ECE 6341

Spring 2016

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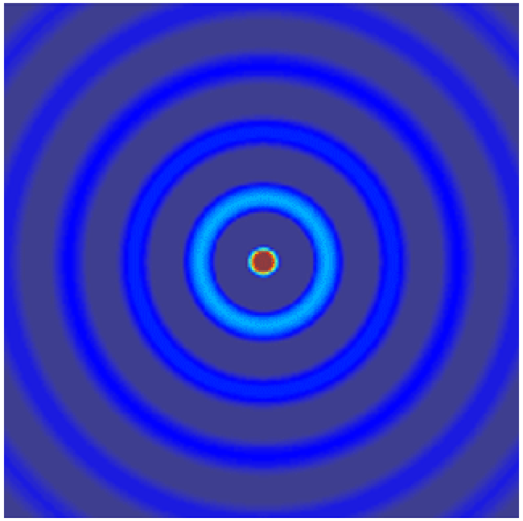
ECE Dept.



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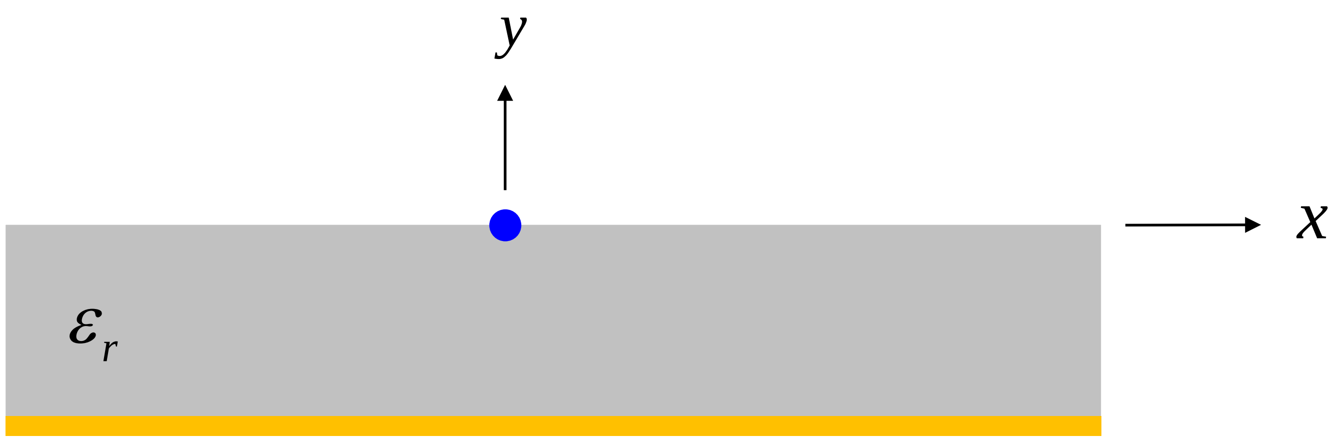
Line Source on a Grounded Slab



$$E_z = -j\omega A_z$$

$$A_z = \frac{\mu_0 I_0}{4\pi j} \int_{-\infty}^{+\infty} \frac{1}{k_{y0}} \left[1 + \Gamma^{TE}(k_x)\right] e^{-jk_{y0}y} e^{-jk_x x} dk_x$$

$$\Gamma^{TE}(k_x) = \frac{Z_{in}^{TE}(k_x) - Z_0^{TE}(k_x)}{Z_{in}^{TE}(k_x) + Z_0^{TE}(k_x)}$$

$$Z_{in}^{TE}(k_x) = jZ_1^{TE} \tan(k_{y1} h)$$

(even function of k_{y1})

$$k_{y0} = \left(k_0^2 - k_x^2\right)^{1/2} \qquad k_{y1} = \left(k_1^2 - k_x^2\right)^{1/2}$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{y0}}$$

$$Z_1^{TE} = \frac{\omega\mu_0}{k_{y1}}$$

There are branch points only at $k_x = \pm k_0$

Steepest-Descent Path Physics

Steepest-descent transformation:

$$k_x = k_0 \sin\zeta \qquad k_{y0} = k_0 \cos\zeta$$

- There are no branch points in the ζ plane ($\cos\zeta$ is analytic).

Both sheets of the k_x plane get mapped into a single sheet of the ζ plane.

Steepest-Descent Path Physics

Examine k_{y0} to see where the ζ plane is proper and improper:

$$k_{y0} = k_0 \cos\left(\zeta_r + j\zeta_i\right)$$
$$= k_0 \left[\cos\zeta_r \cosh\zeta_i - j\sin\zeta_r \sinh\zeta_i\right]$$

$$\operatorname{Im} k_{y0} = -k_0 \sin\zeta_r \sinh\zeta_i$$

Proper : $\operatorname{Im} k_{y0} < 0$

Improper : $\operatorname{Im} k_{y0} > 0$

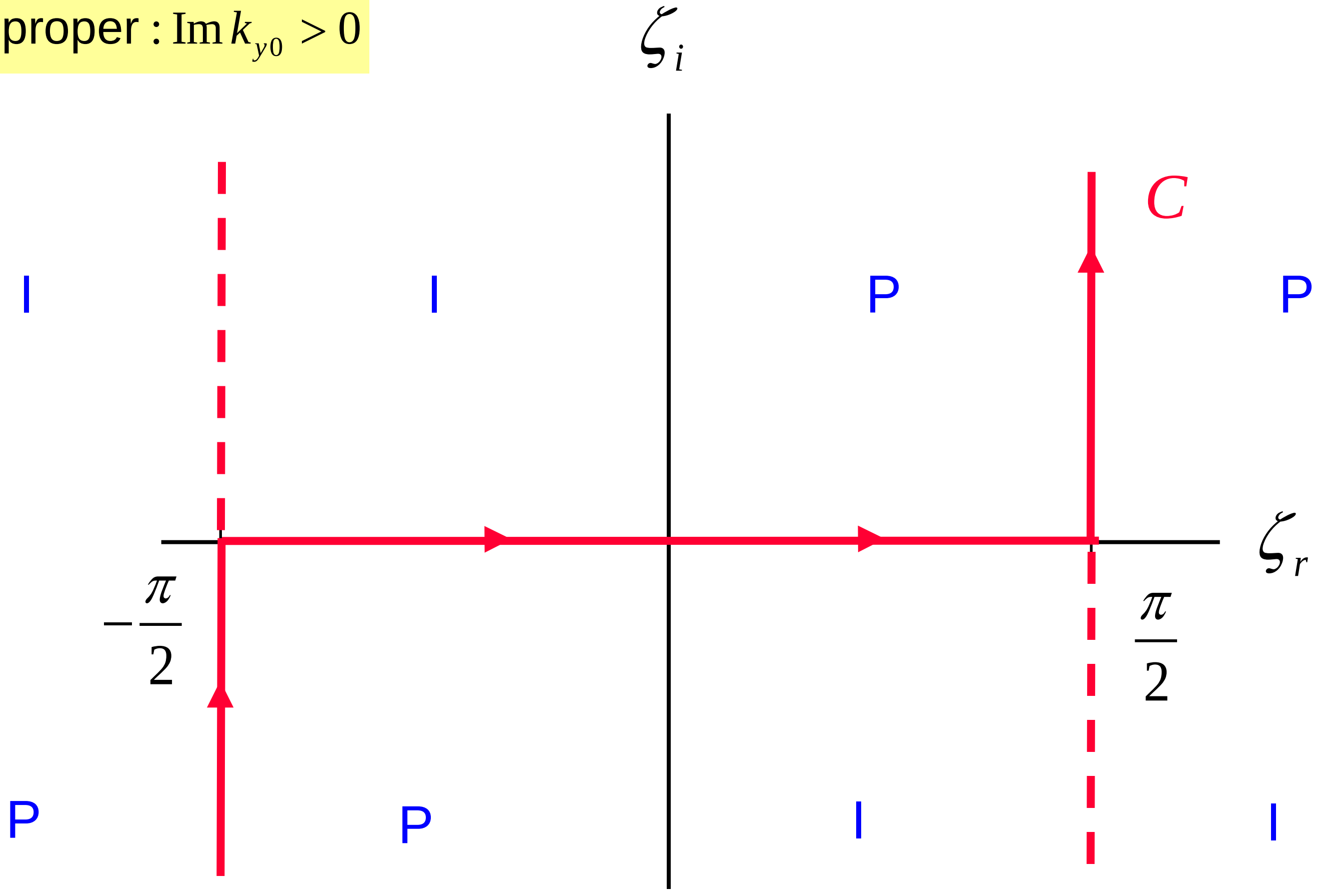
SDP Physics (cont.)

$$\operatorname{Im} k_{y0} = -k_0 \sin \zeta_r \sinh \zeta_i$$

Proper : $\operatorname{Im} k_{y0} < 0$

Improper : $\operatorname{Im} k_{y0} > 0$

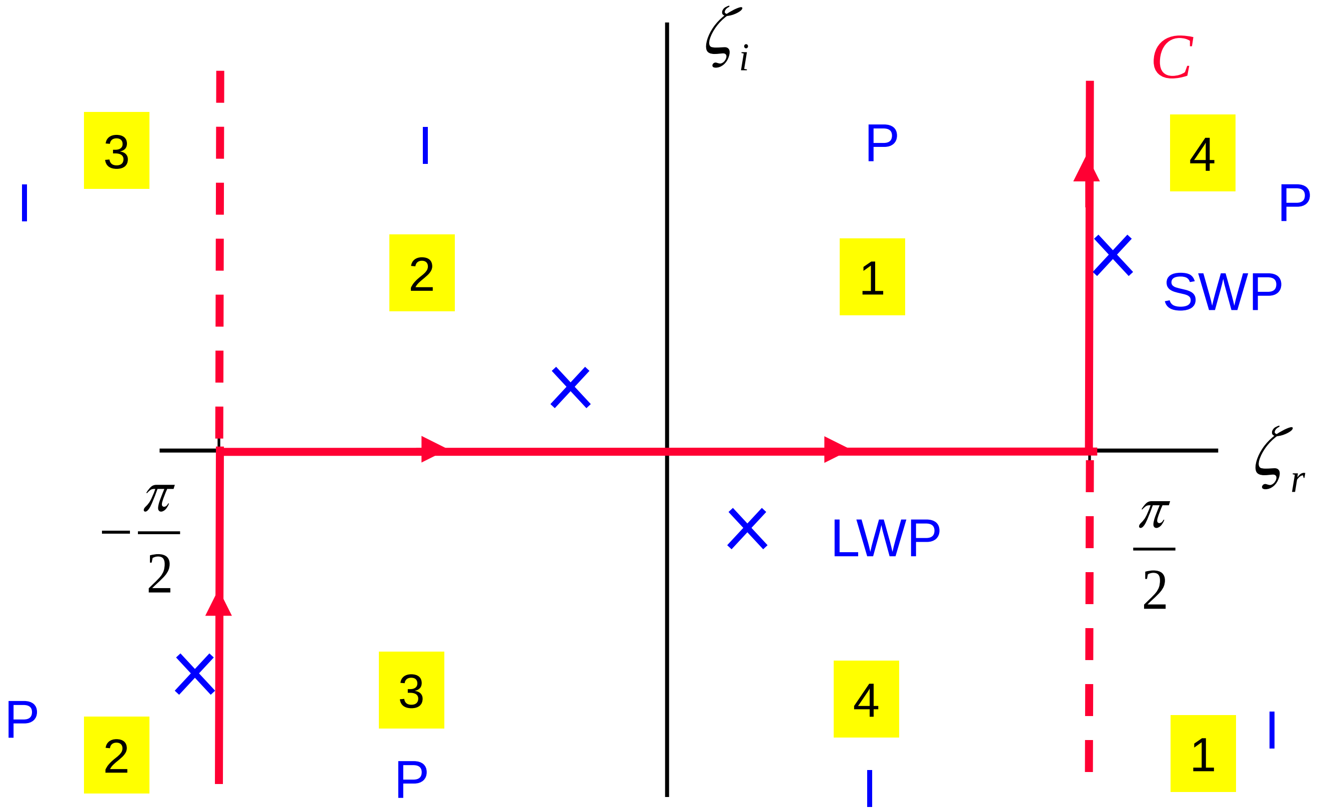
P: proper
I: improper

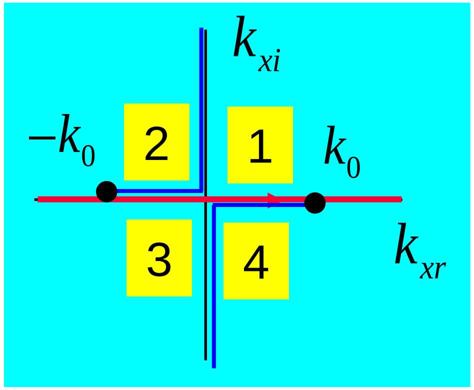


SDP Physics (cont.)

Mapping of quadrants in k_x plane

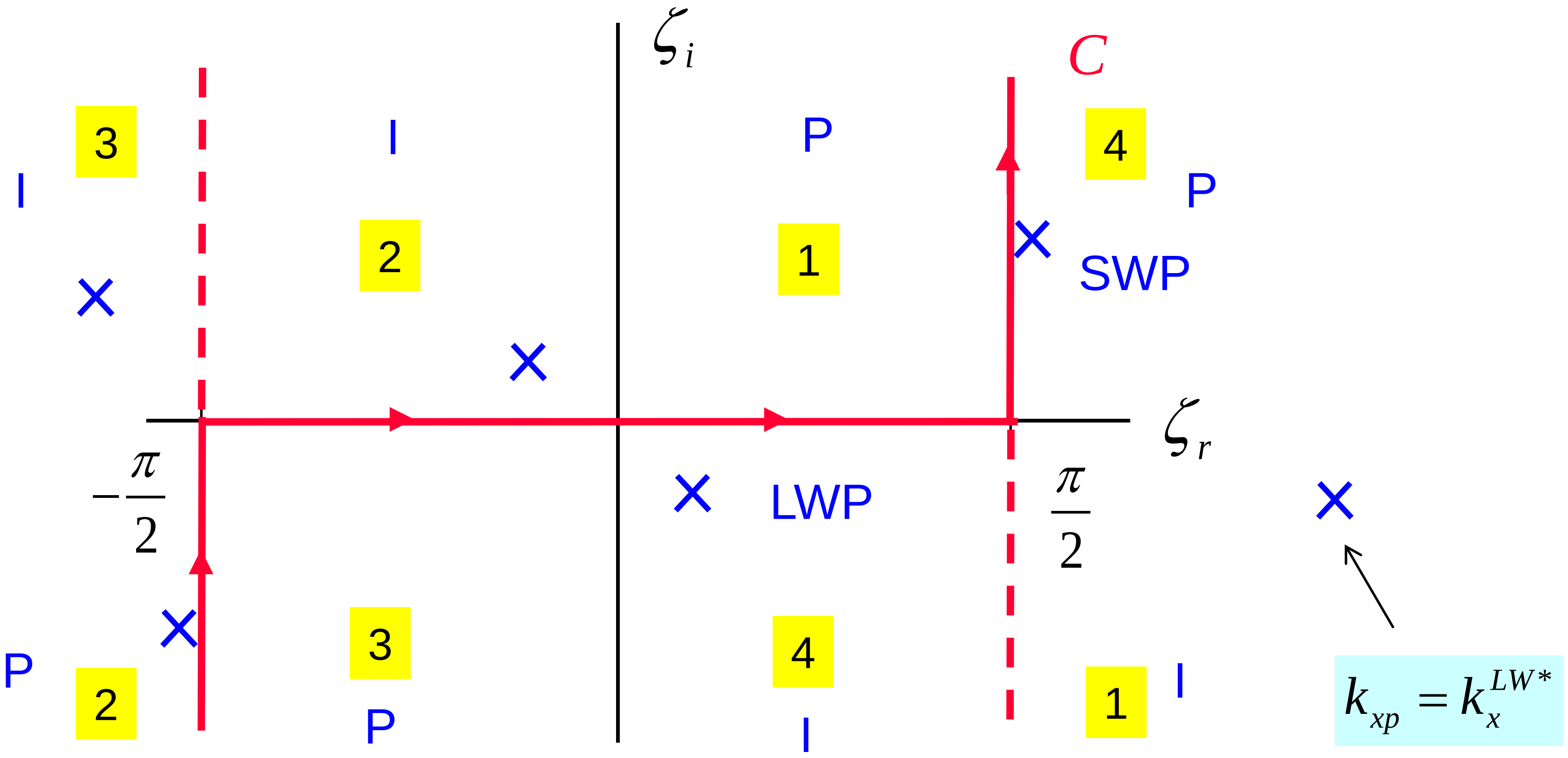
$$k_x = k_0 \sin\zeta = k_0\left[\sin\zeta_r \cosh\zeta_i + j\cos\zeta_r \sinh\zeta_i\right]$$





SDP Physics (cont.)

Non-physical "growing" LW poles (conjugate solution) also exist.



The conjugate pole is symmetric about the $\pi/2$ line:

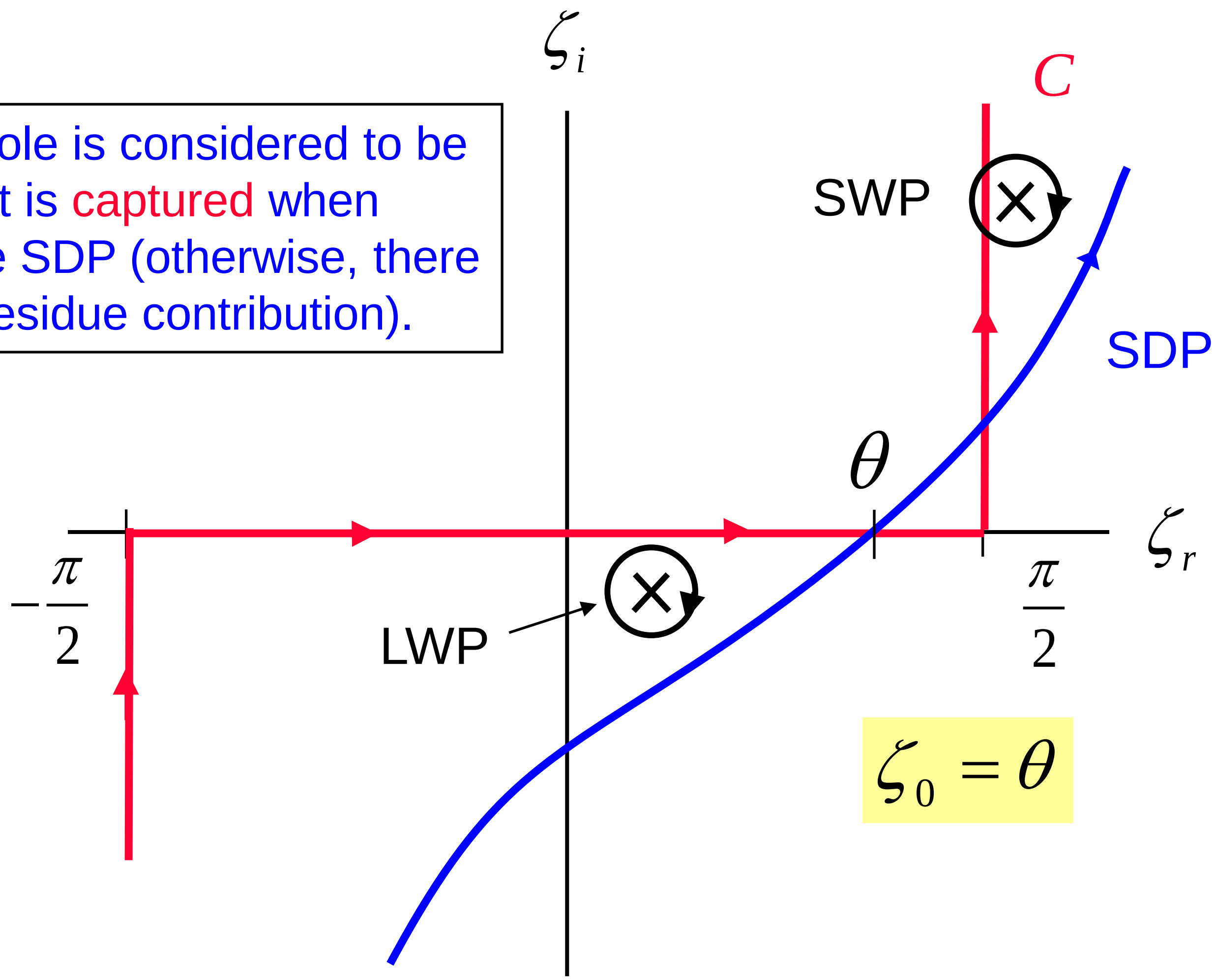
$$\zeta_r^{conj} = \pi/2 + (\pi/2 - \zeta_r) = \pi - \zeta_r$$

$$\zeta_i^{conj} = \zeta_i$$

SDP Physics (cont.)

SDP: $\cos(\zeta_r - \theta)\cosh\zeta_i = 1$

A leaky-wave pole is considered to be physical if it is captured when deforming to the SDP (otherwise, there is no direct residue contribution).



$\zeta_0 = \theta$

SDP Physics (cont.)

Comparison of Fields on interface ($\theta = \pi / 2$):

LWP: $$E_z = -2\pi j \left(\mathrm{Res}\right) e^{-jk_x^{LW} x}$$ (exists if pole is <u>captured</u>)

$$k_x^{LW} = \beta - j\alpha$$

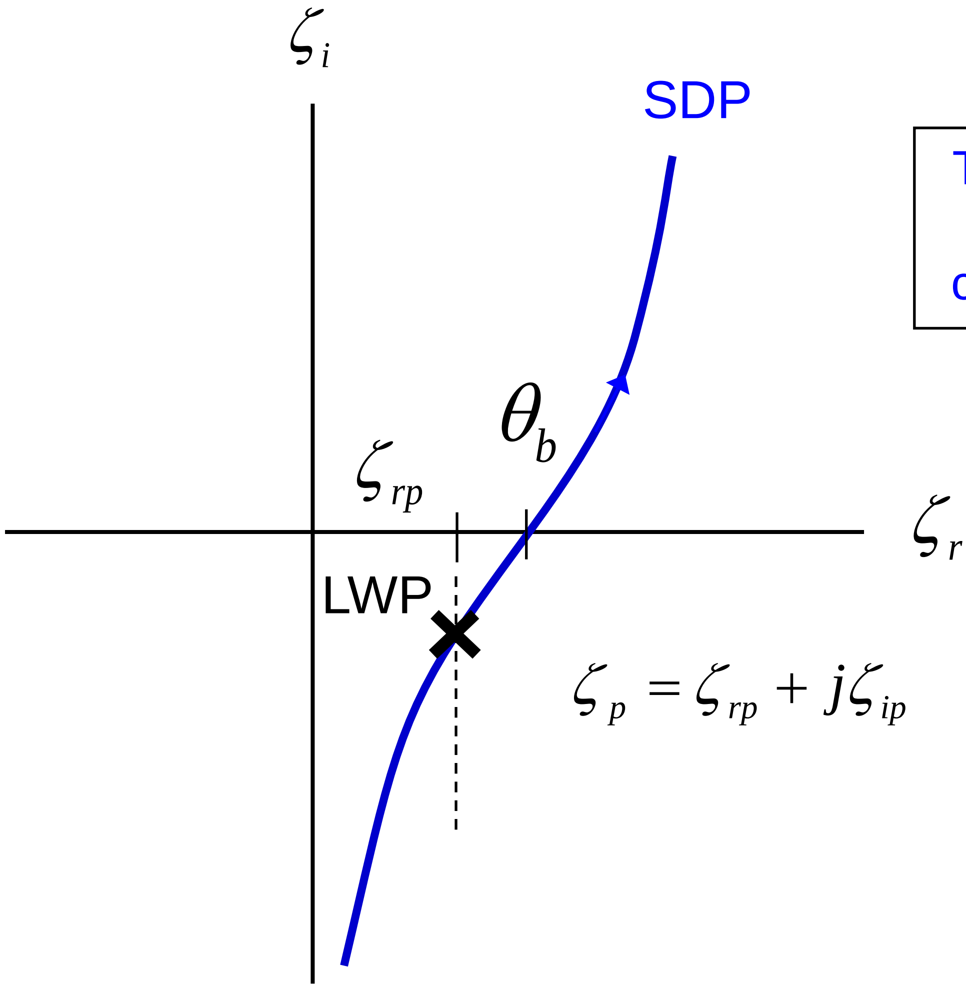
SDP: $$E_z \sim A\left(\frac{e^{-jk_0 x}}{x^{3/2}}\right)$$ (from higher-order steepest-descent method)

The leaky-wave field is <u>important</u> if:

1) The pole is captured (the pole is said to be “physical”).
2) The residue is strong enough.
3) The attenuation constant α is small.

SDP Physics (cont.)

LWP captured: $\theta > \theta_b$



The angle θ_b represents the boundary for which the leaky-wave pole is captured (the leaky-wave field exists).

Note:

$\theta_b > \zeta_{rp}$

SDP Physics (cont.)

Behavior of LW field:

$$E_z = \int_{-\infty}^{+\infty} F(k_x) e^{-jk_{y0}y} e^{jk_x x} dk_x$$

$$= \int_C F(\zeta) e^{-j(k_0\rho)\cos(\zeta-\theta)} k_0 \cos\zeta \, d\zeta$$

$$E_z^{LW} = -2\pi j \,\mathrm{Res}\, F(\zeta_p)(k_0 \cos\zeta_p) \underbrace{e^{-j(k\rho)\cos(\zeta_p-\theta)}}_{\psi}$$

In rectangular coordinates:

$$E_z^{LW} = A e^{jk_{xp}x} e^{-jk_{y0p}y}$$ (It is an inhomogeneous plane-wave field.)

where $k_{xp} = k_x^{LW} = \beta - j\alpha$

SDP Physics (cont.)

Examine the exponential term: $\psi = e^{-j(k_0\rho)\cos(\zeta_p - \theta)}$

$$\cos(\zeta_p - \theta) = \cos\left[(\zeta_{rp} - \theta) + j\zeta_{ip}\right]$$
$$= \cos(\zeta_{rp} - \theta)\cosh\zeta_{ip} - j\sin(\zeta_{rp} - \theta)\sinh\zeta_{ip}$$

Hence

$$|\psi| = e^{-(k_0\rho)\sin(\zeta_{rp} - \theta)\sinh\zeta_{ip}}$$
$$= e^{-(k_0\rho)|\sinh\zeta_{ip}|\,\sin(\theta - \zeta_{rp})} \qquad \text{since} \quad \zeta_{ip} < 0$$

SDP Physics (cont.)

$$\left|\psi\right| = e^{-(k_0\rho)\left|\sinh\zeta_{ip}\right|\,\sin\left(\theta-\zeta_{rp}\right)}$$

Radially decaying:

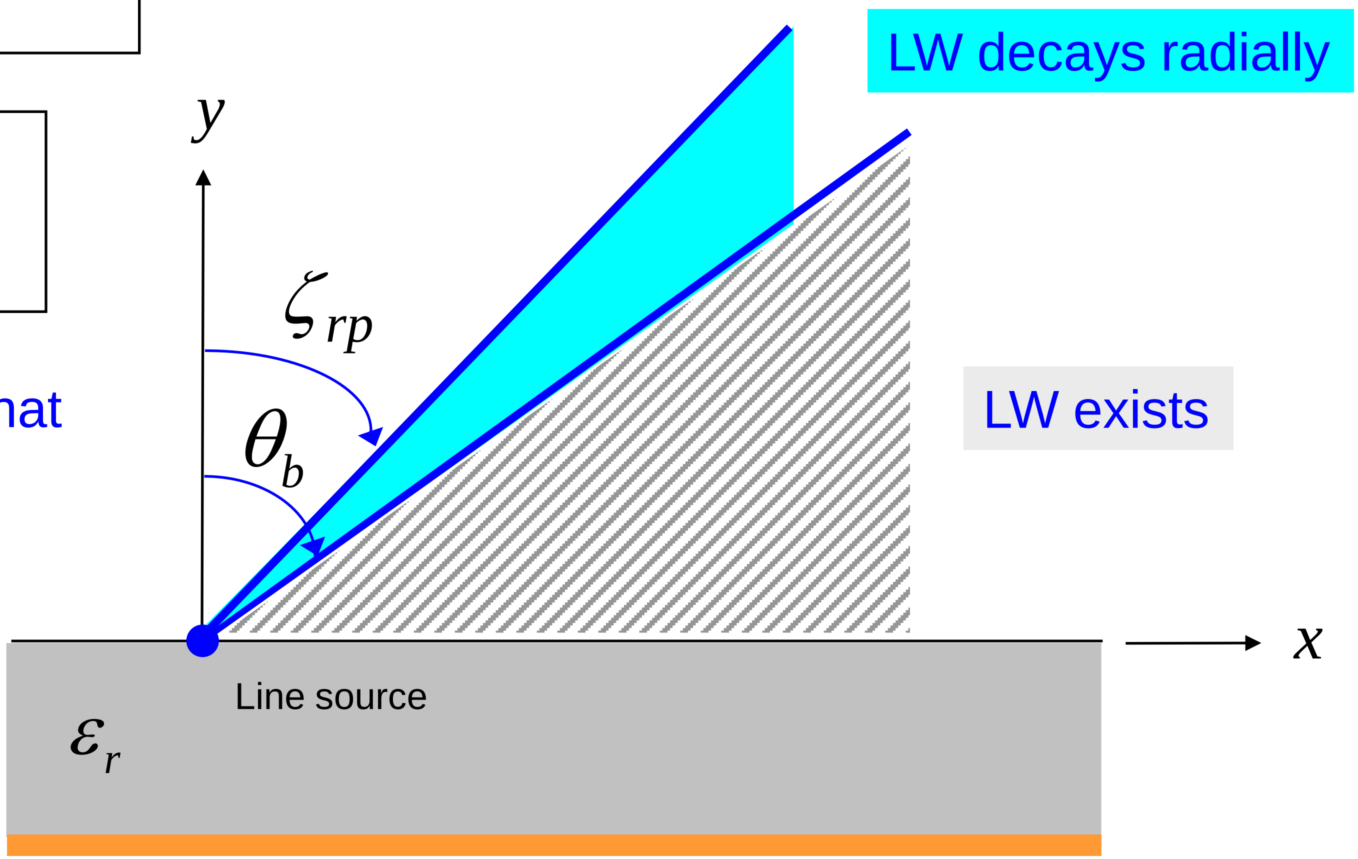
$$\theta > \zeta_{rp}$$

LW exists:

$$\theta > \theta_b$$

Also, recall that

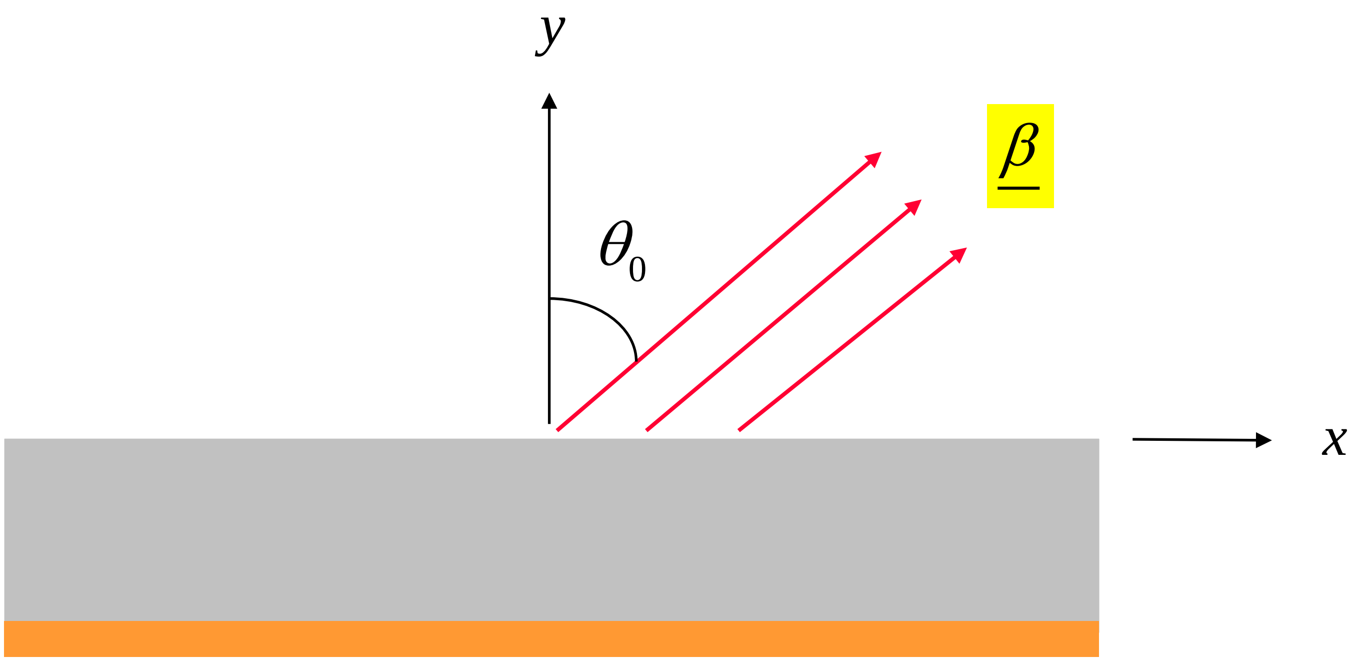
$$\theta_b > \zeta_{rp}$$



Power Flow

Power flows in the direction of the $\underline{\beta}$ vector.

$$
\begin{aligned}
\underline{\beta} &= \operatorname{Re} \underline{k} = \operatorname{Re}\left(\underline{\hat{x}}\, k_x + \underline{\hat{y}} k_{y0}\right) \\
&= \operatorname{Re}\left(\underline{\hat{x}}\left[k_0 \sin\left(\zeta_{rp} + j\,\zeta_{ip}\right)\right] + \underline{\hat{y}}\left[k_0 \cos\left(\zeta_{rp} + j\,\zeta_{ip}\right)\right]\right) \\
&= k_0\left(\underline{\hat{x}} \sin\zeta_{rp} \cosh\zeta_{ip} + \underline{\hat{y}} \cos\zeta_{rp} \cosh\zeta_{ip}\right)
\end{aligned}
$$



Power Flow (cont.)

$$\underline{\beta} = k_0 \left(\underline{\hat{x}} \sin \zeta_{rp} \cosh \zeta_{ip} + \underline{\hat{y}} \cos \zeta_{rp} \cosh \zeta_{ip} \right)$$

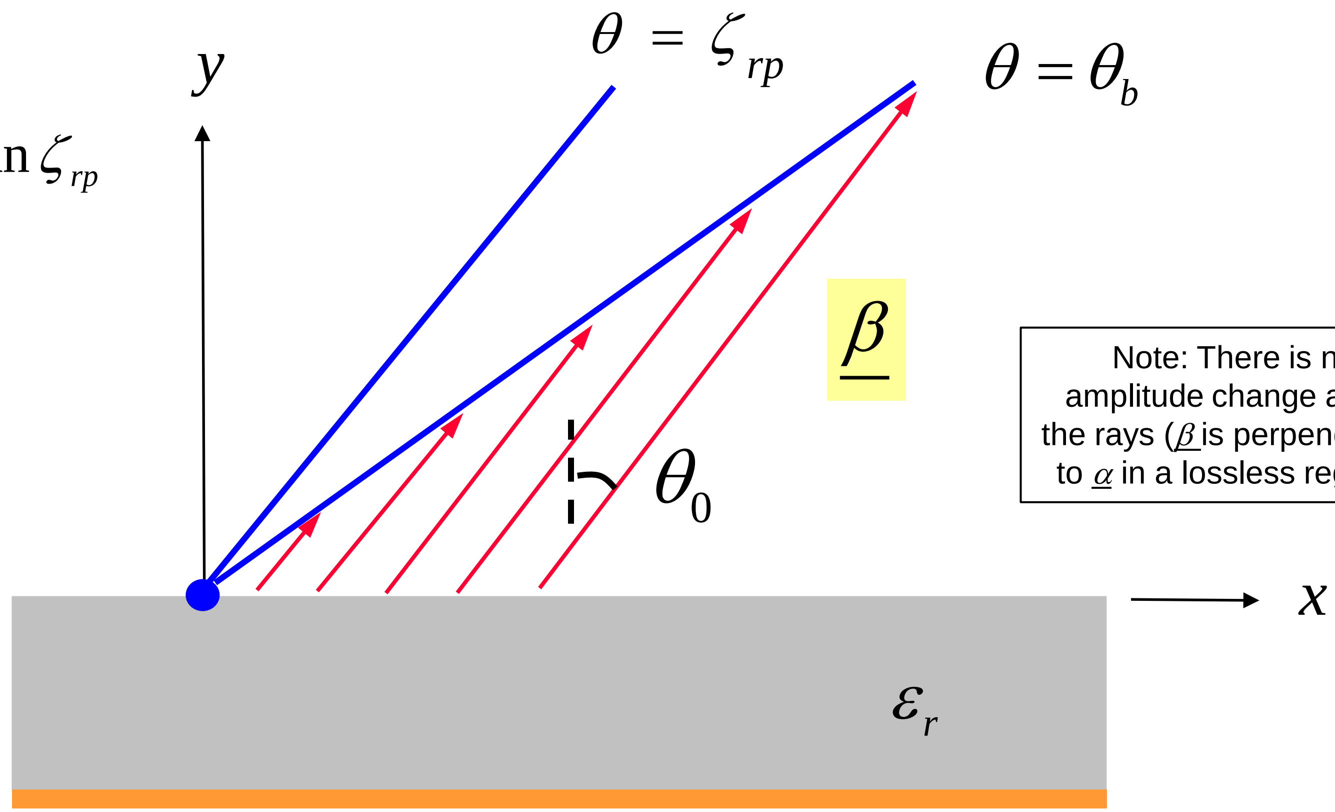
Also, $$\underline{\beta} = \left|\underline{\beta}\right| \left(\underline{\hat{x}} \sin \theta_0 + \underline{\hat{y}} \cos \theta_0 \right)$$

Note that

$$\tan \theta_0 = \frac{\beta_x}{\beta_y} = \tan \zeta_{rp}$$

Hence

$$\theta_0 = \zeta_{rp}$$



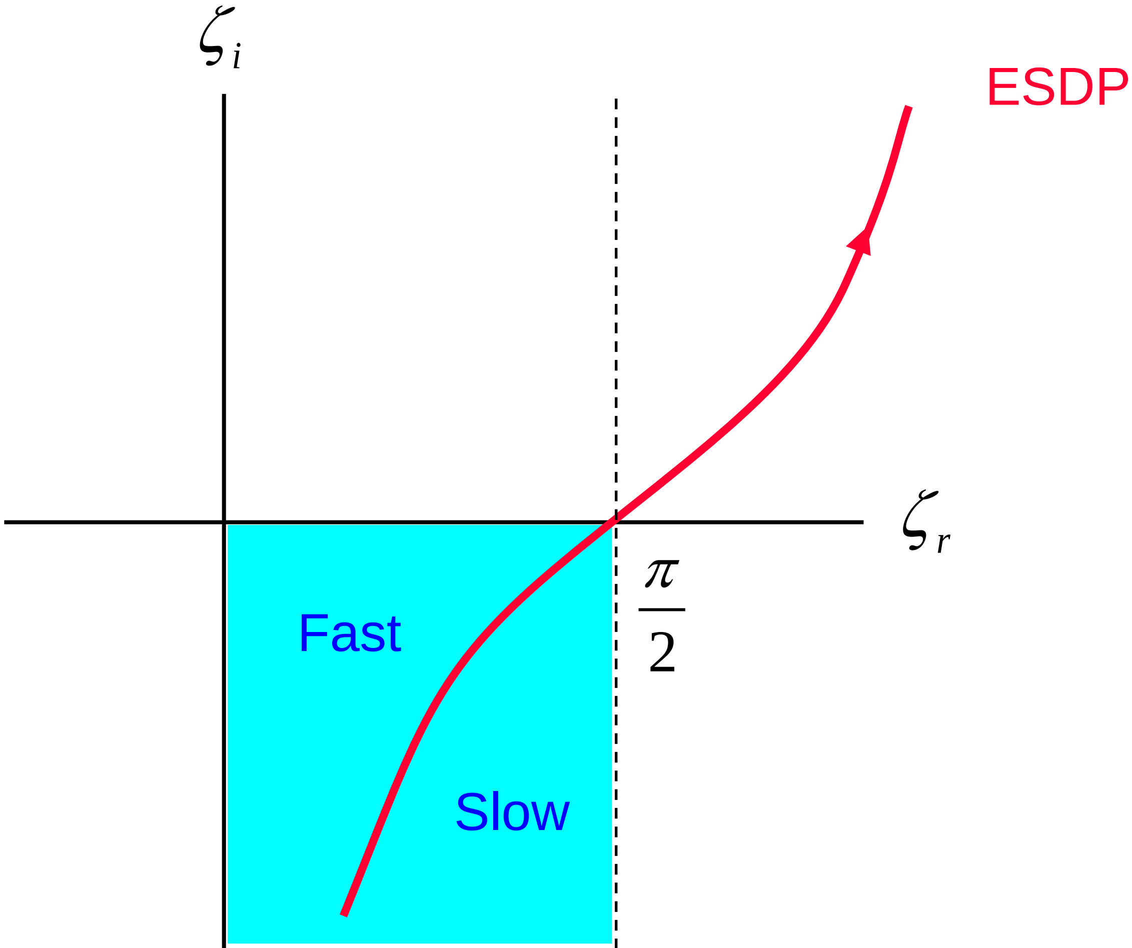
Note: There is no amplitude change along the rays ($\underline{\beta}$ is perpendicular to $\underline{\alpha}$ in a lossless region).

ESDP (Extreme SDP)

The ESDP is the SDP for $\theta = \pi / 2$.

$$\zeta_0 = \theta = \pi / 2$$

The ESDP is important for evaluating the fields on the interface (which determines the far-field pattern).



We can show that the ESDP divides the LW region into slow-wave and fast-wave regions.

ESDP (cont.)

To see this: $\cos\left(\zeta_r - \theta\right)\cosh\zeta_i = 1$ (SDP)

$\sin\zeta_r \cosh\zeta_i = 1$ (ESDP)

Recall that
$$\begin{aligned} k_{xp} &= k_0 \sin\zeta_p \\ &= k_0 \sin\left(\zeta_{rp} + j\zeta_{ip}\right) \end{aligned}$$

Hence
$$\begin{aligned} \beta &= \operatorname{Re} k_{xp} \\ &= k_0 \sin\zeta_{rp} \cosh\zeta_{ip} \end{aligned}$$

ESDP (cont.)

Hence $$\frac{\beta}{k_0} = \sin\zeta_{rp}\cosh\zeta_{ip}$$

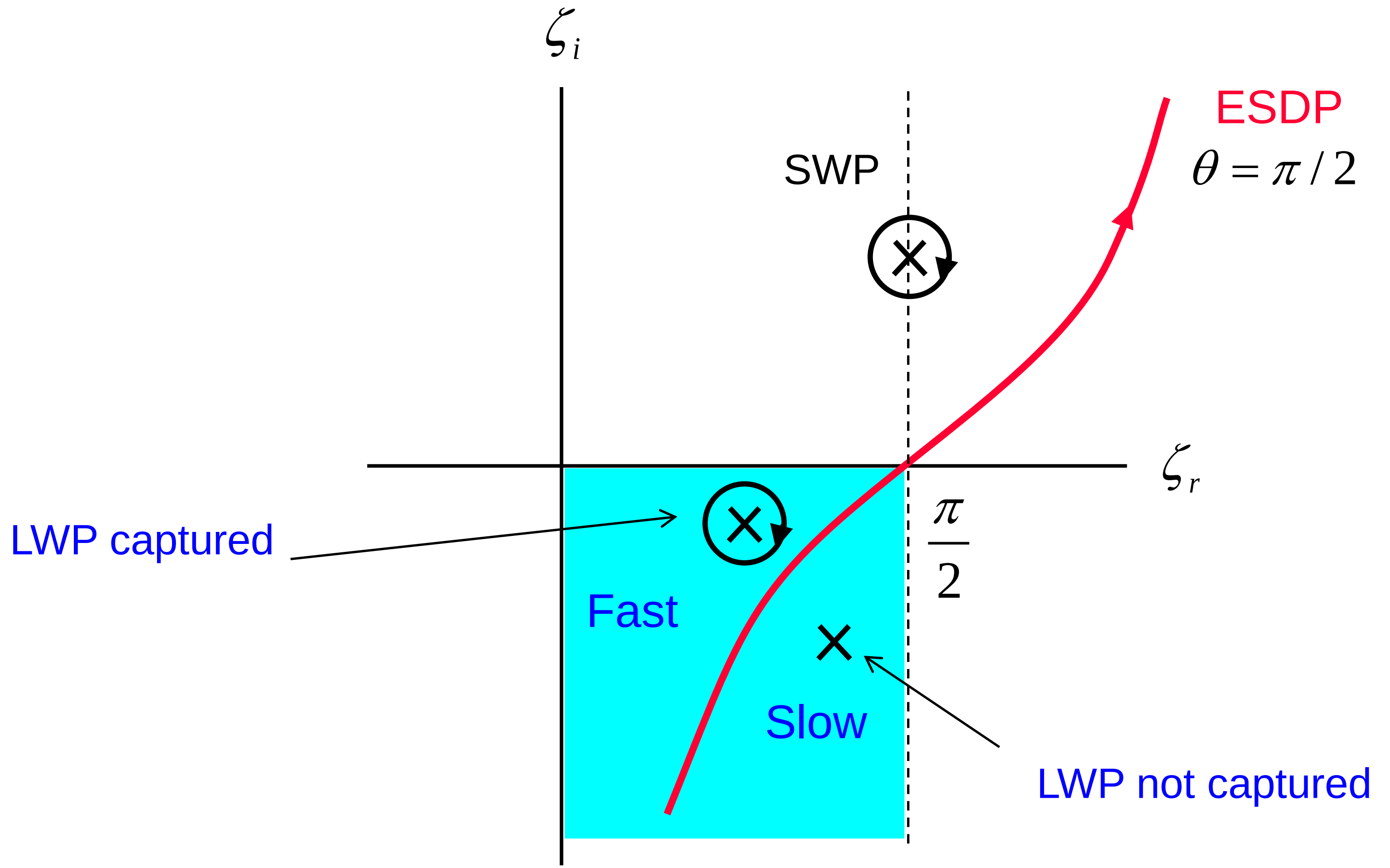
Fast-wave region: $\frac{\beta}{k_0} < 1$ $\quad \sin\zeta_{rp}\cosh\zeta_{ip} < 1$

Slow-wave region: $\frac{\beta}{k_0} > 1$ $\quad \sin\zeta_{rp}\cosh\zeta_{ip} > 1$

Compare with ESDP: $\sin\zeta_r\cosh\zeta_i = 1$

ESDP (cont.)

The ESDP thus establishes that for fields on the interface, a leaky-wave pole is physical (captured) if it is a fast wave.



SDP in k_x Plane

We now examine the shape of the SDP in the k_x plane.

$$
\begin{aligned}
k_x &= k_0 \sin\zeta \\
&= k_0 \sin(\zeta_r + j\zeta_i)
\end{aligned}
$$

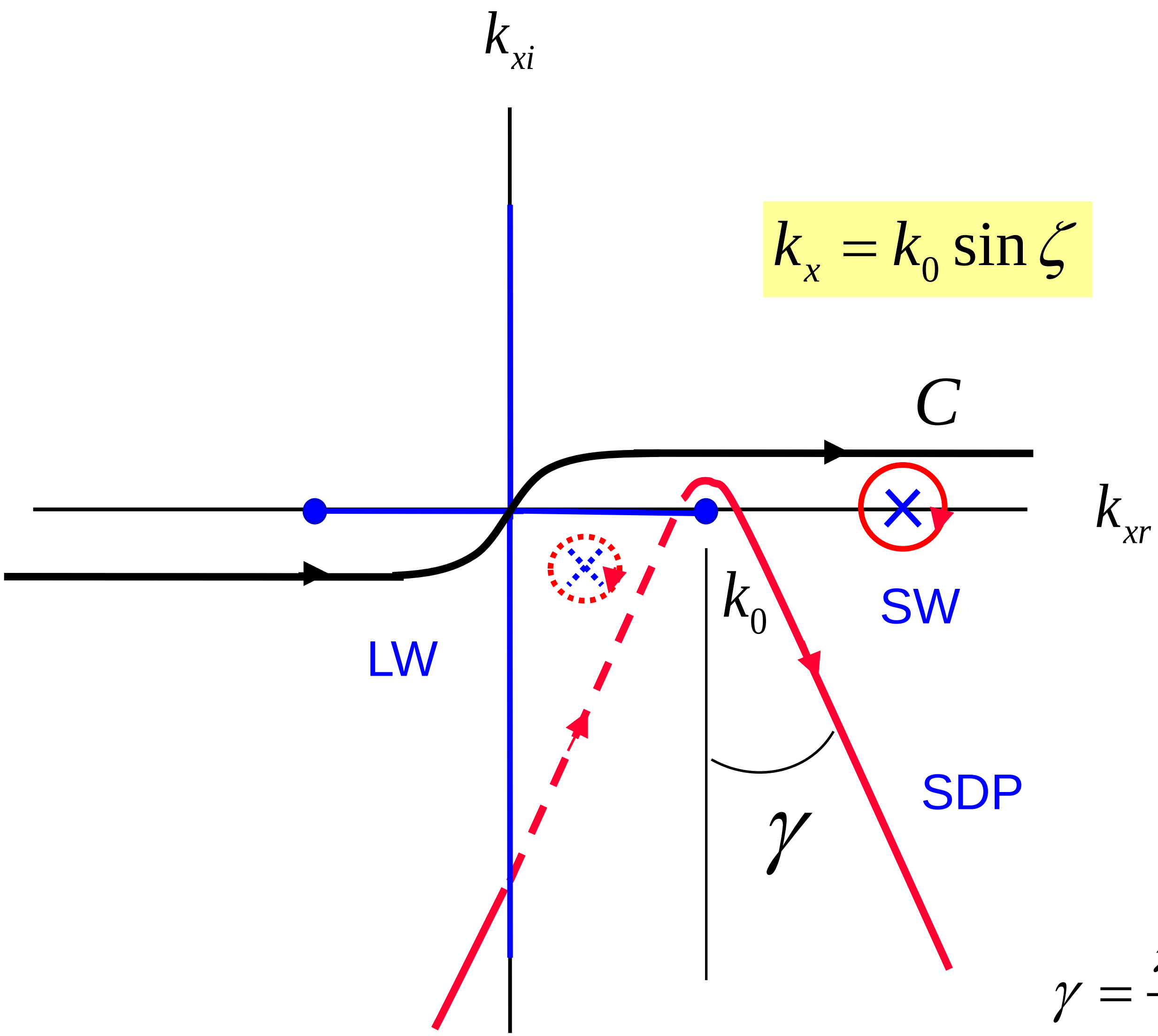
so that

$$
\begin{aligned}
k_{xr} &= k_0 \sin\zeta_r \cosh\zeta_i \\
k_{xi} &= k_0 \cos\zeta_r \sinh\zeta_i
\end{aligned}
$$

SDP: $\cos(\zeta_r - \theta)\cosh\zeta_i = 1$

The above equations allow us to numerically plot the shape of the SDP in the k_x plane.

SDP in k_x Plane (cont.)



$$k_x = k_0 \sin \zeta$$

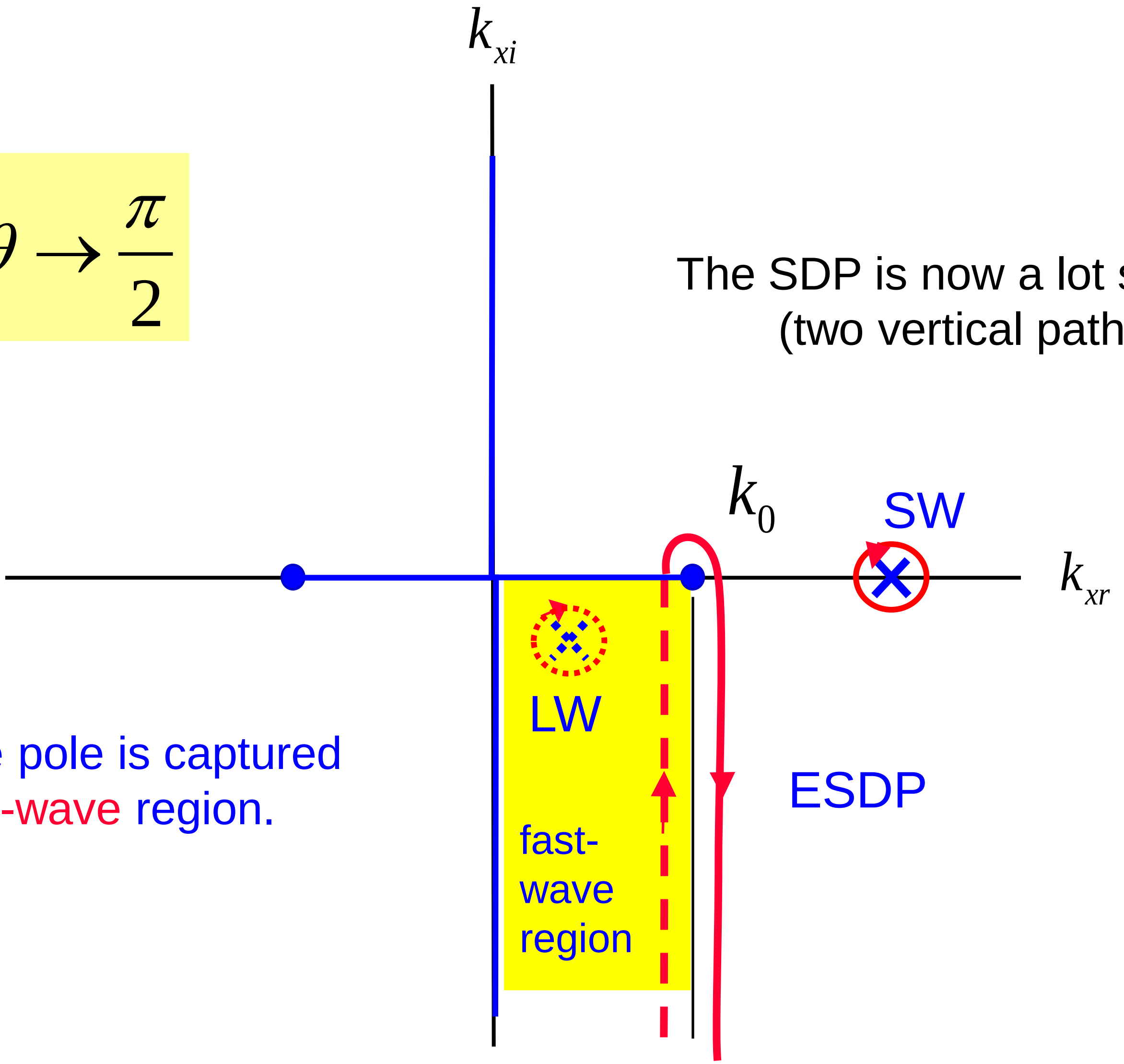
$$\gamma = \frac{\pi}{2} - \theta$$

(Please see the appendix for a proof.)

Fields on Interface

$$\theta \to \frac{\pi}{2}$$

The SDP is now a lot simpler
(two vertical paths)!

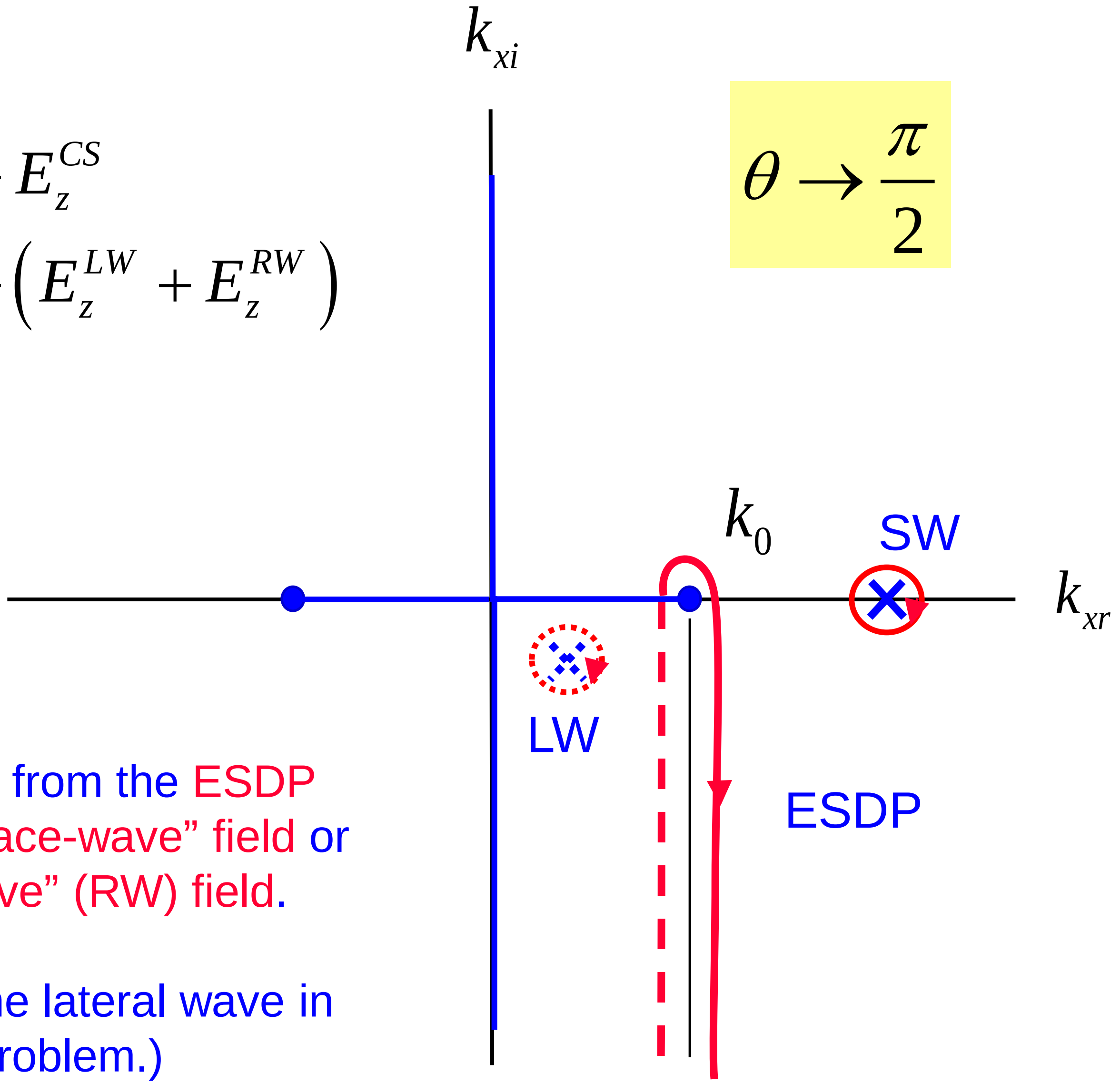


The leaky-wave pole is captured
if it is in the fast-wave region.

Fields on Interface (cont.)

$$E_z = E_z^{SW} + E_z^{CS}$$
$$= E_z^{SW} + \left(E_z^{LW} + E_z^{RW}\right)$$

$$\theta \to \frac{\pi}{2}$$



The contribution from the ESDP is called the "space-wave" field or the "residual-wave" (RW) field.

(It is similar to the lateral wave in the half-space problem.)

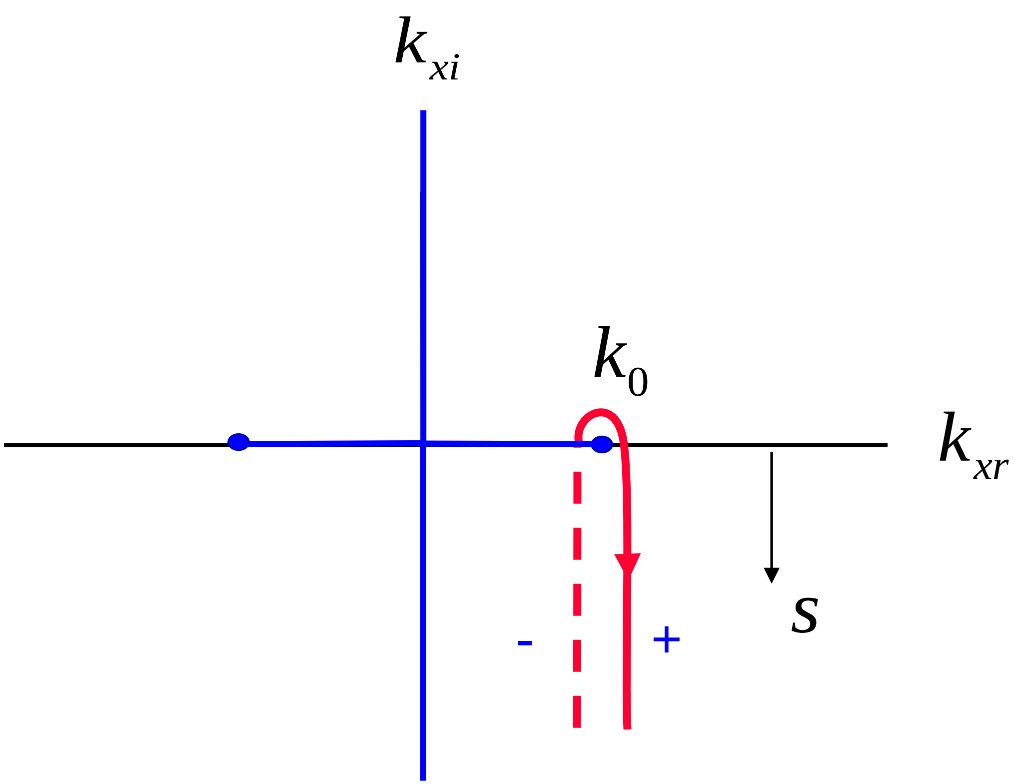
Asymptotic Evaluation of "Residual-Wave" Field

$$E_z^{RW} = \int_{EDSP} F(k_x) e^{-jk_x x} dk_x \qquad (y=0)$$

Use

$$k_x = k_0 - js$$

$$dk_x = -j\, ds$$



Asymptotic Evaluation of "Residual-Wave" Field (cont.)

$$E_z^{RW} = -je^{-jk_0x}\int_0^{+\infty} F^+\left(k_0 - js\right)e^{-sx}ds$$
$$+\left(-je^{-jk_0x}\right)\int_{\infty}^{0} F^-\left(k_0 - js\right)e^{-sx}ds$$

$$E_z^{RW} = -je^{-jk_0x}\int_0^{+\infty} F^+\left(k_0 - js\right)e^{-sx}ds$$
$$-\left(-je^{-jk_0x}\right)\int_0^{\infty} F^-\left(k_0 - js\right)e^{-sx}ds$$

Define $H\left(s\right) \equiv F^+\left(s\right) - F^-\left(s\right)$

Asymptotic Evaluation of "Residual-Wave" Field (cont.)

Then

$$E_z^{RW} = -je^{-jk_0 x} \int_0^{\infty} H(s) e^{-sx} ds$$

$$\Omega = x \quad \text{for} \quad x \to \infty$$

Assume $H(s) \sim A s^{\alpha}$ as $s \to 0$

We then have

Watson's lemma (alternative form):

$$E_z^{RW} \sim -je^{-jk_0 x} \left[\frac{A\Gamma(\alpha+1)}{x^{\alpha+1}} \right]$$

Asymptotic Evaluation of "Residual-Wave" Field (cont.)

It turns out that for the line-source problem at an interface,

$$\alpha = 1/2$$

Hence

$$E_z^{RW} \sim -j\,A\,\Gamma\left(\frac{3}{2}\right)\left[\frac{e^{-jk_0 x}}{x^{3/2}}\right]$$

Note that the wavenumber is that of free space.

Note: For a dipole source we have $E_z^{RW} \sim A_1 \dfrac{e^{-jk_0\rho}}{\rho^2}$

Discussion of Asymptotic Methods

We have now seen two ways to asymptotically evaluate the fields on an interface as $x \to \infty$ for a line source on a grounded substrate:

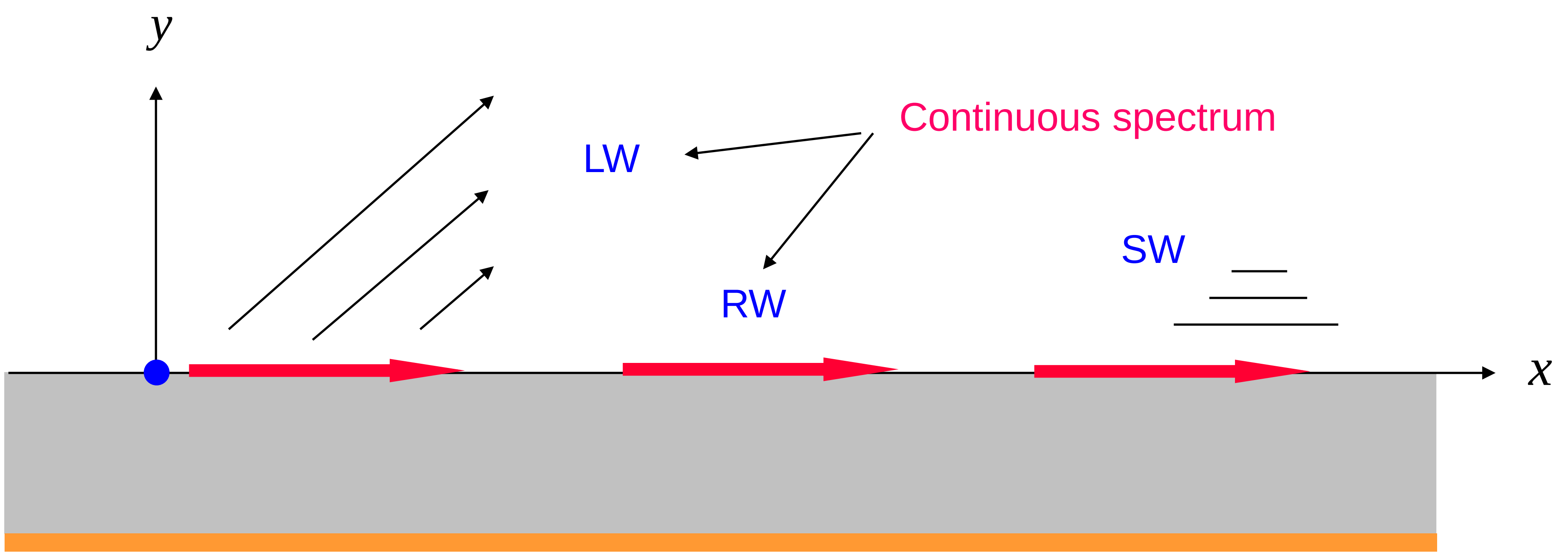
1) Steepest-descent (ζ) plane

There are no branch points in the steepest-descent plane. The function $f(\zeta)$ is analytic at the saddle point $\zeta_0 = \theta = \pi/2$, but is zero there. The fields on the interface correspond to a higher-order saddle-point evaluation.

2) Wavenumber (k_x) plane

The SDP becomes an integration along a vertical path that descends from the branch point at $k_x = k_0$. The integrand is not analytic at the endpoint of integration (branch point) since there is a square-root behavior at the branch point. Watson's lemma is used to asymptotically evaluate the integral.

Summary of Waves



$$E_z^{LW} = A_{LW}\, e^{-jk_x^{LW} x}$$

$$k_x^{LW} = \beta^{LW} - j\alpha^{LW}$$

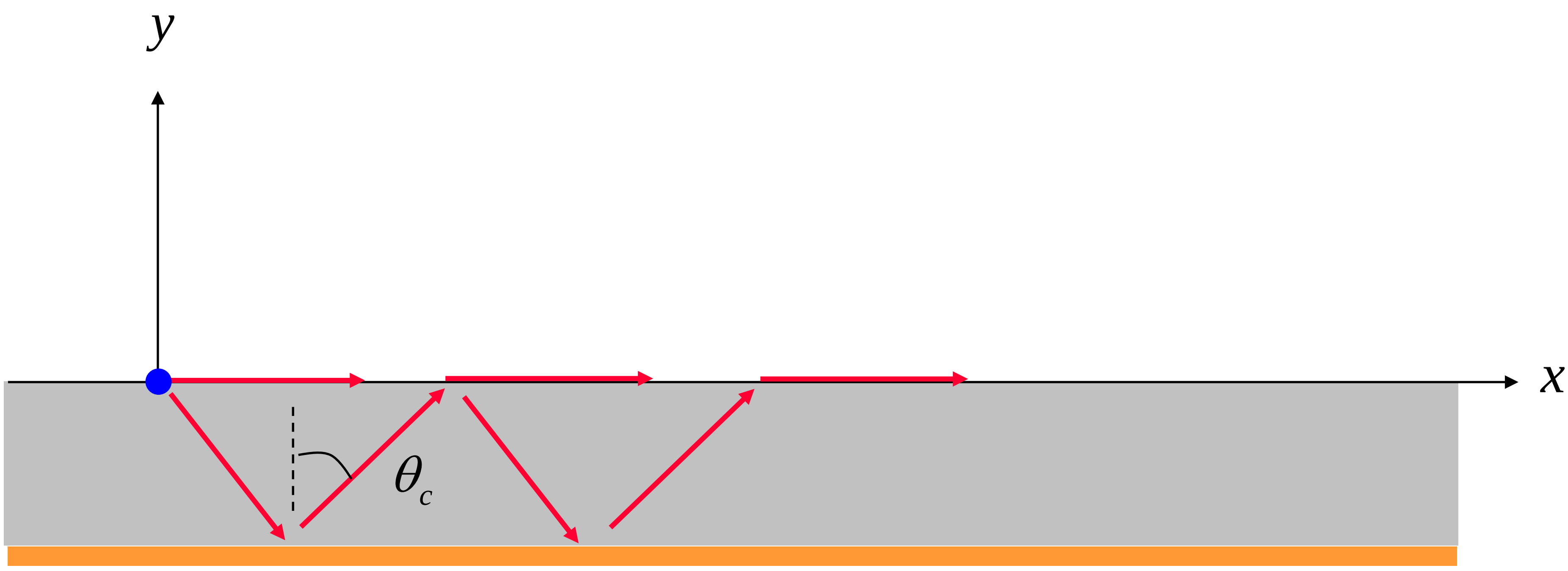
$$E_z^{RW} \sim A_{RW} \left[\frac{e^{-jk_0 x}}{x^{3/2}} \right]$$

$$E_z^{SW} = A_{SW}\, e^{-jk_x^{SW} x}$$

$$k_x^{SW} = \beta^{SW}$$

Interpretation of RW Field

The residual-wave (RW) field is actually a sum of lateral-wave fields.



Appendix: Proof of Angle Property

Proof of angle property:

$$\gamma = \frac{\pi}{2} - \theta$$

$$\tan\gamma = \frac{(k_{xr} - k_0)}{-k_{xi}} \sim -\frac{k_{xr}}{k_{xi}}$$

$$\sim -\tan\zeta_r \qquad (\zeta_i \to \infty)$$

The last identity follows from

$$k_{xr} = k_0 \sin\zeta_r \cosh\zeta_i$$
$$k_{xi} = k_0 \cos\zeta_r \sinh\zeta_i$$

$$\Rightarrow \quad \frac{k_{xr}}{k_{xi}} \sim \tan\zeta_r$$

Hence $\gamma \sim -\zeta_r$ or $\gamma \sim \pi - \zeta_r$

Proof (cont.)

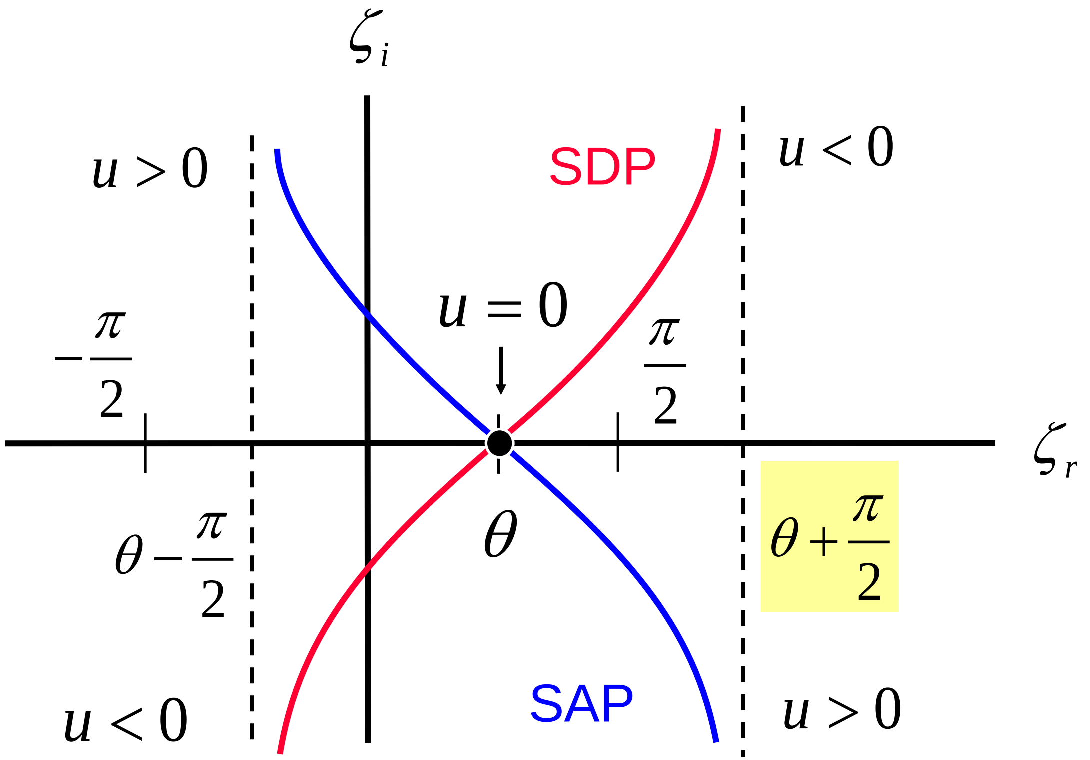
On SDP:

As

$$\zeta_i \to \infty$$

$$\zeta_r \to \theta + \frac{\pi}{2}$$

(the asymptote)



Hence $\gamma \sim -\left(\theta + \frac{\pi}{2}\right)$ or $\gamma \sim \pi - \left(\theta + \frac{\pi}{2}\right)$

$$= \frac{\pi}{2} - \theta$$

Proof (cont.)

To see which choice is correct:

ESDP: $\theta = \dfrac{\pi}{2}$

In the k_x plane, this corresponds to a vertical line for which $\gamma = 0$

Hence $$\gamma = \frac{\pi}{2} - \theta$$