\zeta-\theta)} d\zeta$$

Recall

$$f(\zeta_0) = f(\theta) = f\left(\frac{\pi}{2}\right) = 0$$

$$\psi = \frac{\mu_0 I_0}{2\pi j} I(\Omega)$$

From the higher-order steepest-descent method, we have:

$$I(\Omega) \sim \frac{1}{4} h''(0) e^{\Omega g(z_0)} \sqrt{\pi} \left(\frac{1}{\Omega^{3/2}} \right)$$

Example (cont.)

Recall: $\underline{E} = -j\omega\underline{A} - \cancel{\nabla\Phi}$

so $E_z = -j\omega A_z = -j\omega\psi$

We then have

$$E_z \sim -j\omega\left(\frac{\mu_0 I_0}{2\pi j}\right)\left(\frac{1}{4}h''(0)\right)e^{-j(k_0\rho)}\sqrt{\pi}\,\frac{1}{(k_0\rho)^{3/2}}$$

Example (cont.)

We have

$$h''(0) = f''(\zeta_0)\left(\frac{2}{\left|g''(\zeta_0)\right|}\right)^{3/2} e^{j3\theta_{SDP}}$$

$$+3f'(\zeta_0)\sqrt{\frac{2}{\left|g''(\zeta_0)\right|}}e^{j\theta_{SDP}}\left(\frac{2}{3}\frac{g'''(\zeta_0)}{\left(g''(\zeta_0)\right)^2}\right)$$

with

$$\zeta_0 = \theta = \pi / 2 \qquad \theta_{SDP} = \frac{\pi}{4}$$

$$f(\zeta) = \frac{\cos\zeta}{\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}} \qquad g(\zeta) = -j\cos(\zeta - \theta)$$

Example (cont.)

We have

$$g(\zeta) = -j\cos(\zeta - \theta)$$

$$\zeta_0 = \theta = \pi / 2$$

so

$$g(\zeta_0) = -j$$

$$g'(\zeta_0) = 0$$

$$g''(\zeta_0) = j$$

$$g'''(\zeta_0) = 0$$

Example (cont.)

We therefore have

$$h''(0) = f''(\zeta_0)\, 2^{3/2} e^{j3\pi/4}$$

with

$$f(\zeta) = \frac{\cos\zeta}{\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}}$$

Example (cont.)

We next calculate the derivates of the *f* function:

$$f(\zeta)=\frac{\cos\zeta}{\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}}$$

$$f'(\zeta)=\frac{-\sin\zeta\left(\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}\right)+\sin\zeta\cos\zeta\left(1+\left(n_1^2-\sin^2\zeta\right)^{-1/2}\cos\zeta\right)}{\left(\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}\right)^2}$$

$$f''(\zeta)=\frac{D(\zeta)N'(\zeta)-N(\zeta)D'(\zeta)}{D^2(\zeta)}$$

$$N(\zeta)=-\sin\zeta\left(\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}\right)+\sin\zeta\cos\zeta\left(1+\left(n_1^2-\sin^2\zeta\right)^{-1/2}\cos\zeta\right)$$

$$D(\zeta)=\left(\cos\zeta+\left(n_1^2-\sin^2\zeta\right)^{1/2}\right)^2$$

Example (cont.)

We then have

$$f''(\zeta) = \frac{D(\zeta)N'(\zeta) - N(\zeta)D'(\zeta)}{D^2(\zeta)}$$

where

$$\begin{aligned}
N'(\zeta) = -\cos\zeta\left(\cos\zeta + \left(n_1^{\;2} - \sin^2\zeta\right)^{1/2}\right) \\
-\sin\zeta\left(-\sin\zeta\right)\left(1 + \left(n_1^{\;2} - \sin^2\zeta\right)^{-1/2}\cos\zeta\right) \\
+\left(\cos\zeta - \sin^2\zeta\right)\left(1 + \left(n_1^{\;2} - \sin^2\zeta\right)^{-1/2}\cos\zeta\right) \\
+\left(\sin\zeta\cos\zeta\right)\left[-\left(n_1^{\;2} - \sin^2\zeta\right)^{-1/2}\sin\zeta + \cos\zeta\left(n_1^{\;2} - \sin^2\zeta\right)^{-3/2}\sin\zeta\cos\zeta\right]
\end{aligned}$$

$$D'(\zeta) = 2\left(\cos\zeta + \left(n_1^2 - \sin^2\zeta\right)^{1/2}\right)\left(-\sin\zeta - \left(n_1^2 - \sin^2\zeta\right)^{-1/2}\sin\zeta\cos\zeta\right)$$

Example (cont.)

At the saddle point we have

$$f'(\zeta_0) = -\frac{1}{\sqrt{n_1^2 - 1}}$$

$$N(\zeta_0) = -\sqrt{n_1^2 - 1}$$

$$N'(\zeta_0) = 0$$

$$D(\zeta_0) = n_1^2 - 1$$

$$D'(\zeta_0) = -2\sqrt{n_1^2 - 1}$$

$$f''(\zeta_0) = \frac{-2}{n_1^2 - 1}$$

Example (cont.)

We then have

$$h''(0) = \left(\frac{-2}{n_1^2 - 1} \right) 2^{3/2} e^{j3\pi/4}$$

The field along the interface is then

$$E_z \sim -j\omega \left(\frac{\mu_0 I_0}{2\pi j} \right) \left(\frac{1}{4} \left(\frac{-2}{n_1^2 - 1} \right) 2^{3/2} e^{j3\pi/4} \right) e^{-j(k_0\rho)} \sqrt{\pi} \frac{1}{(k_0\rho)^{3/2}}$$

Note: Far away from the source, the wave along the interface of the lossy earth travels with a wavenumber of free space.

Example (cont.)

Summary

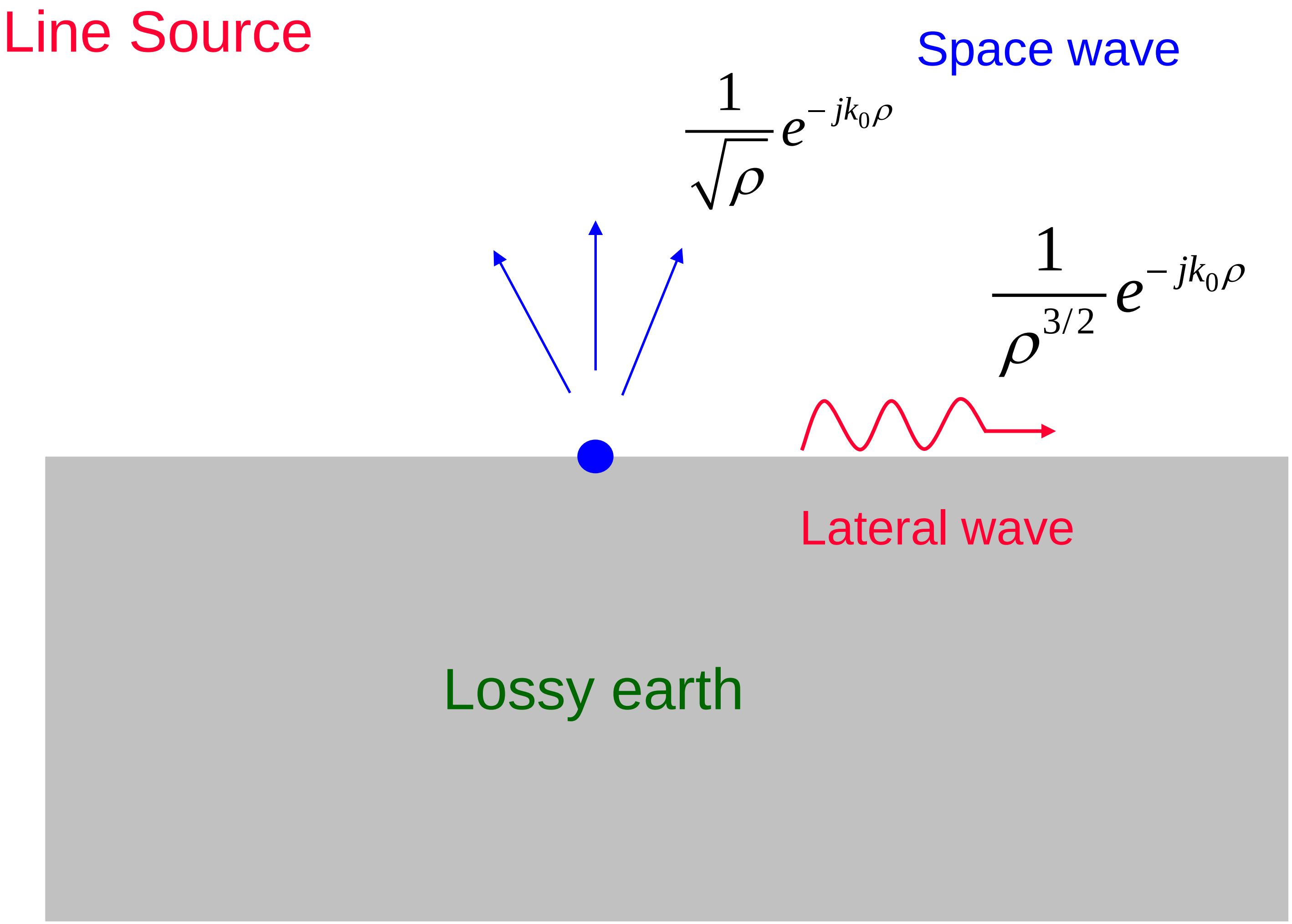
The field in space ($\theta < \pi/2$) is

$$E_z \sim -j\omega\left(\frac{\mu_0 I_0}{2\pi j}\right)\left(\frac{\cos\theta}{\cos\theta+\left(n_1^2-\sin^2\theta\right)^{1/2}}\right)e^{-j(k_0\rho)}\sqrt{\frac{2\pi}{k_0\rho}}\,e^{j\pi/4}$$

The field along the interface ($\theta = \pi/2$) is

$$E_z \sim -j\omega\left(\frac{\mu_0 I_0}{2\pi j}\right)\left(\frac{1}{4}\left(\frac{-2}{n_1^2-1}\right)2^{3/2}e^{j3\pi/4}\right)e^{-j(k_0\rho)}\sqrt{\pi}\,\frac{1}{(k_0\rho)^{3/2}}$$

Example (cont.)



Line Source

Space wave

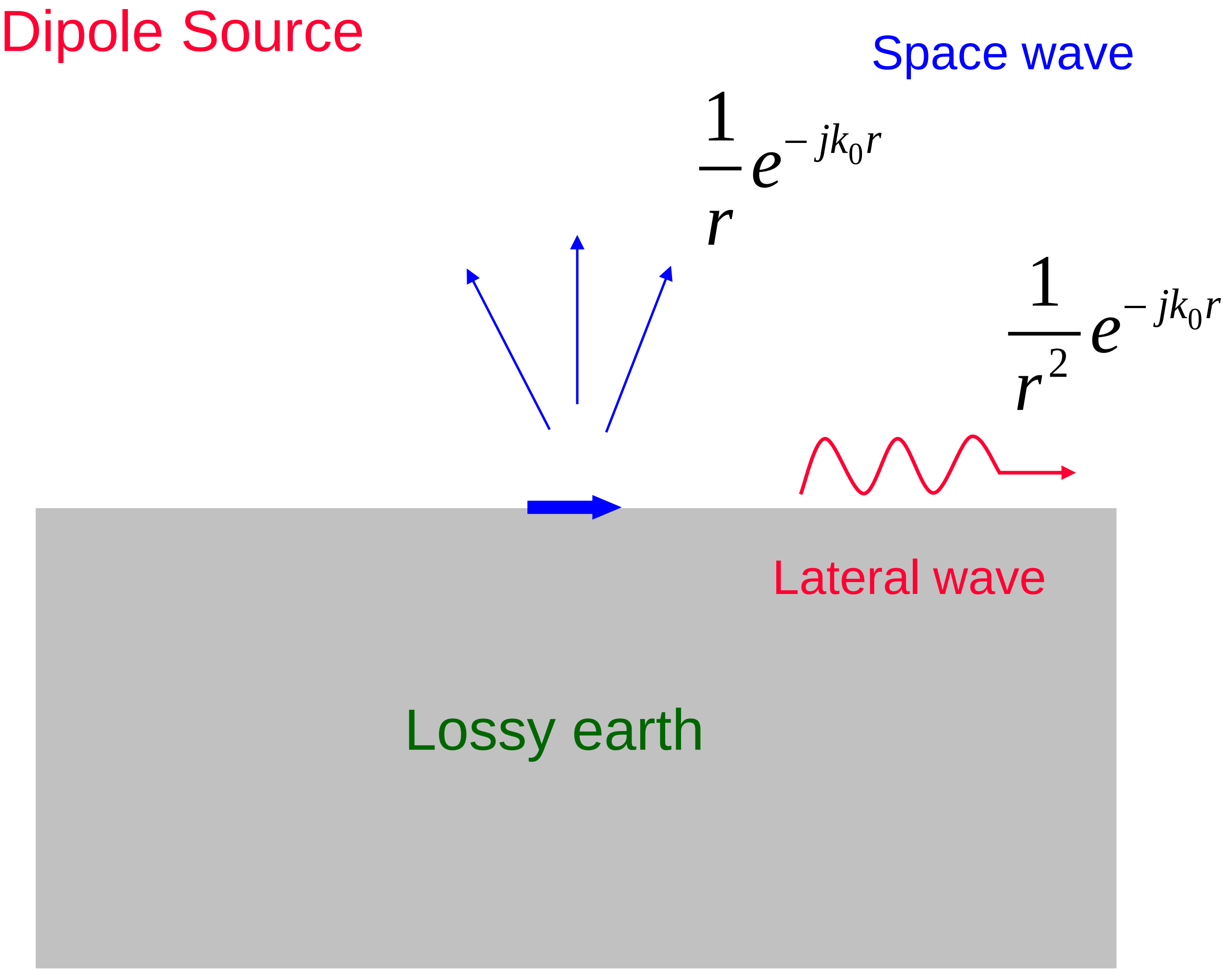
$$\frac{1}{\sqrt{\rho}}e^{-jk_0\rho}$$

$$\frac{1}{\rho^{3/2}}e^{-jk_0\rho}$$

Lateral wave

Lossy earth

Extension to Dipole



Dipole Source

Space wave

$$\frac{1}{r}e^{-jk_0r}$$

$$\frac{1}{r^2}e^{-jk_0r}$$

Lateral wave

Lossy earth

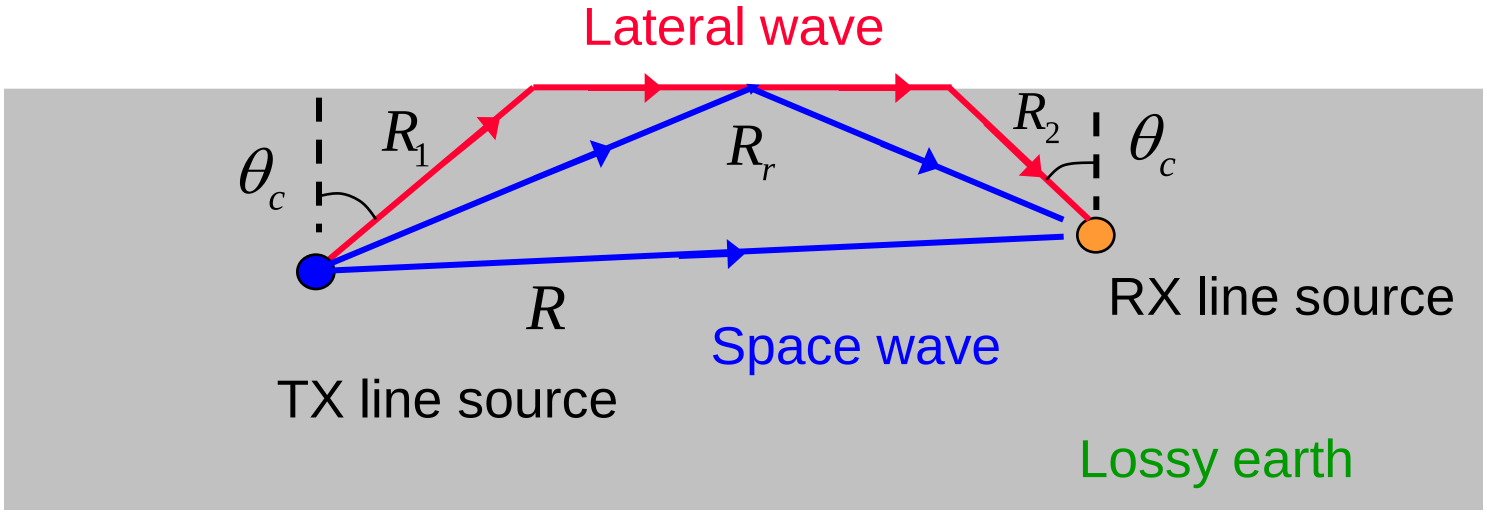
Important Geophysical Problem

The field is asymptotically evaluated for $R \to \infty$

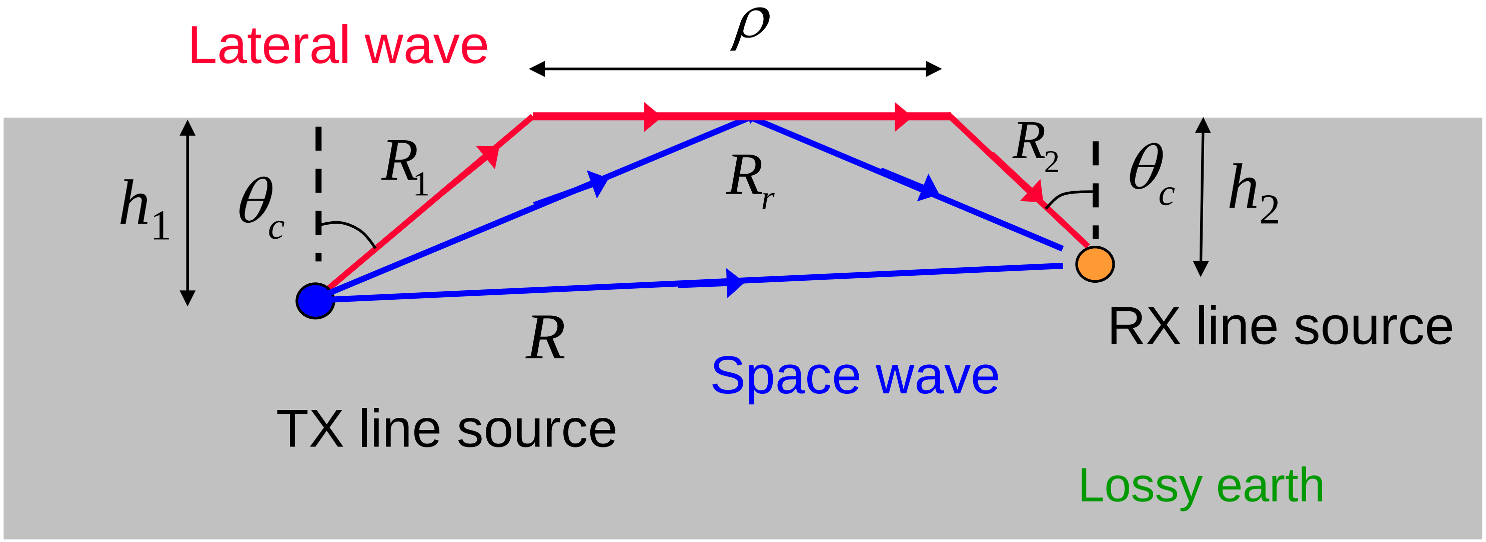
Two types of wave fields are important for large distances:

- Space wave
- Lateral wave

Note: More will be said about these waves in the next chapter on “Radiation Physics of Layered Media.”



Important Geophysical Problem (cont.)



Space wave:

$$\frac{1}{\sqrt{R}}e^{-jk_1R}+\Gamma_{TE}\frac{1}{\sqrt{R_r}}e^{-jk_1R_r}$$

Lateral wave:

$$\frac{1}{\rho^{3/2}}e^{-jk_0\rho}\left[\frac{e^{-jk_1R_1}}{\sqrt{R_1}}\frac{e^{-jk_1R_2}}{\sqrt{R_2}}\right]$$

This will be the dominant field for a lossy earth (k_1 is complex).

Note: Amplitude terms have been suppressed here.