ECE 6341

Spring 2016

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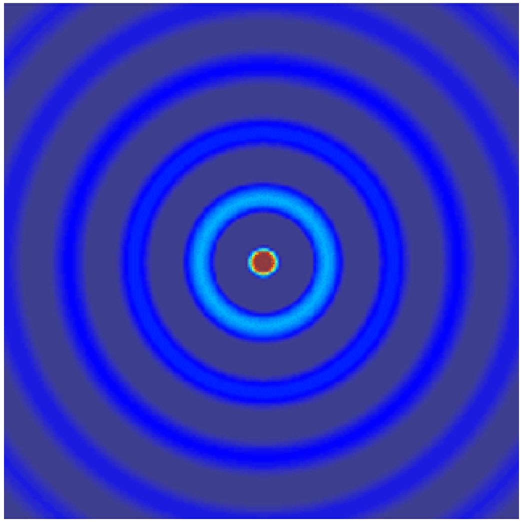
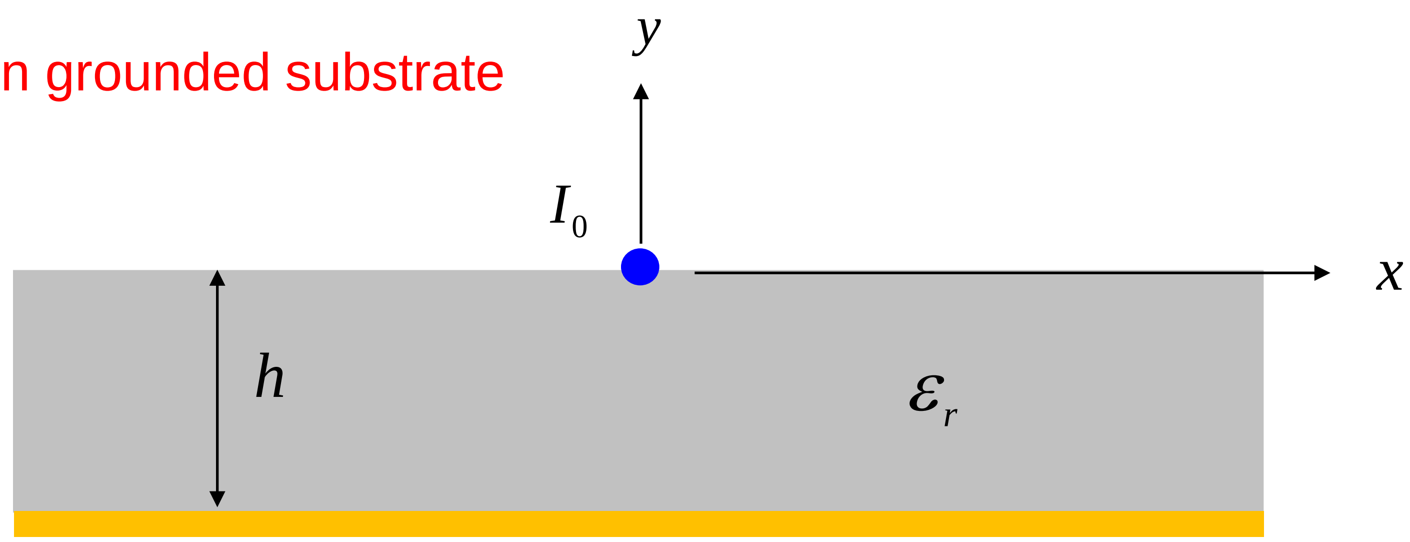
ECE Dept.



Notes 36

Radiation Physics in Layered Media

Line source on grounded substrate



Note: TM$_z$ and also TE$_y$ (since $\frac{\partial}{\partial z} = 0$)

$$\underline{E} = \underline{\hat{z}}\, E_z(x, y)$$

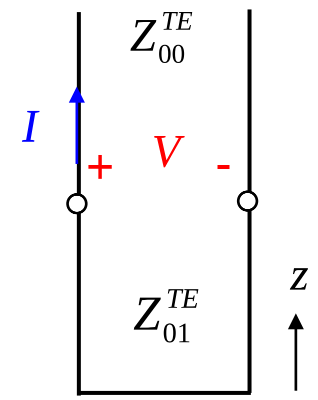
For $y > 0$:

$$\psi = A_z$$

$$\psi = \frac{\mu_0 I_0}{4\pi j} \int_{-\infty}^{+\infty} \frac{1}{k_{y0}} \left[1 + \Gamma^{TE}(k_x)\right] e^{-jk_{y0} y} e^{-jk_x x} \, dk_x$$

Reflection Coefficient

$$\Gamma^{TE}(k_x) = \frac{Z_{in}^{TE}(k_x) - Z_0^{TE}(k_x)}{Z_{in}^{TE}(k_x) + Z_0^{TE}(k_x)}$$



where

$$Z_{in}^{TE}(k_x) = jZ_1^{TE}\tan\left(k_{y1}h\right)$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{y0}} \qquad k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$

$$Z_1^{TE} = \frac{\omega\mu_0}{k_{y1}} \qquad k_{y1} = \left(k_1^2 - k_x^2\right)^{1/2}$$

Poles

$$\Gamma^{TE}\left(k_x\right)=\frac{Z_{in}^{TE}\left(k_x\right)-Z_0^{TE}\left(k_x\right)}{Z_{in}^{TE}\left(k_x\right)+Z_0^{TE}\left(k_x\right)}$$

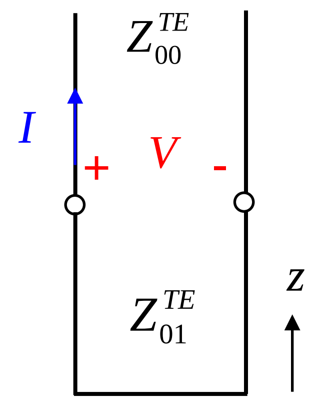
Poles: $k_x = k_{xp}$

$$Z_{in}^{TE}\left(k_{xp}\right)=-Z_0^{TE}\left(k_{xp}\right)$$

This is the same equation as the TRE for finding the wavenumber of a surface wave:

$$Z_{in}^{TE}\left(k_x^{SW}\right)=-Z_0^{TE}\left(k_x^{SW}\right)$$

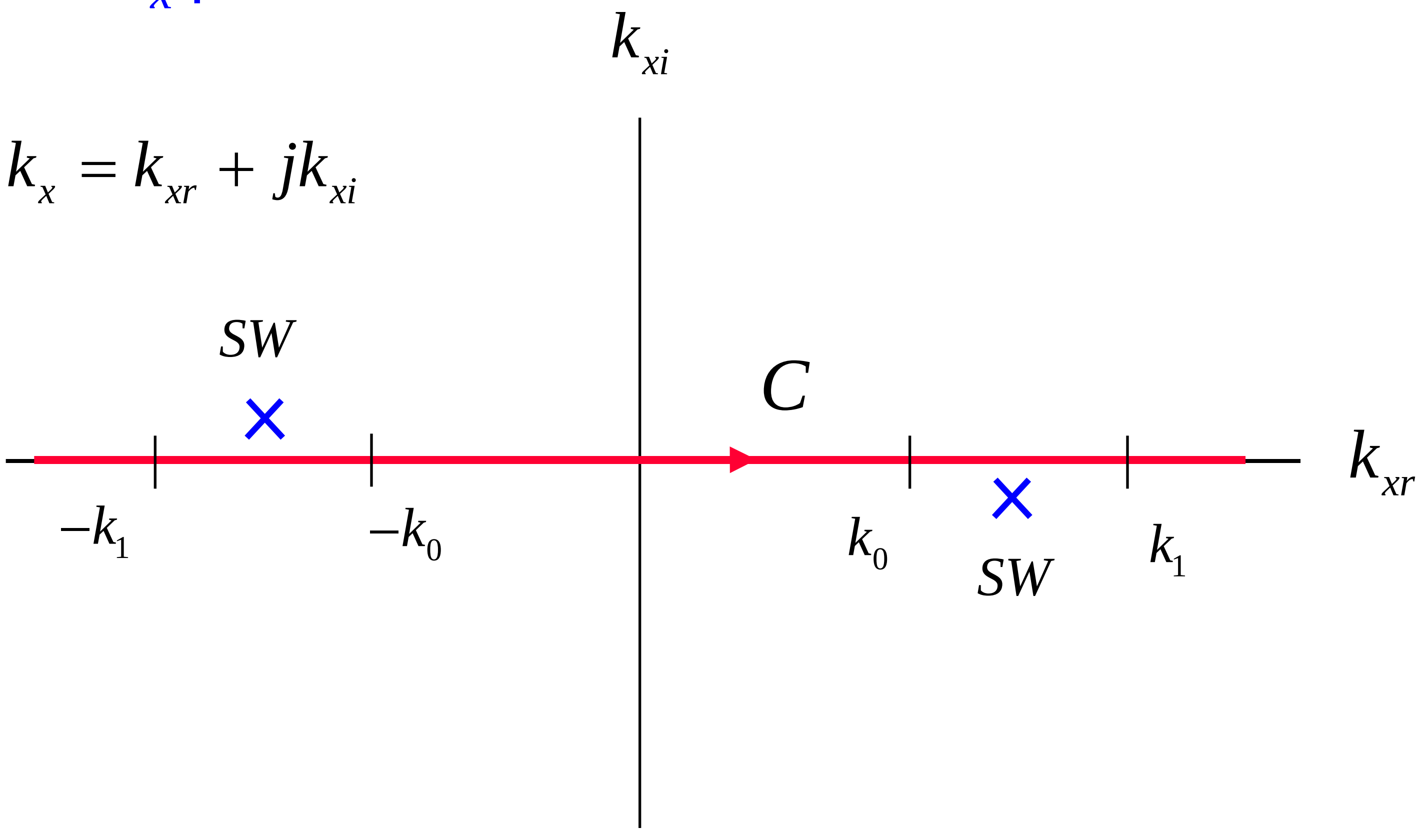
$\Rightarrow$ k_{xp} = roots of TRE = k_x^{SW}



Poles (cont.)

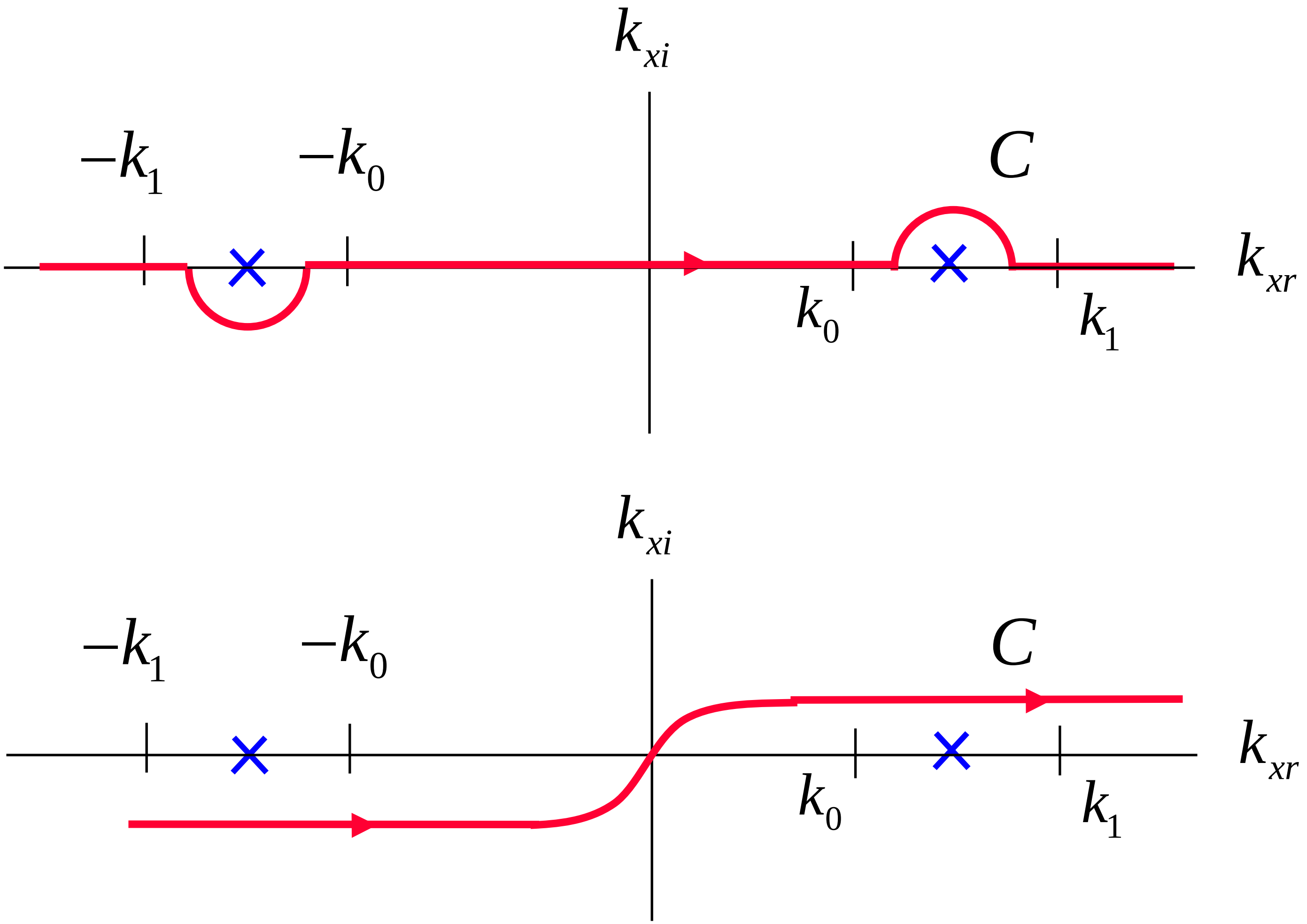
Complex k_x plane

$$k_x = k_{xr} + jk_{xi}$$



If a slight loss is added, the SW poles are shifted off the real axis as shown.

Poles (cont.)



For the lossless case, two possible paths are shown here.

Review of Branch Cuts and Branch Points

In the next few slides we review the basic concepts of branch points and branch cuts.

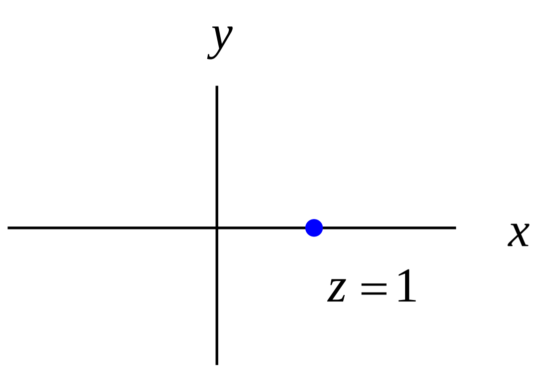
Branch Cuts and Points (cont.)

Consider $f(z) = z^{1/2}$ $\quad z = x + jy = r\,e^{j\theta}$

$$z^{1/2} = \left(r\,e^{j\theta}\right)^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

Choose $z = 1$

$\Rightarrow r = 1$



$\theta = 0: \quad z^{1/2} = 1$

$\theta = 2\pi: \quad z^{1/2} = -1$

$\theta = 4\pi: \quad z^{1/2} = 1$

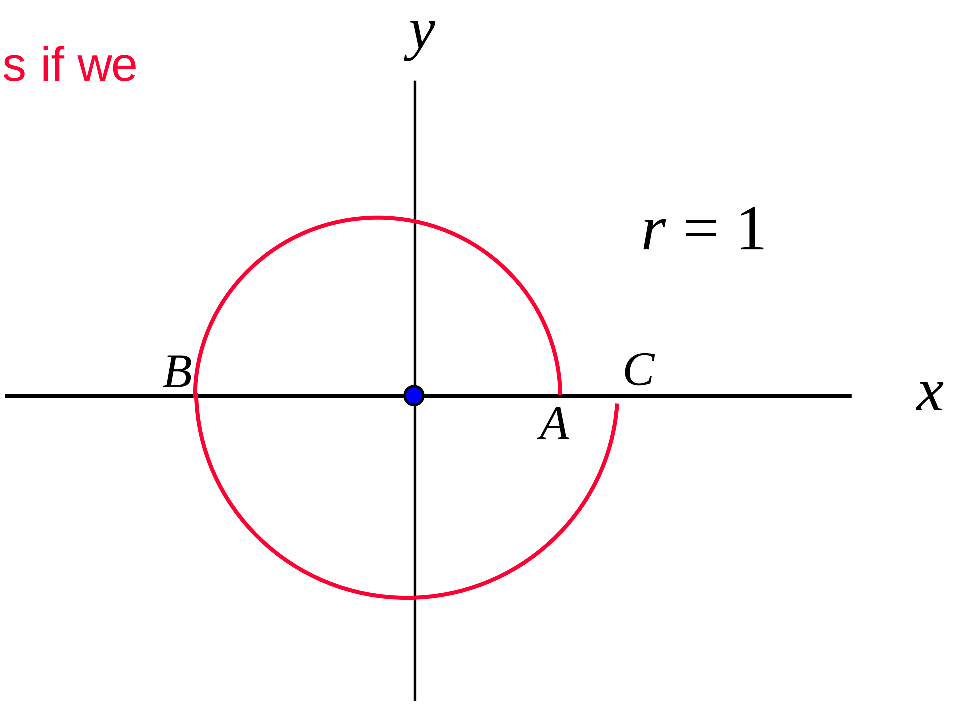
There are two possible values.

Branch Cuts and Points (cont.)

The concept is illustrated for $f(z) = z^{1/2}$ $\qquad z = r\,e^{j\theta}$

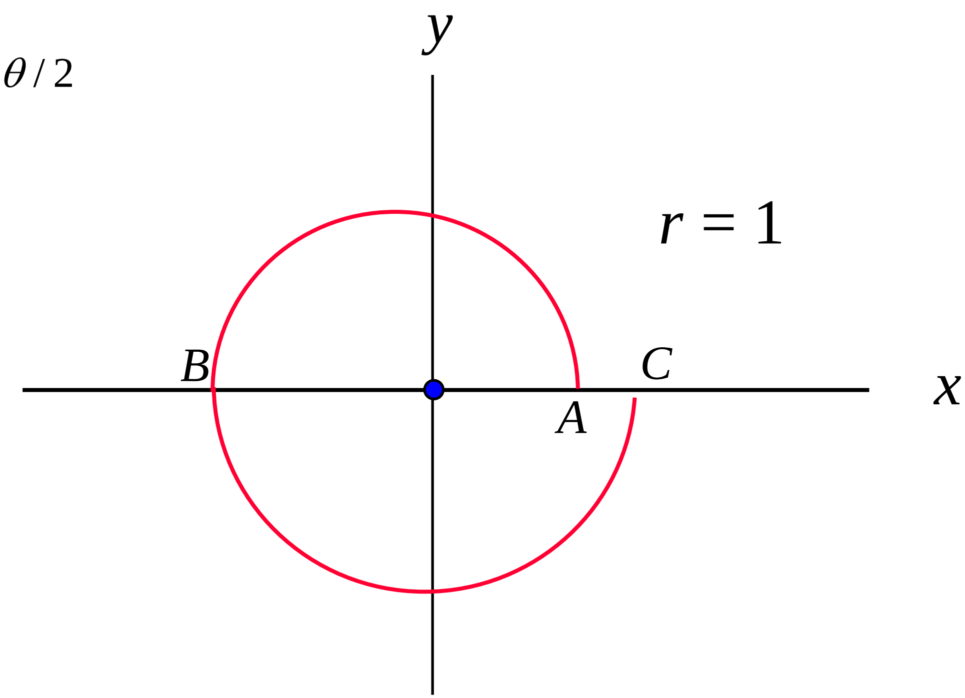
$$z^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

Consider what happens if we encircle the origin:



Branch Cuts and Points (cont.)

$$z^{1/2} = \sqrt{r}\, e^{j\theta/2}$$



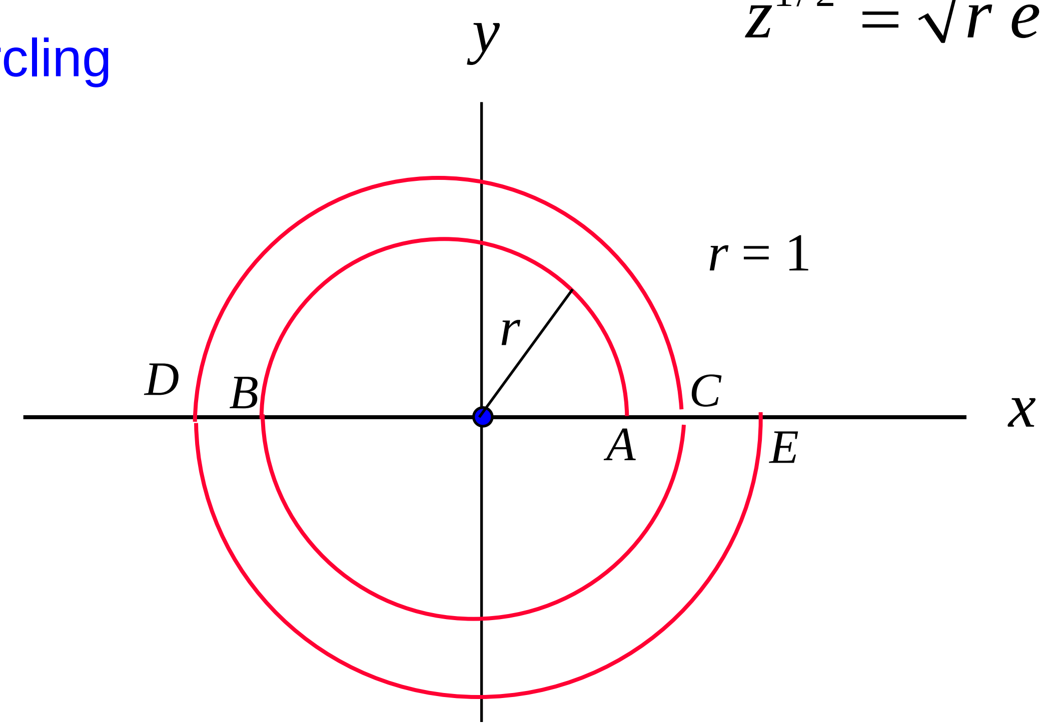
point	θ	$z^{1/2}$
A	0	1
B	π	$+j$
C	2π	-1

We don't get back the same result!

Branch Cuts and Points (cont.)

$$z^{1/2} = \sqrt{r}\, e^{j\theta/2}$$

Now consider encircling the origin twice:



point	θ	$z^{1/2}$
A	0	1
B	π	$+j$
C	2π	-1
D	3π	$-j$
E	4π	+1

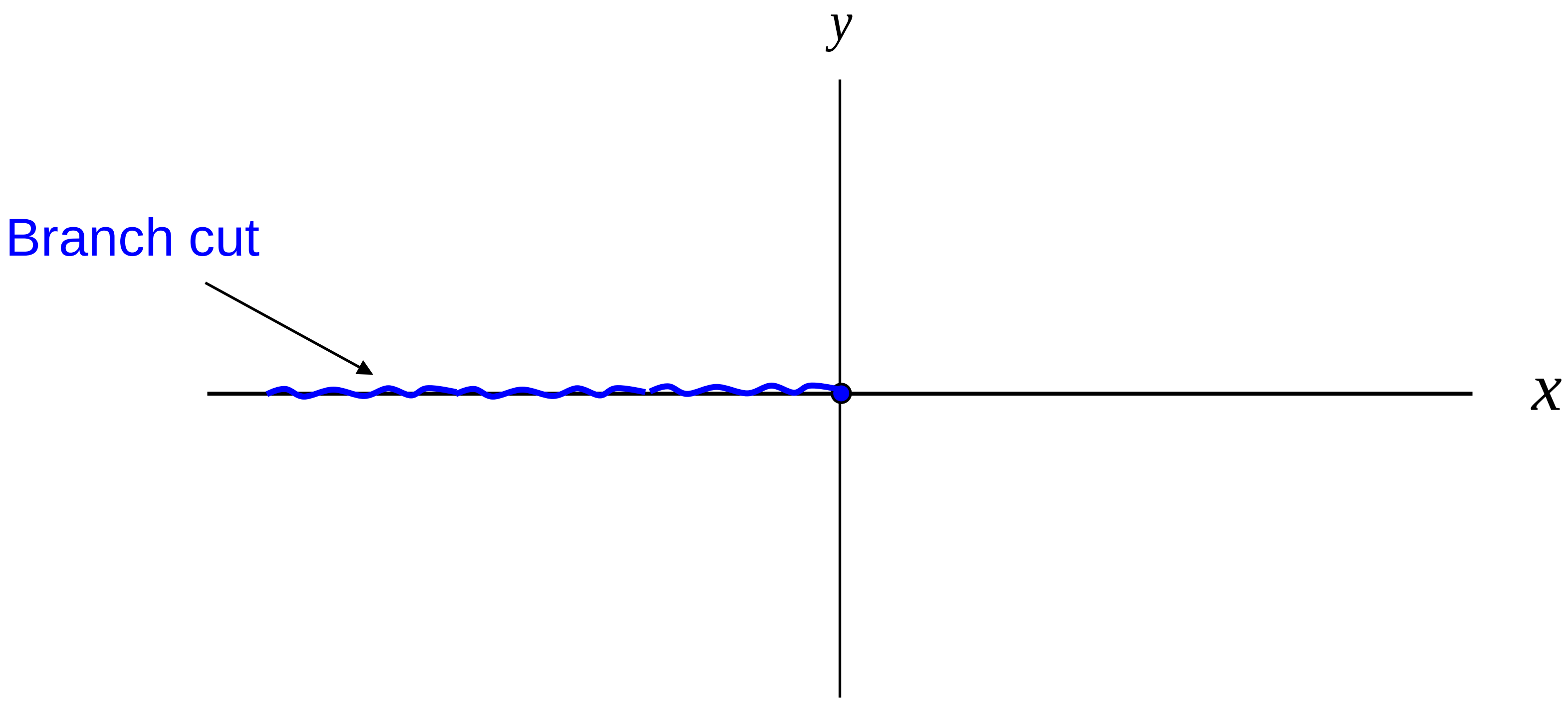
We now get back the same result!

Hence the square-root function is a double-valued function.

Branch Cuts and Points (cont.)

The origin is called a branch point: we are not allowed to encircle it if we wish to make the square-root function *single-valued*.

In order to make the square-root function single-valued, we must put a “barrier” or “branch cut”.



Here the branch cut was chosen to lie on the negative real axis (an arbitrary choice).

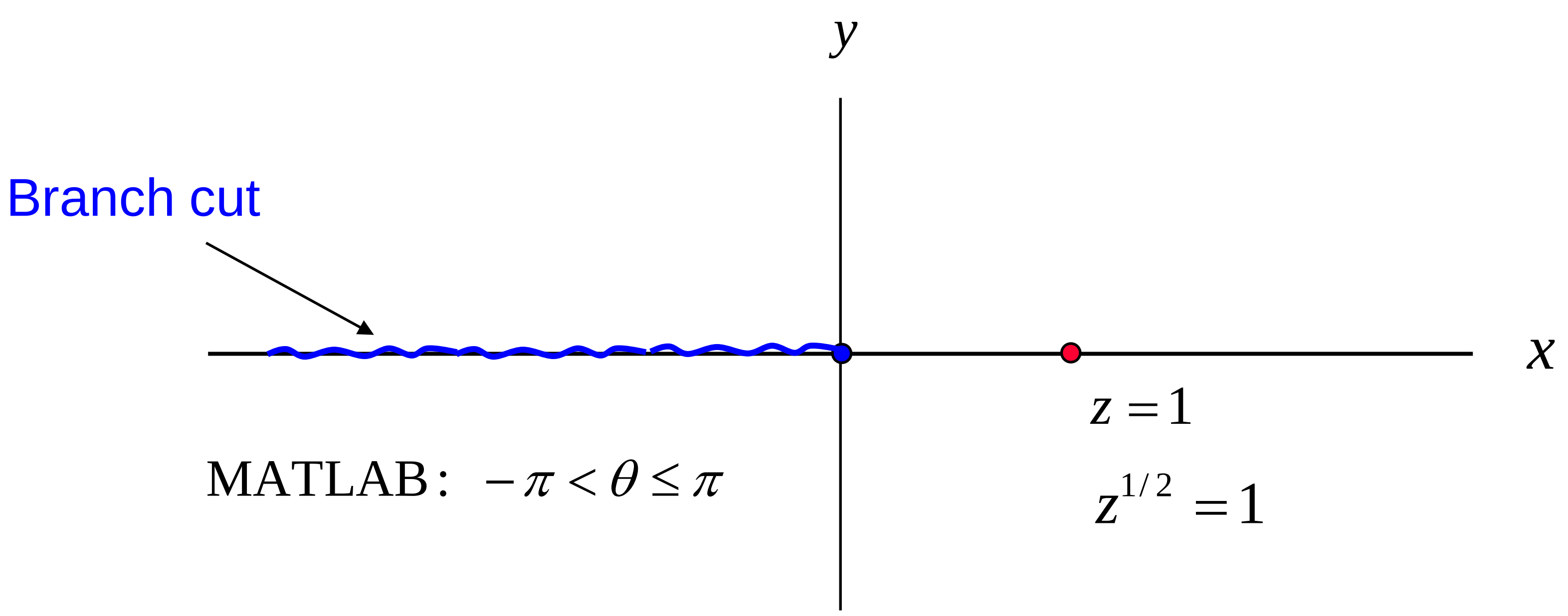
Branch Cuts and Points (cont.)

We must now choose what "branch" of the function we want.

$$z = r\, e^{j\theta} \qquad z^{1/2} = \sqrt{r}\, e^{j\theta/2}$$

$$-\pi < \theta < \pi$$

This is the "principle" branch, denoted by $\sqrt{z}$.

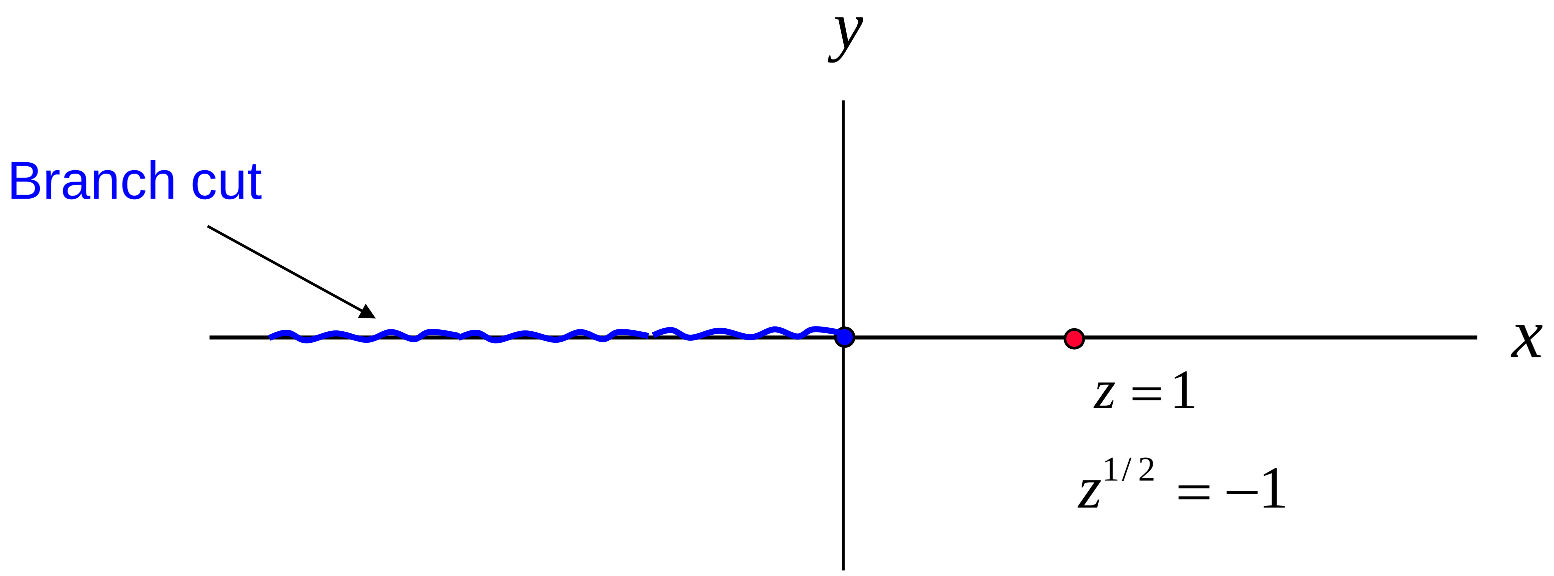


Branch Cuts and Points (cont.)

Here is the other choice of branch.

$$z = r\,e^{j\theta} \qquad z^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

$$\pi < \theta < 3\pi$$

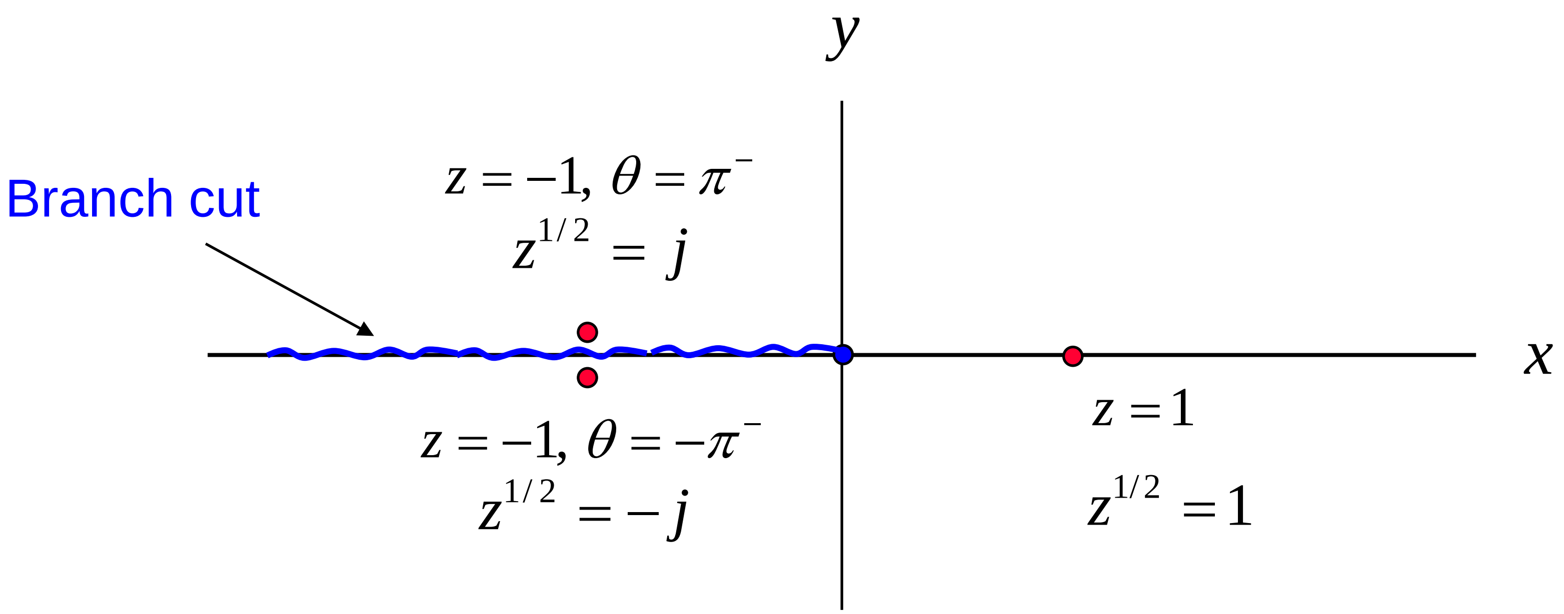


Branch Cuts and Points (cont.)

Note that the function is discontinuous across the branch cut.

$$z = r\,e^{j\theta} \qquad z^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

$$-\pi < \theta < \pi$$



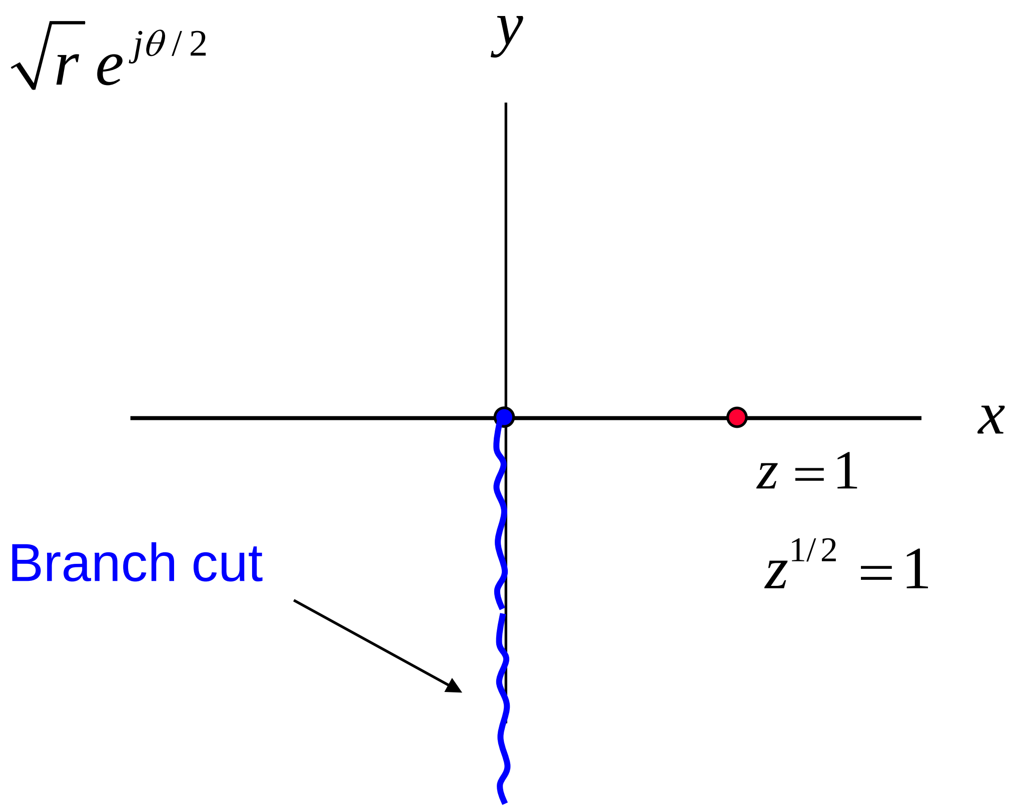
Branch Cuts and Points (cont.)

The <u>shape</u> of the branch cut is arbitrary.

$$z = r\,e^{j\theta}$$

$$z^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

$$-\pi/2 < \theta < 3\pi/2$$



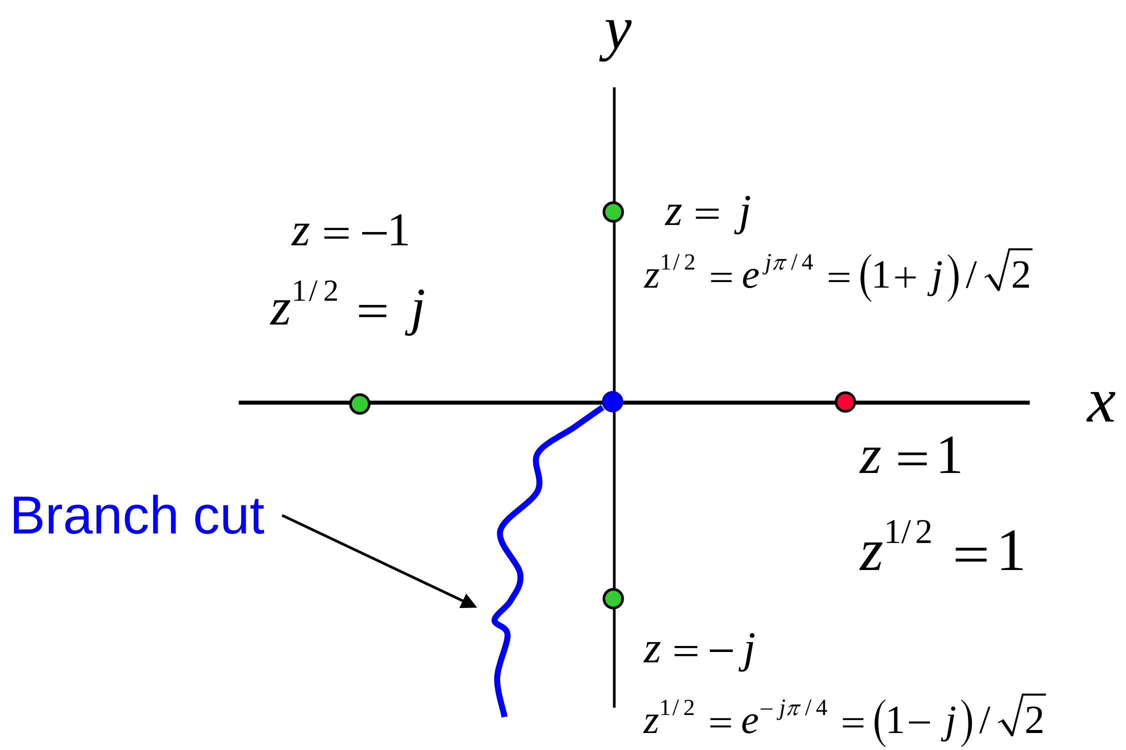
Branch Cuts and Points (cont.)

The branch cut does not have to be a straight line.

$$z = r\,e^{j\theta}$$

$$z^{1/2} = \sqrt{r}\,e^{j\theta/2}$$

In this case the branch is determined by requiring that the square-root function (and hence the angle θ) change continuously as we start from a specified value (e.g., $z = 1$).



Branch Cuts and Points (cont.)

Consider this function:

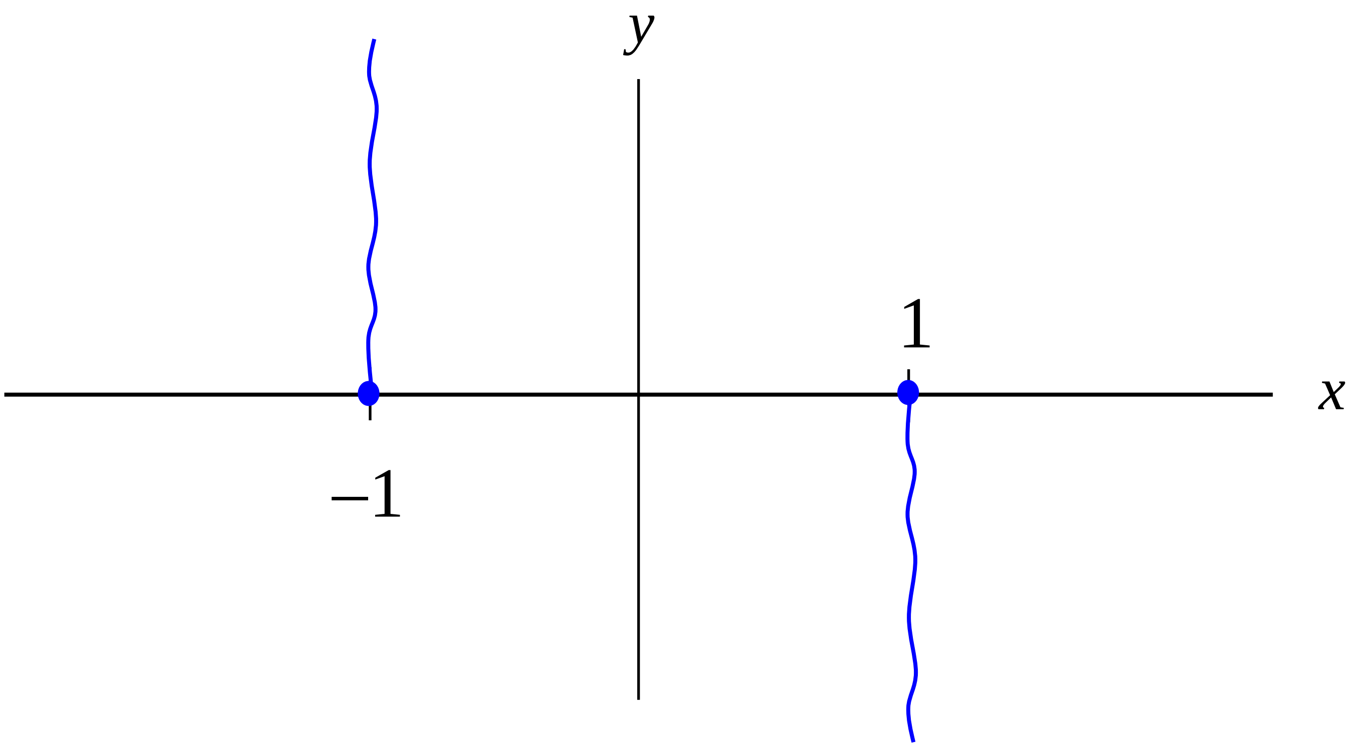
$$f(z) = \left(z^2 - 1\right)^{1/2}$$

(similar to our wavenumber function)

What do the branch points and branch cuts look like for this function?

Branch Cuts and Points (cont.)

$$f(z) = \left(z^2 - 1\right)^{1/2} = \left(z-1\right)^{1/2}\left(z+1\right)^{1/2} = \left(z-1\right)^{1/2}\left(z-(-1)\right)^{1/2}$$

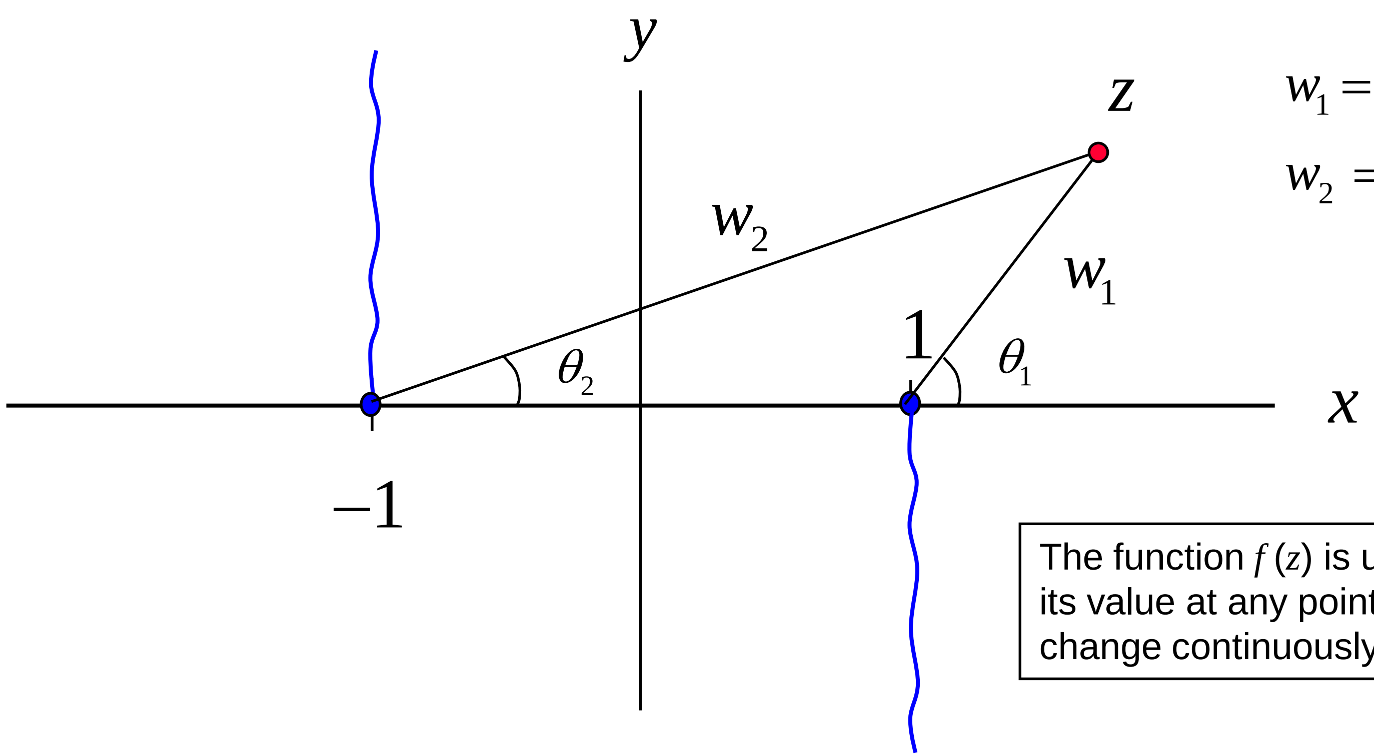


There are <u>two</u> branch cuts: we are not allowed to encircle either branch point.

Branch Cuts and Points (cont.)

Geometric interpretation

$$f(z) = (z-1)^{1/2}\,(z-(-1))^{1/2} = w_1^{1/2}\,w_2^{1/2}$$



$$w_1 = z - 1 = r_1\,e^{j\theta_1}$$

$$w_2 = z - (-1) = r_2\,e^{j\theta_2}$$

The function $f(z)$ is unique once we specify its value at any point. (The function must change continuously away from this point.)

$$f(z) = \left(\sqrt{r_1}\,e^{j\theta_1/2}\right)\left(\sqrt{r_2}\,e^{j\theta_2/2}\right)$$

Riemann Surface

The Riemann surface is a set of multiple complex planes connected together.

The function $z^{1/2}$ has a surface with two sheets.

The function $z^{1/2}$ is continuous everywhere on this surface (there are no branch cuts). It also assumes all possible values on the surface.

Georg Friedrich Bernhard Riemann (September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis and differential geometry, some of them enabling the later development of general relativity.

Riemann Surface

The concept of the Riemann surface is illustrated for

$$f(z) = z^{1/2} \qquad z = r\,e^{j\theta}$$

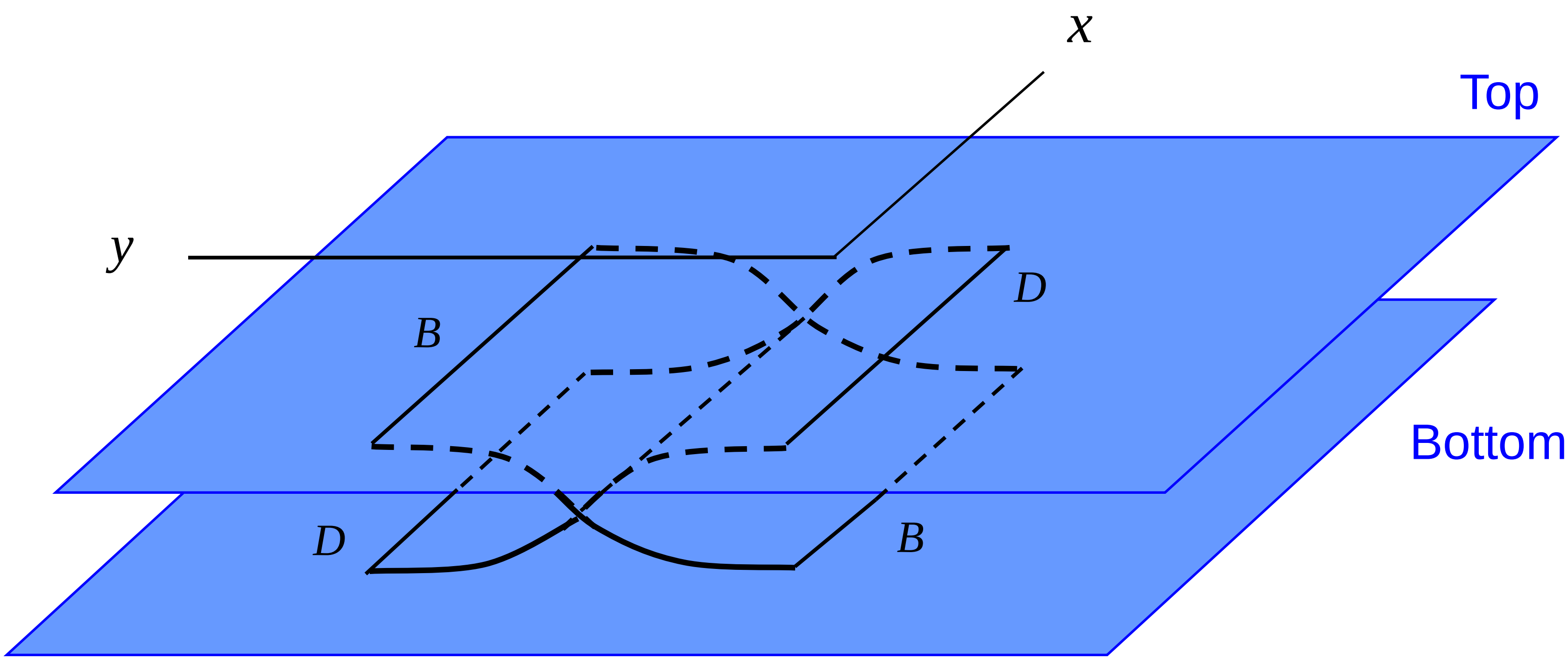
Consider this choice:

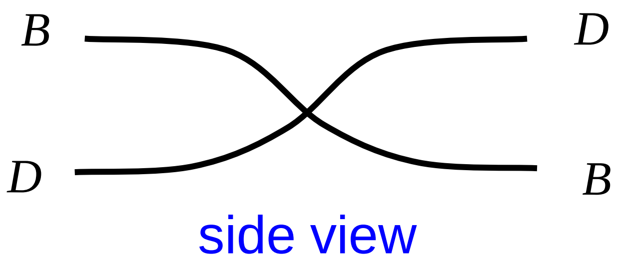
Top sheet: $-\pi < \theta < \pi \quad (\sqrt{1} = 1)$

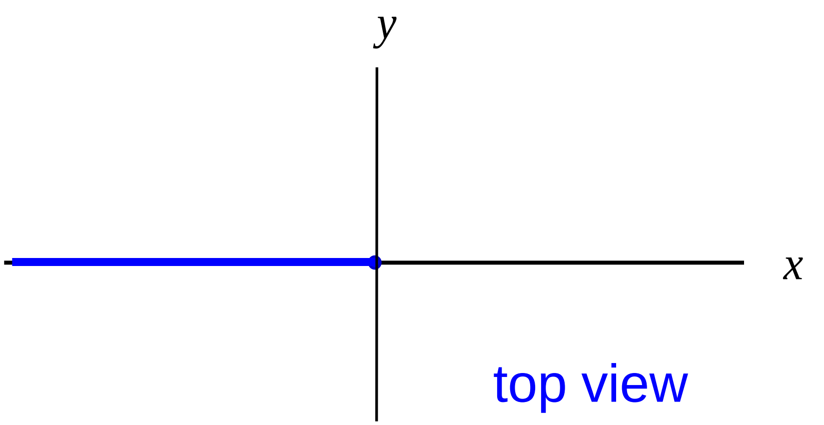
Bottom sheet: $\pi < \theta < 3\pi \quad (\sqrt{1} = -1)$

For a single complex plane, this would correspond to a branch cut on the negative real axis.

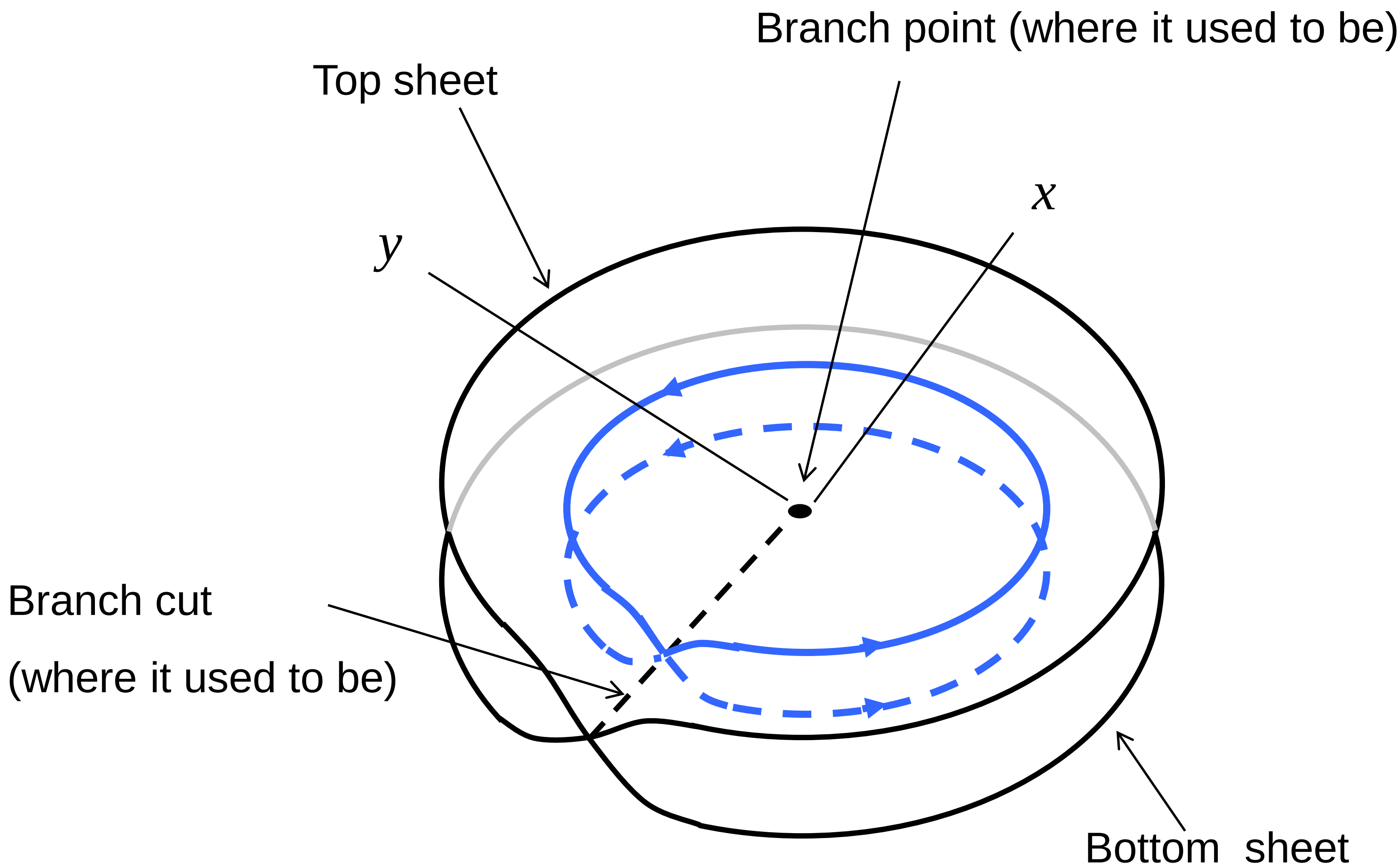
Riemann Surface (cont.)



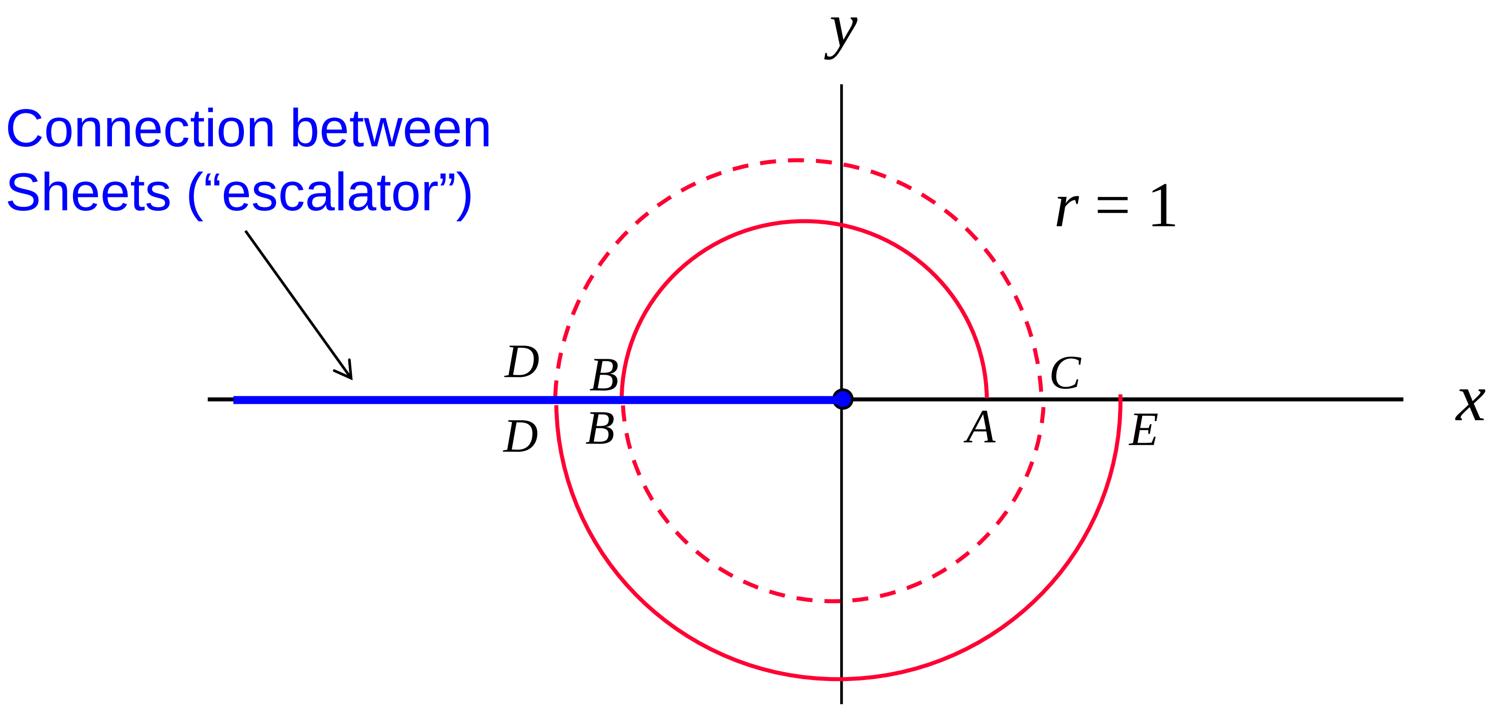




Riemann Surface (cont.)



Riemann Surface (cont.)



point	θ	$z^{1/2}$
A	0	1
B	π	$+j$
C	2π	-1
D	3π	$-j$
E	4π	+1

Branch Cuts in Radiation Problem

Now we return to the problem (line source over grounded slab):

$$\psi = A_z$$

$$\psi = \frac{\mu_0 I_0}{4\pi j} \int_{-\infty}^{+\infty} \frac{1}{k_{y0}} \left[1 + \Gamma^{TE}\left(k_x\right) \right] e^{-jk_{y0}y} e^{-jk_x x} dk_x$$

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$

Note: There are no branch points from k_{y1}:

$$k_{y1} = \left(k_1^{\,2} - k_x^2\right)^{1/2} \qquad Z_{in}^{TE}\left(k_x\right) = jZ_1^{TE}\tan\left(k_{y1}h\right) \qquad Z_1^{TE} = \frac{\omega\mu_0}{k_{y1}}$$

(The integrand is an even function of k_{y1}.)

Branch Cuts

$$k_{y0} = \left(k_0{}^2 - k_x^2\right)^{1/2} = \left(k_0 + k_x\right)^{1/2}\left(k_0 - k_x\right)^{1/2}$$

$$= -j\left(k_x - k_0\right)^{1/2}\left(k_x + k_0\right)^{1/2}$$

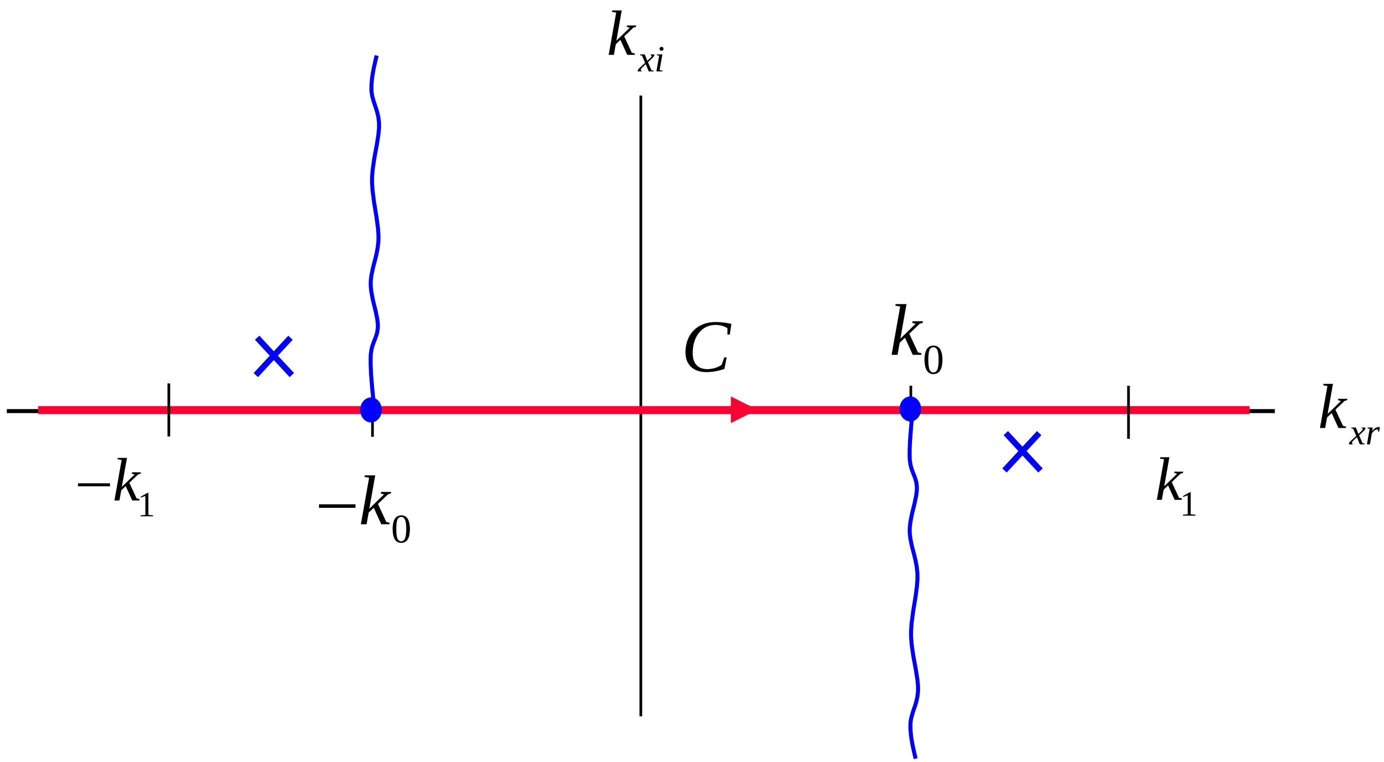
Note: It is arbitrary that we have factored out a $-j$ instead of a $+j$, since we have not yet determined the meaning of the square roots yet.

Branch points appear at $k_x = \pm k_0$

No branch cuts appear at $k_x = \pm k_1$ (The integrand is an even function of k_{y1}.)

Branch Cuts (cont.)

$$k_{y0} = -j\left(k_x - k_0\right)^{1/2}\left(k_x + k_0\right)^{1/2}$$



Branch cuts are lines we are not allowed to cross.

Branch Cuts (cont.)

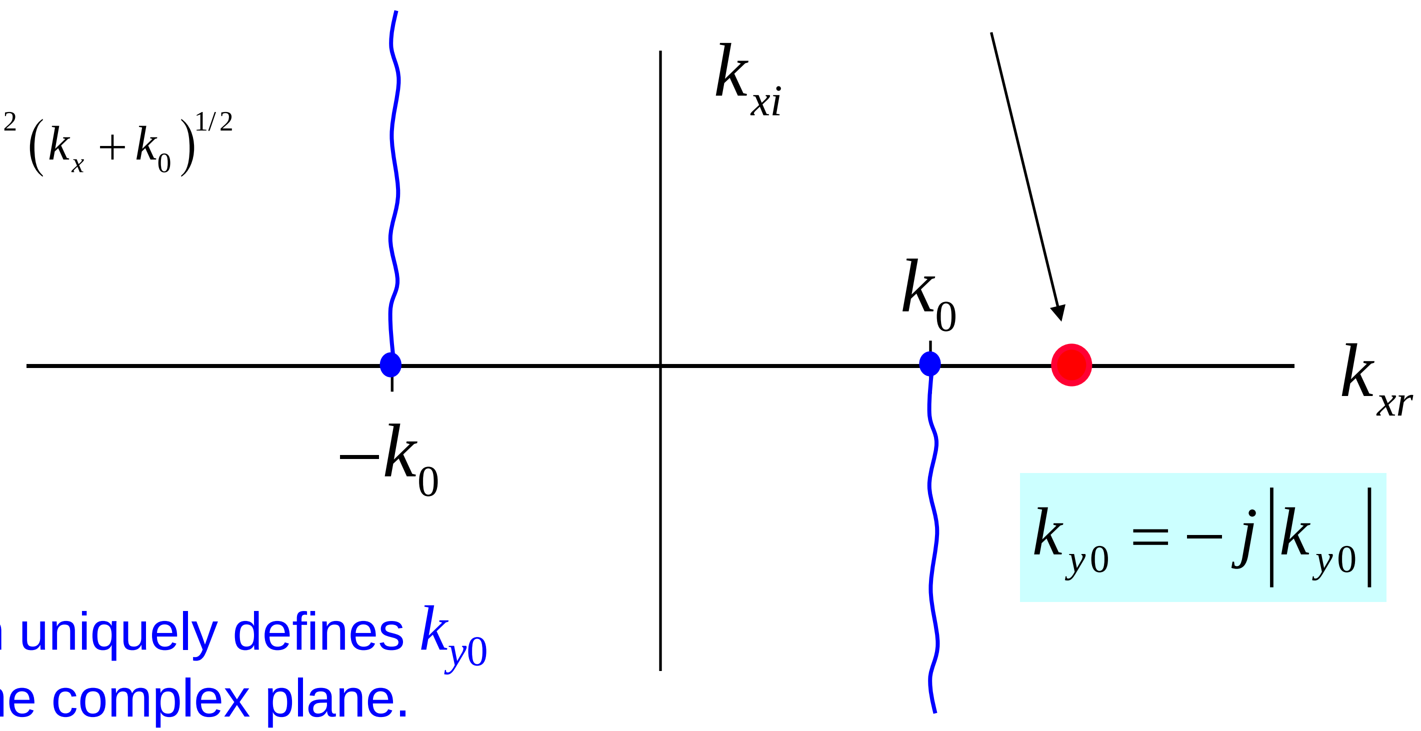
For $\begin{cases} k_x = \text{real} > k_0, \\ k_{y0} = -j\left|k_{y0}\right| \end{cases}$

Choose $\begin{cases} \arg\left(k_x - k_0\right) = 0 \\ \arg\left(k_x + k_0\right) = 0 \end{cases}$

at this point

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$

$$k_{y0} = -j\left(k_x - k_0\right)^{1/2}\left(k_x + k_0\right)^{1/2}$$



$$k_{y0} = -j\left|k_{y0}\right|$$

This choice then uniquely defines k_{y0} everywhere in the complex plane.

Branch Cuts (cont.)

For $\begin{cases} k_x = \text{real}, \\ 0 < k_x < k_0 \end{cases}$ we have $\arg(k_x - k_0) = \pi$, $\arg(k_x + k_0) = 0$

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$

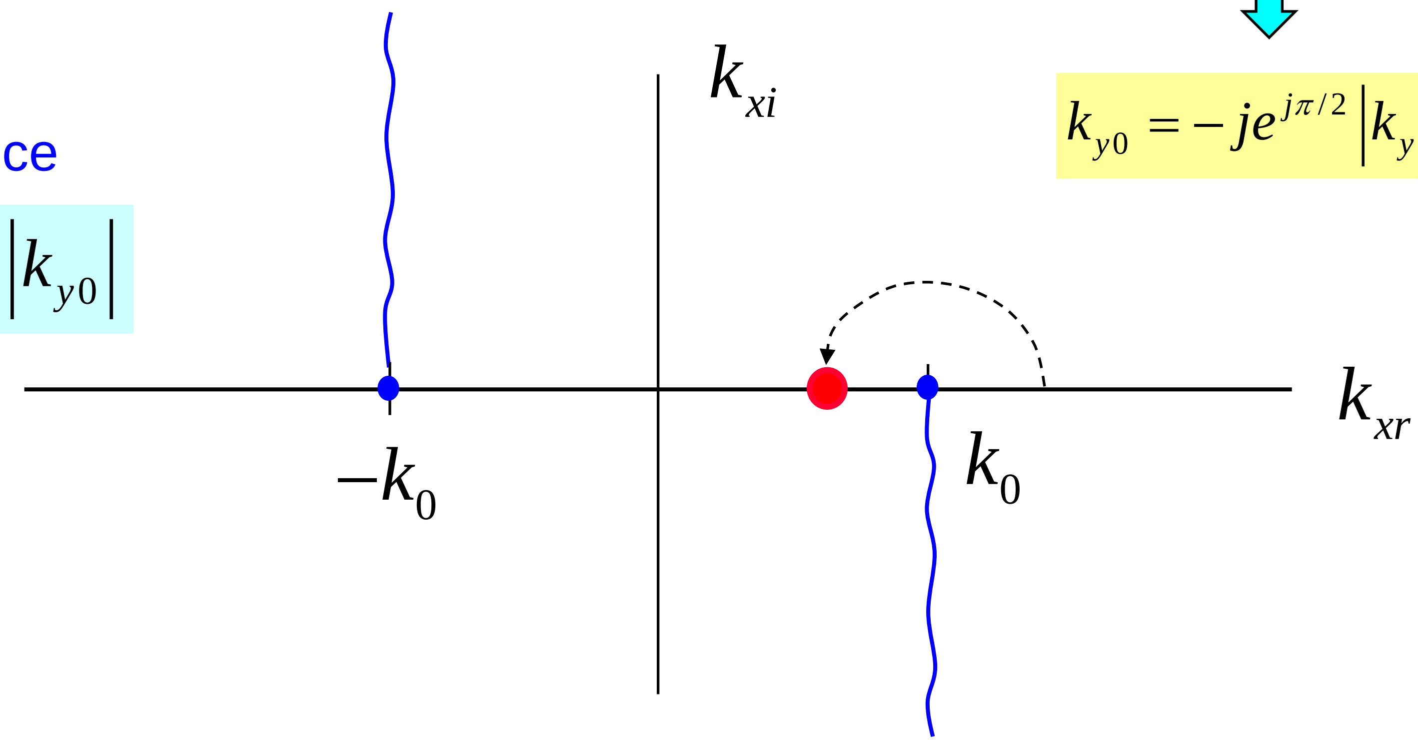
$$k_{y0} = -j\left(k_x - k_0\right)^{1/2}\left(k_x + k_0\right)^{1/2}$$

$$k_{y0} = -j\left[\sqrt{\left|k_x - k_0\right|}\, e^{j\pi/2}\right]\left[\sqrt{\left|k_x + k_0\right|}\, e^{j0/2}\right]$$

$$k_{y0} = -je^{j\pi/2}\left|k_{y0}\right|$$

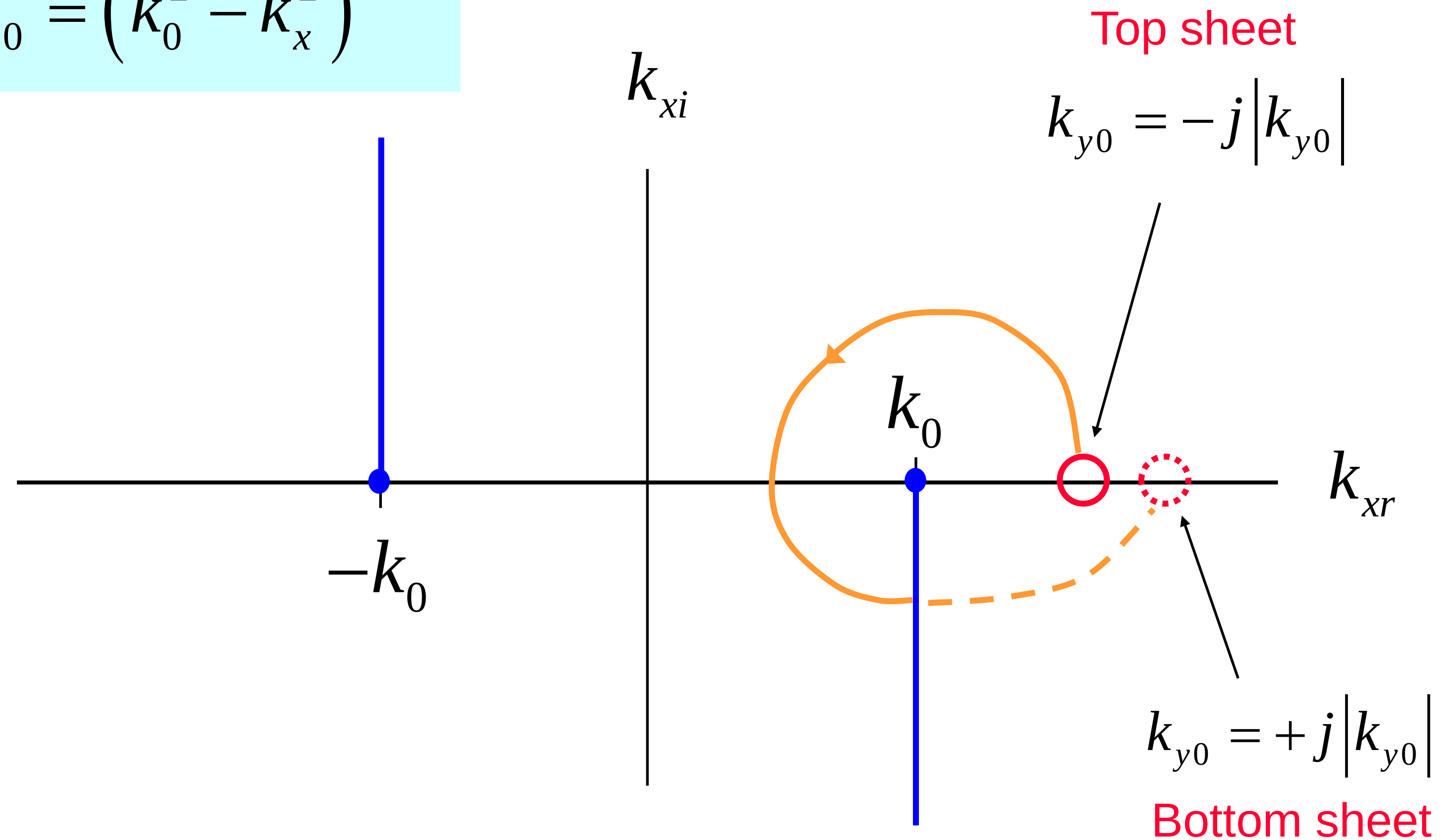
Hence

$$k_{y0} = \left|k_{y0}\right|$$



Riemann Surface

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$



There are two sheets, joined at the blue lines.

The path of integration is on the top sheet.

Proper / Improper Regions

Let

$$k_x = k_{xr} + jk_{xi}$$

$$k_0 = k_0' - jk_0''$$

The goal is to figure out which regions of the complex plane are "proper" and "improper."

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2}$$

"Proper" region: $\operatorname{Im} k_{y0} < 0$

"Improper" region: $\operatorname{Im} k_{y0} > 0$

Boundary: $\operatorname{Im} k_{y0} = 0$

$\Rightarrow$ $k_{y0} = \text{real} \;\Rightarrow\; k_{y0}^2 = k_0^2 - k_x^2 = \text{real} > 0$

Proper / Improper Regions (cont.)

Hence $$\left(k_0' - jk_0''\right)^2 - \left(k_{xr} + jk_{xi}\right)^2 = \text{real} > 0$$

$$\left(k_0'^2 - k_0''^2 - k_{xr}^2 + k_{xi}^2\right) + j\left(-2k_0'k_0'' - 2k_{xr}k_{xi}\right) = \text{real} > 0$$

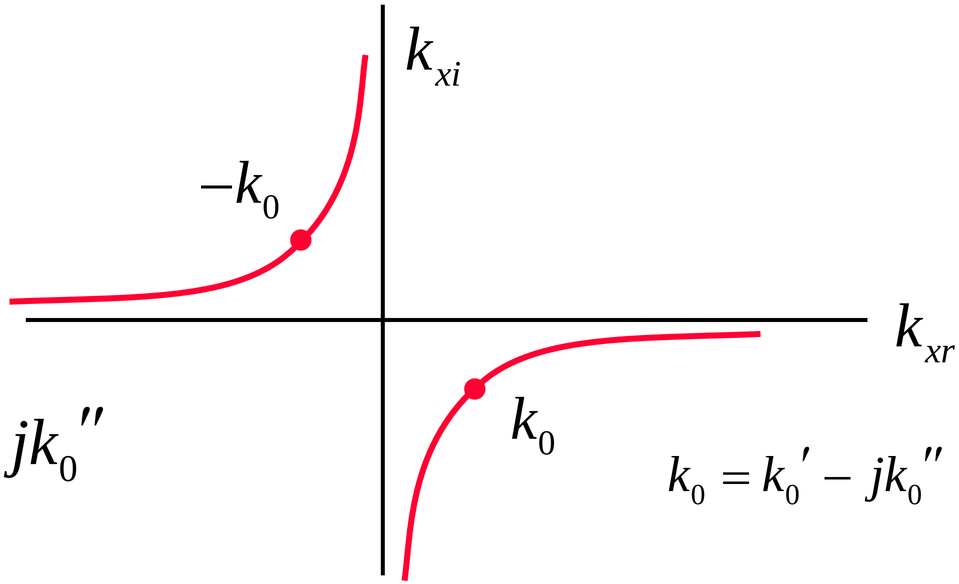
Therefore $$k_{xr}k_{xi} = -k_0'k_0''$$ (hyperbolas)

One point on curve:

$$k_{xr} = k_0'$$

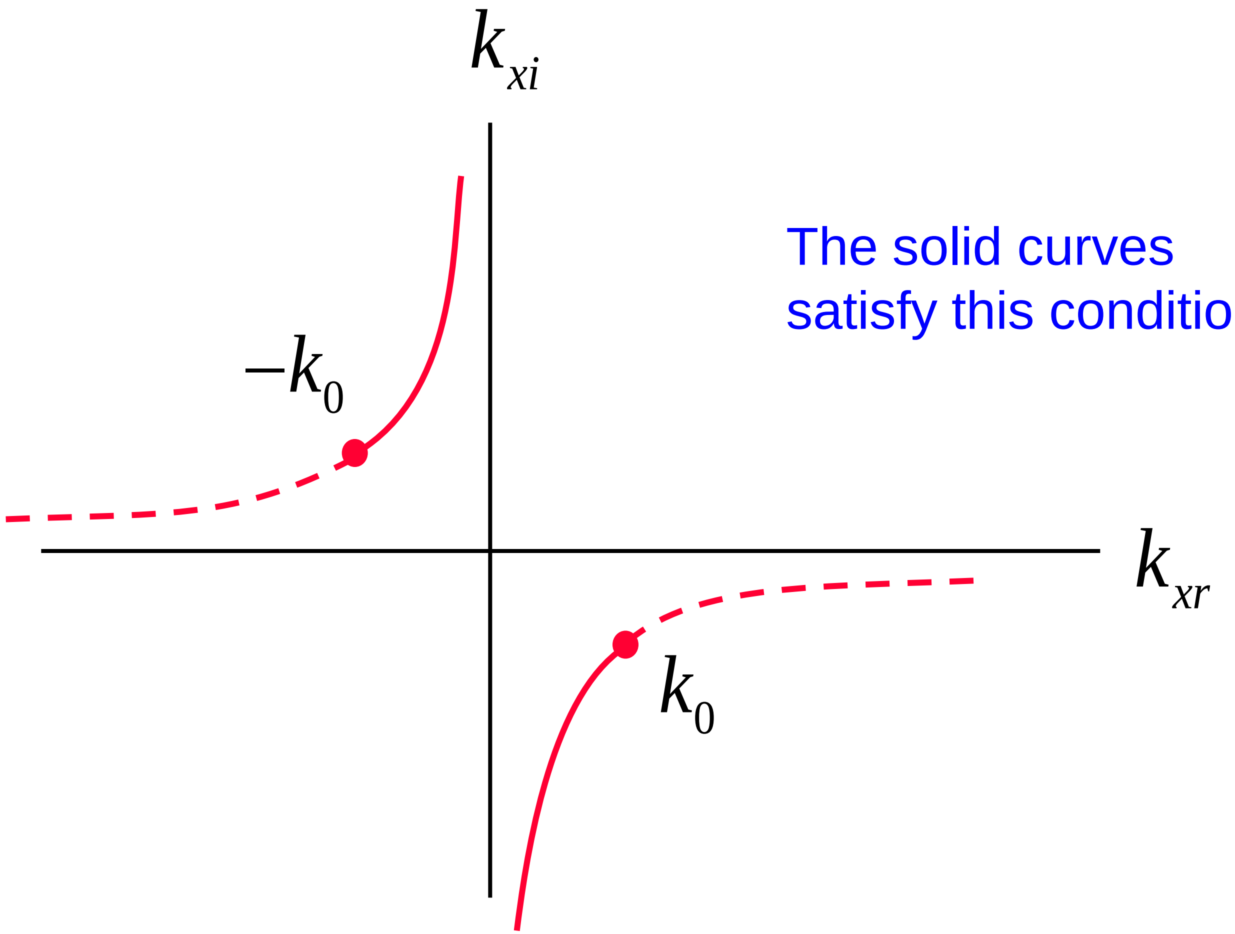
$$k_{xi} = -k_0''$$

$$\Rightarrow \quad k_x = k_0 = k_0' - jk_0''$$



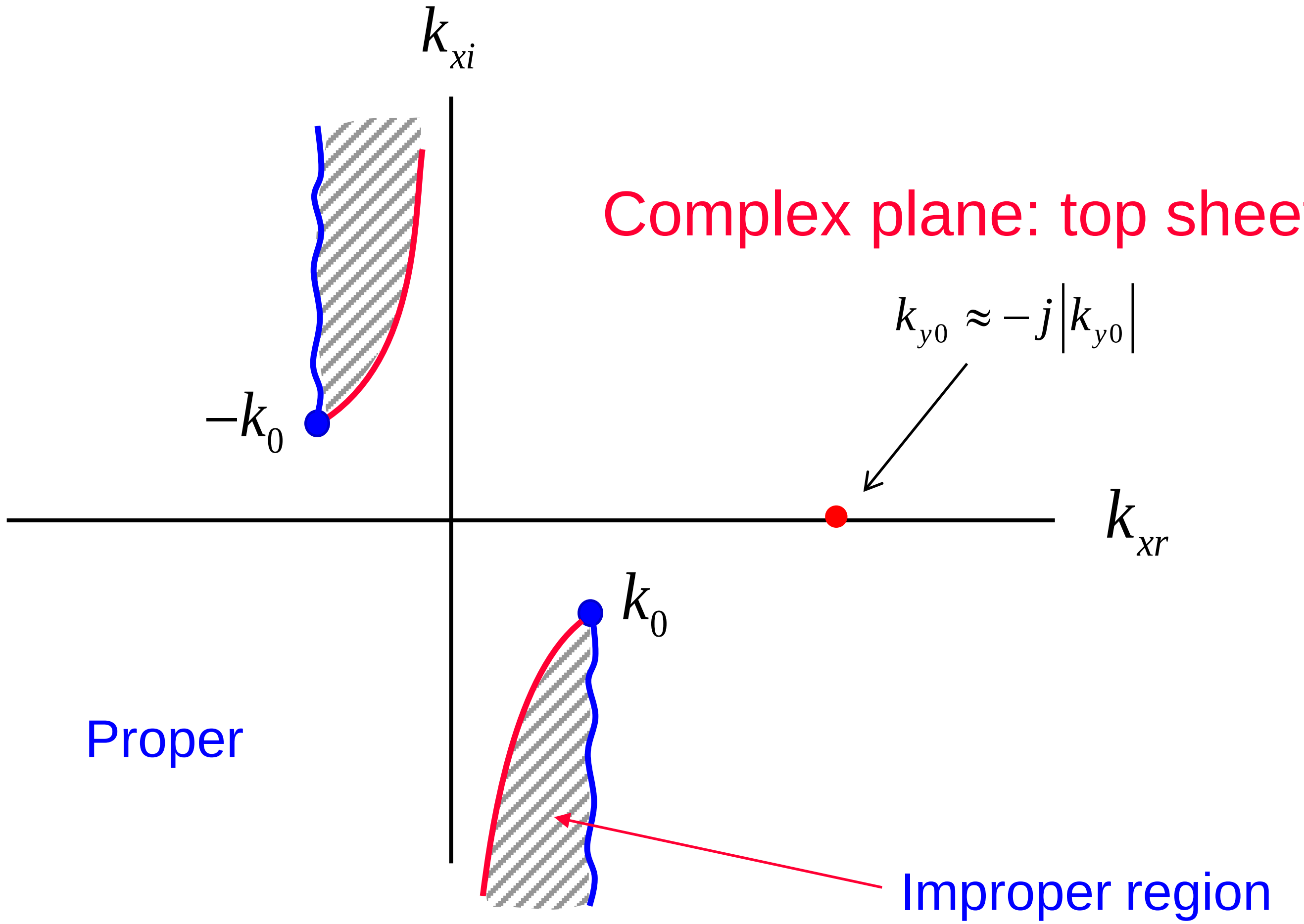
Proper / Improper Regions (cont.)

Also $$k_0'^2 - k_0''^2 - k_{xr}^2 + k_{xi}^2 > 0$$



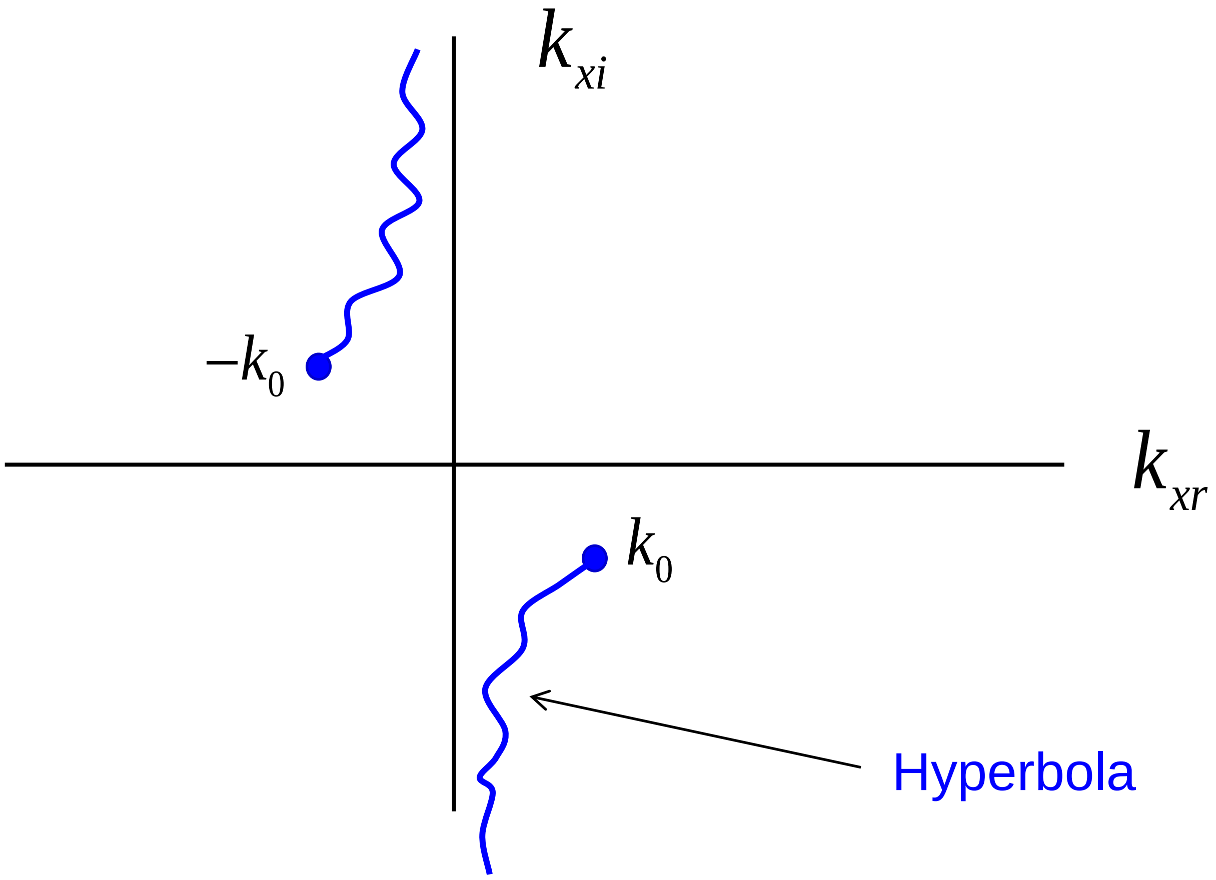
The solid curves satisfy this condition.

Proper / Improper Regions (cont.)



On the complex plane corresponding to the bottom sheet, the proper and improper regions are reversed from what is shown here.

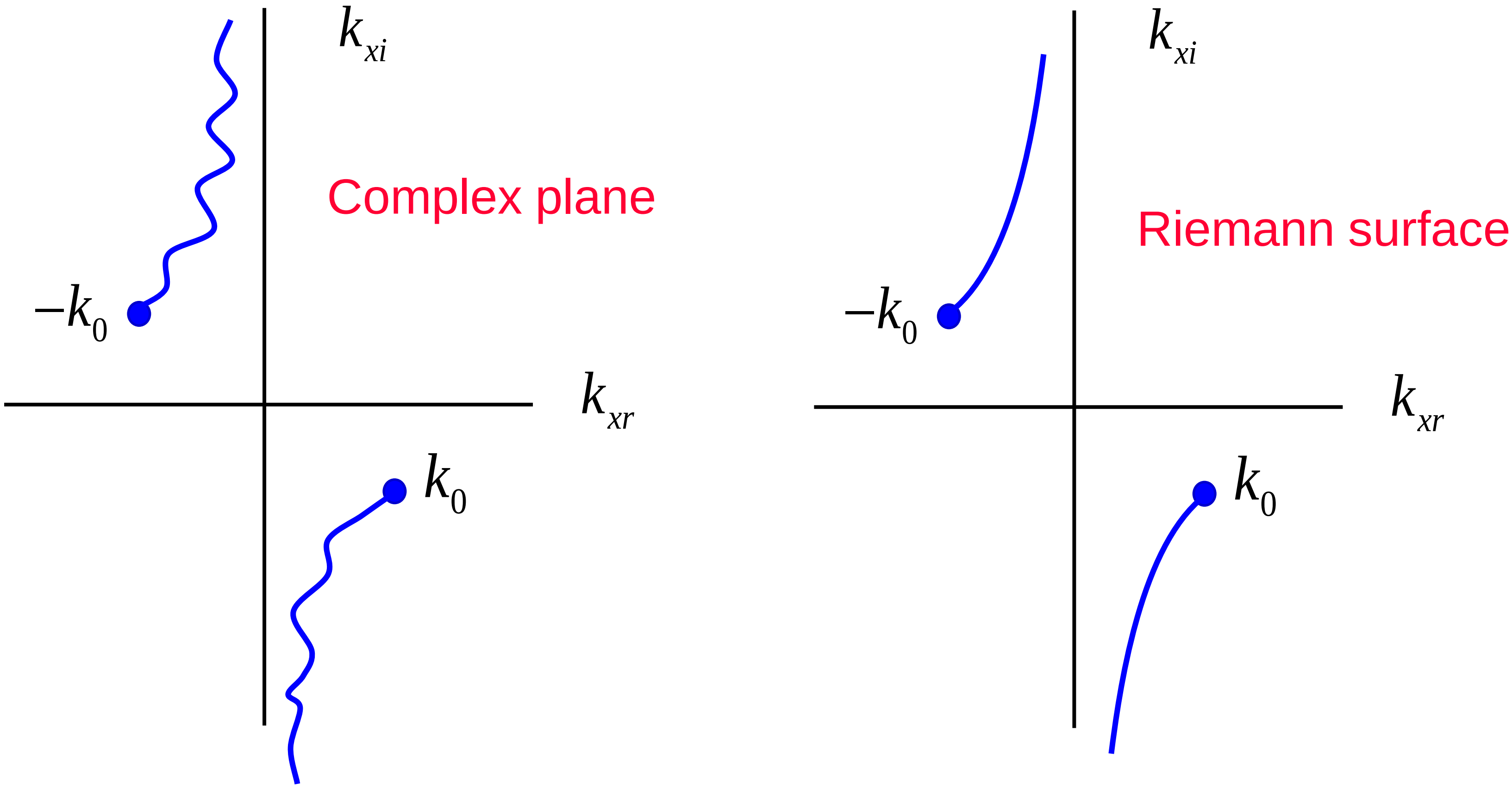
Sommerfeld Branch Cuts



Complex plane corresponding to top sheet: proper everywhere

Complex plane corresponding to bottom sheet: improper everywhere

Sommerfeld Branch Cuts

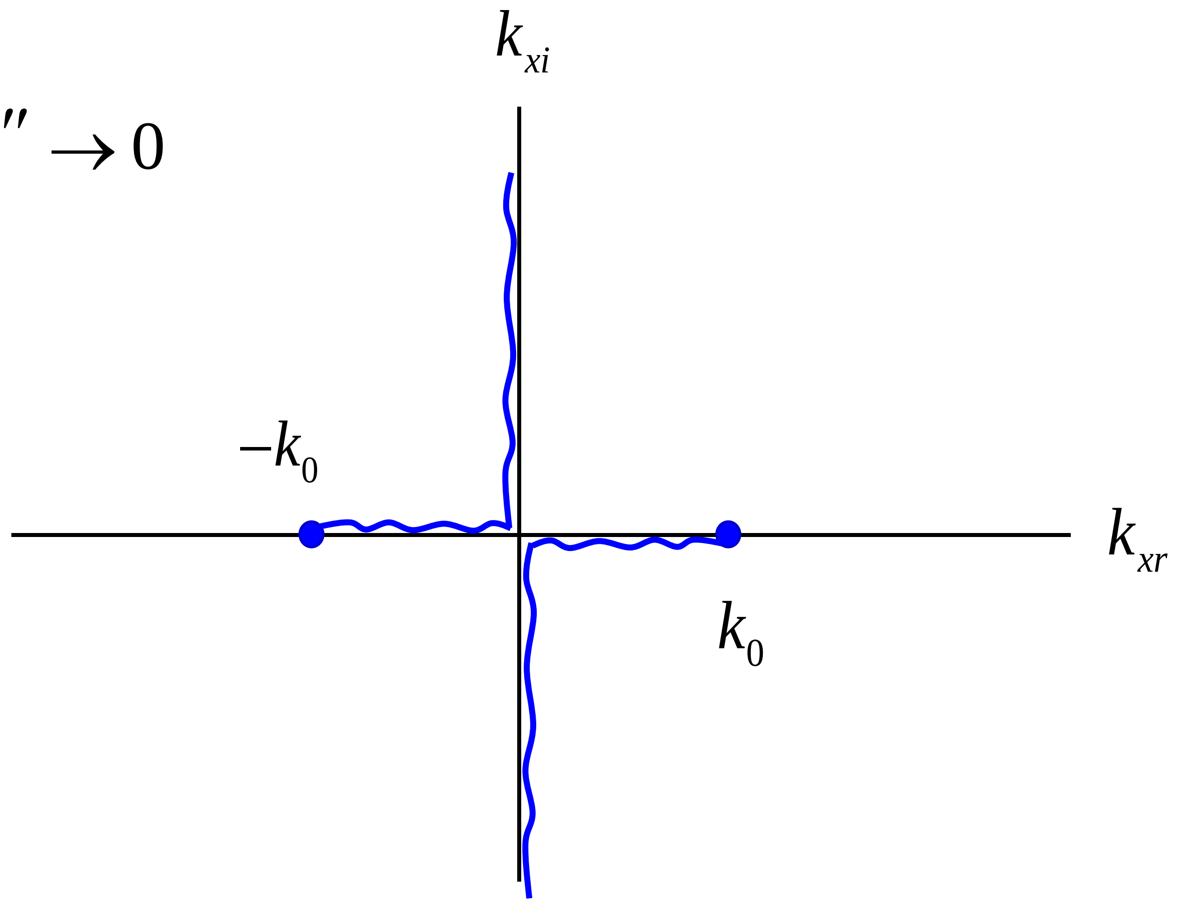


Note: We can think of a two complex planes with branch cuts, or a Riemann surface with hyperbolic-shaped "ramps" connecting the two sheets.

The Riemann surface allows us to show <u>all possible poles</u>, both proper (surface-wave) and improper (leaky-wave).

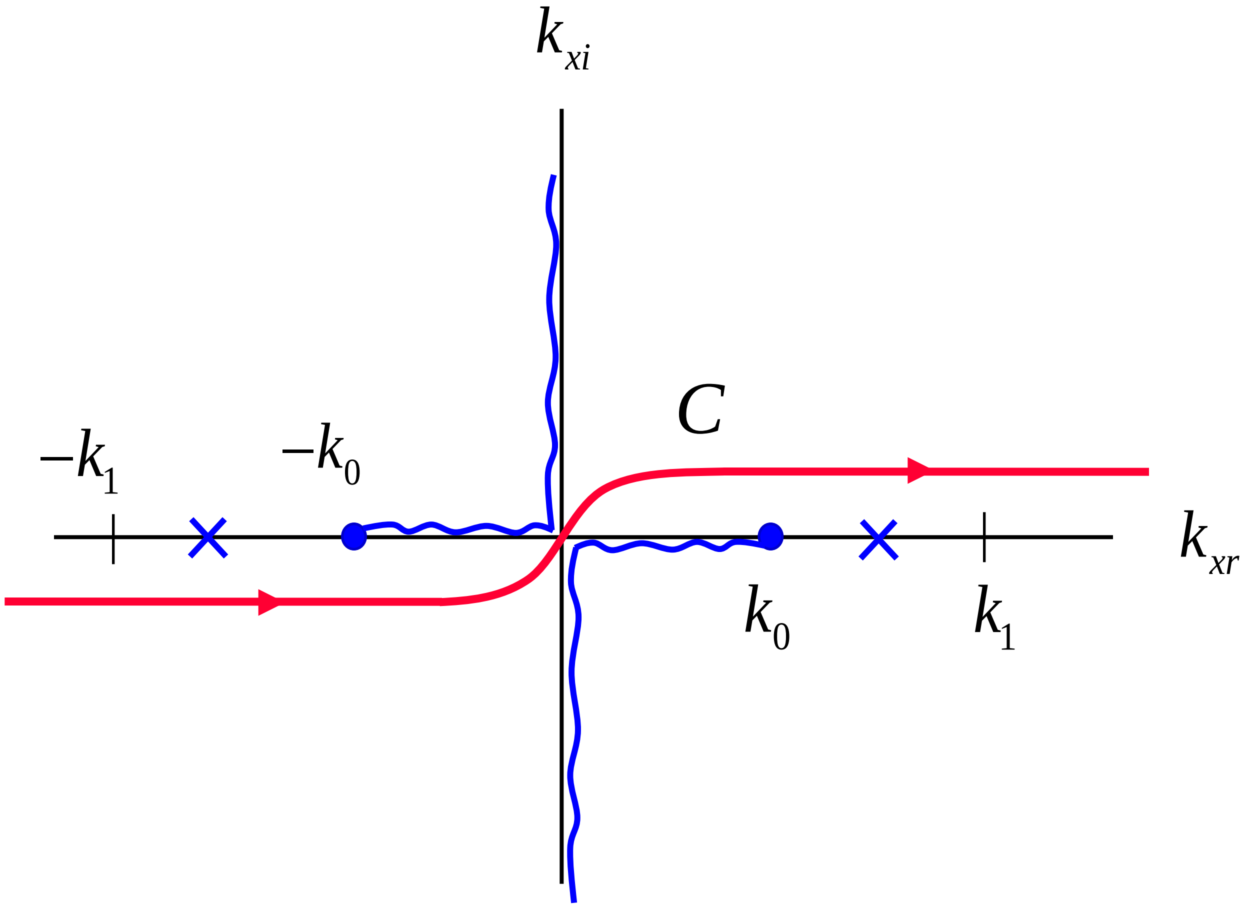
Sommerfeld Branch Cut

Let $k_0'' \to 0$



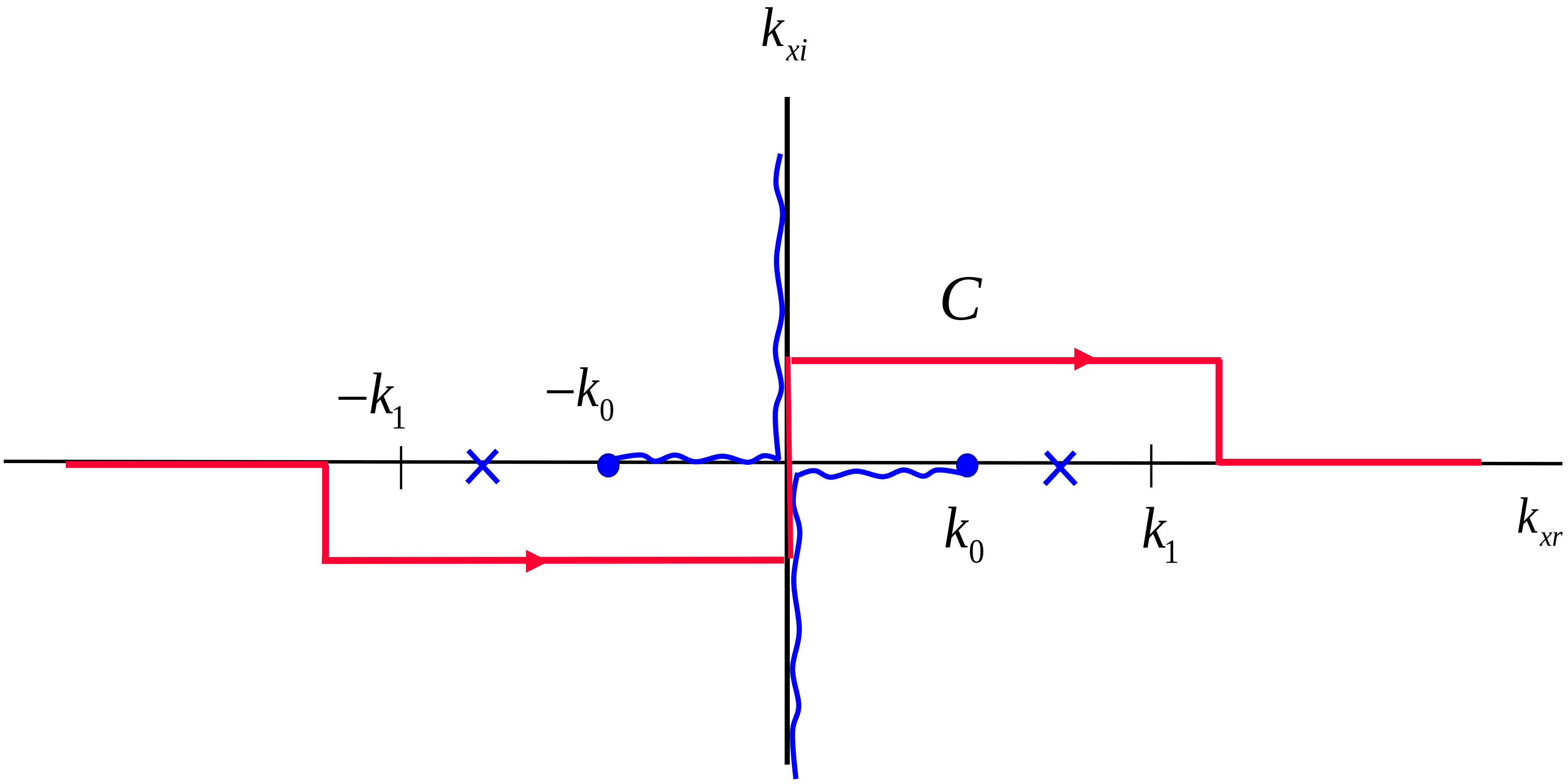
The branch cuts now lie along the imaginary axis, and part of the real axis.

Path of Integration



The path is on the complex plane corresponding to the top Riemann sheet.

Numerical Path of Integration



Leaky-Mode Poles

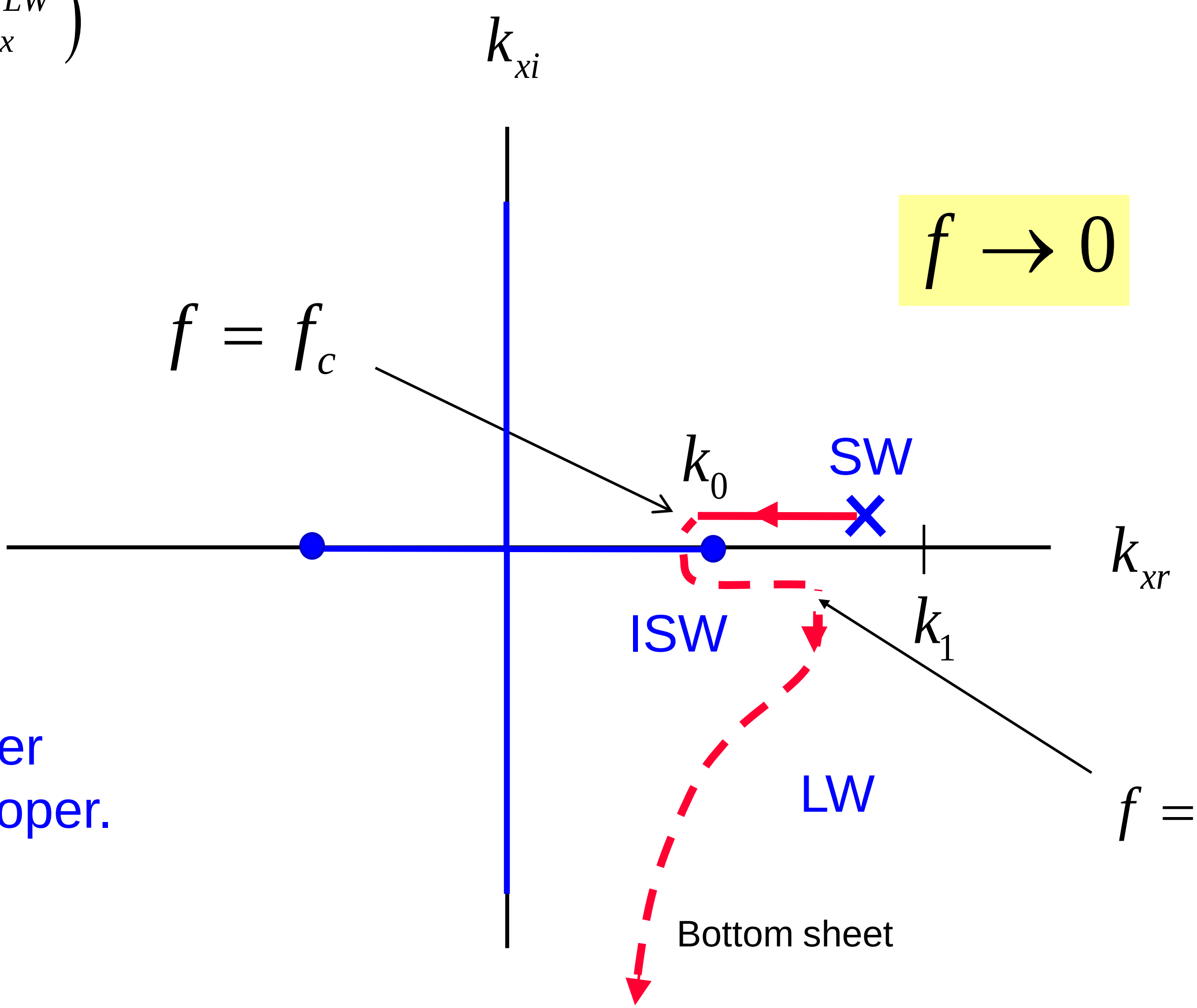
TRE:

$$Z_{in}\left(k_x^{LW}\right) = -Z_0\left(k_x^{LW}\right)$$

$$\operatorname{Im} k_{y0} > 0$$

(improper)

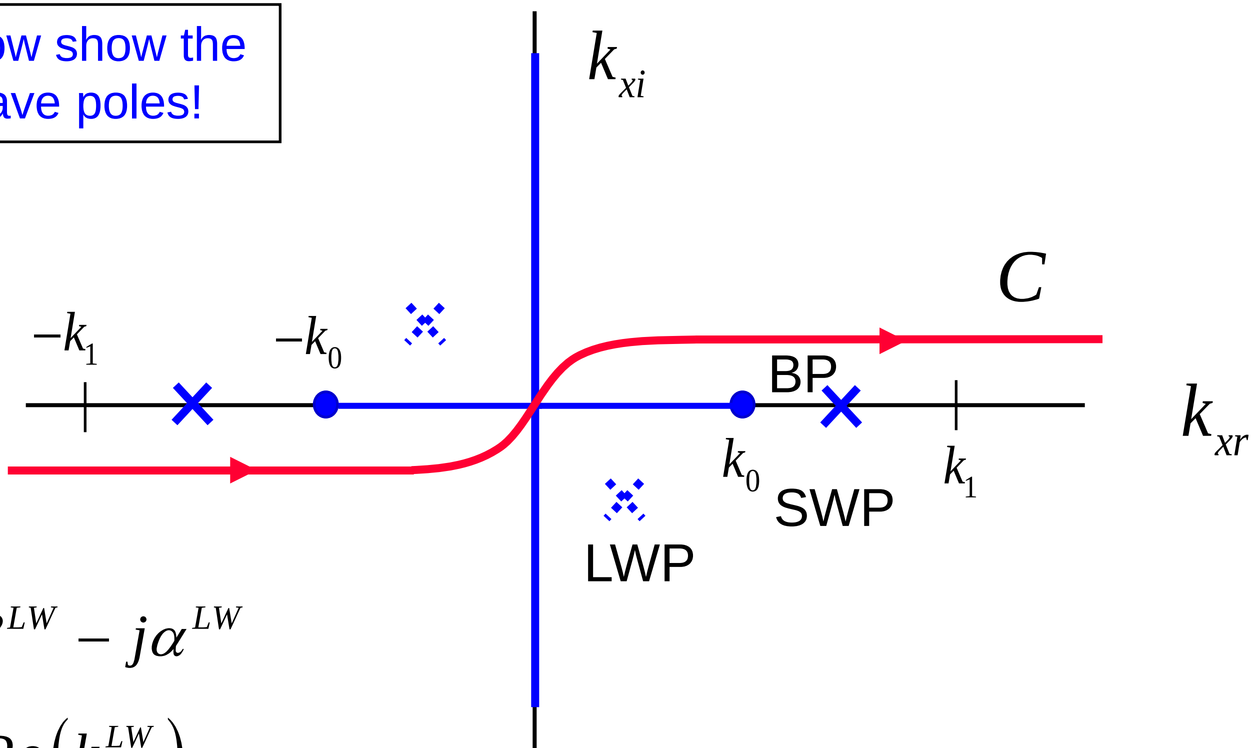
Frequency behavior on the Riemann surface



Note: TM_0 never becomes improper.

Riemann Surface

We can now show the leaky-wave poles!



$$k_{xp}^{LW} = \beta^{LW} - j\alpha^{LW}$$

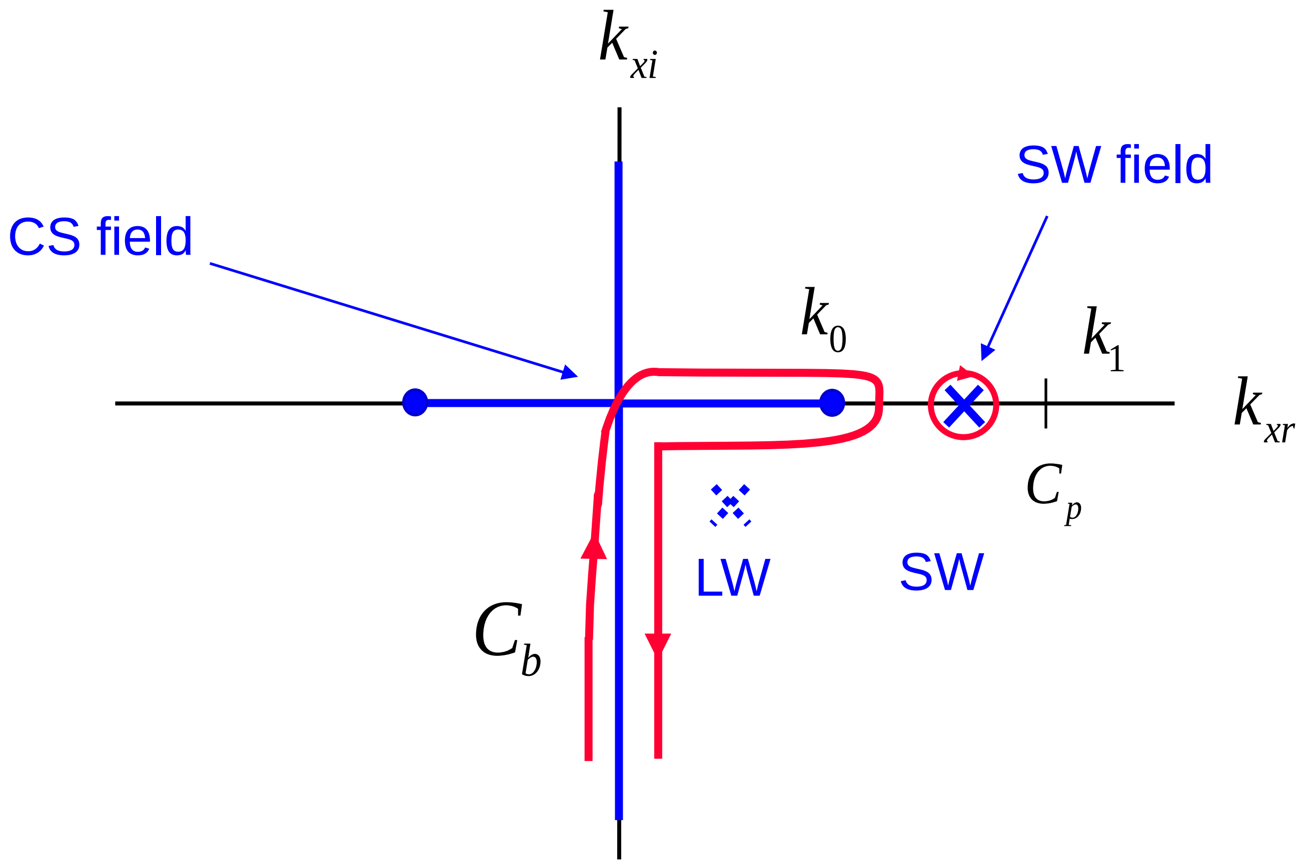
$$\beta^{LW} = \mathrm{Re}\left(k_{xp}^{LW}\right)$$

$$-k_0 \le \beta^{LW} \le k_0$$

$\Rightarrow$

The LW pole is then "close" to the path on the Riemann surface (and it usually makes an important contribution).

SW and CS Fields



Total field = surface-wave (SW) field
+ continuous-spectrum (CS) field

Note: The CS field <u>indirectly</u> accounts for the LW pole.

Leaky Waves

LW poles may be important if

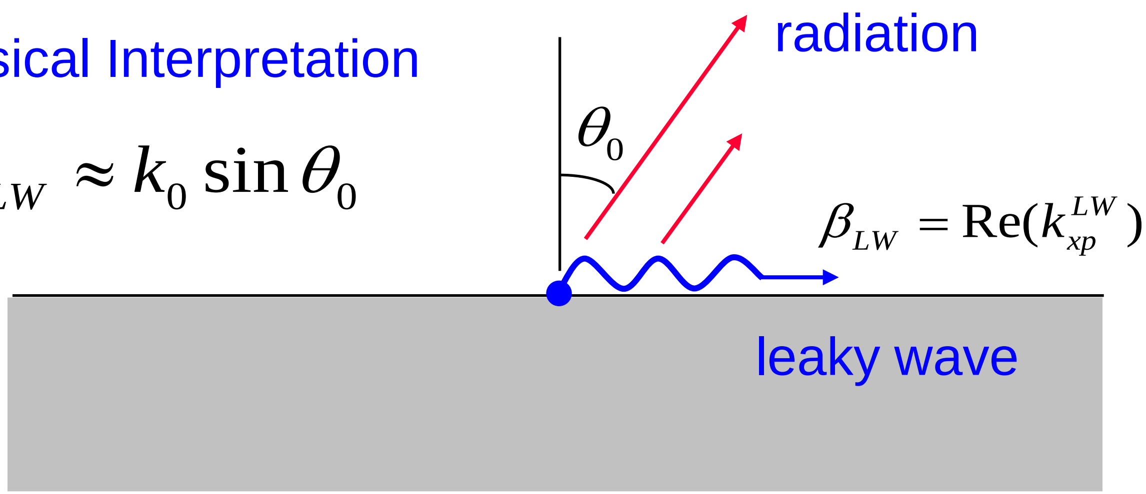
$$-k_0 \le \beta^{LW} \le k_0$$

$$\alpha^{LW} \ll k_0$$

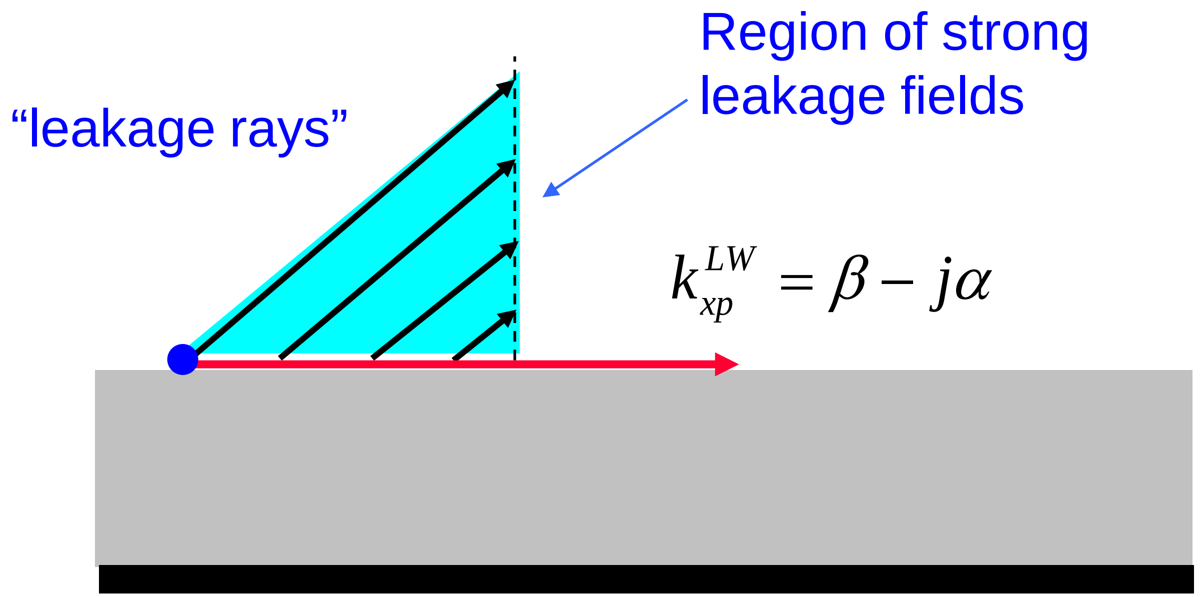
The LW pole is then "close" to the path on the Riemann surface.

Physical Interpretation

$$\beta_{LW} \approx k_0 \sin\theta_0$$



Improper Nature of LWs



The rays are stronger near the beginning of the wave: this gives us exponential growth vertically.

Improper Nature (cont.)

Mathematical explanation of exponential growth (improper behavior):

$$k_{y0}^{LW} = \left(k_0^{\;2} - k_{xp}^{LW\;\;2}\right)^{1/2}$$

$$\Rightarrow \quad \left(k_{y0}^{LW}\right)^2 = k_0^{\;2} - \left(k_{xp}^{LW}\right)^2$$

$$\Rightarrow \quad \left(\beta_y - j\alpha_y\right)^2 = k_0^{\;2} - \left(\beta - j\alpha\right)^2$$

Equate imaginary parts:

$$\beta_y \alpha_y = -\beta\,\alpha$$

$$\alpha_y = -\alpha\left(\frac{\beta}{\beta_y}\right)$$

$$\alpha > 0 \;\rightarrow\; \alpha_y < 0$$ (improper)