ECE 6341

Spring 2016

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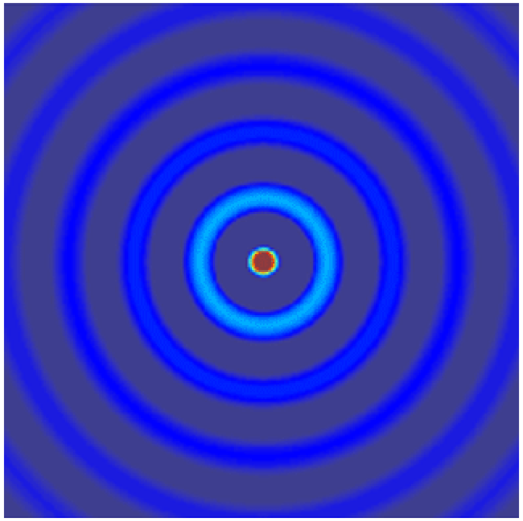
ECE Dept.

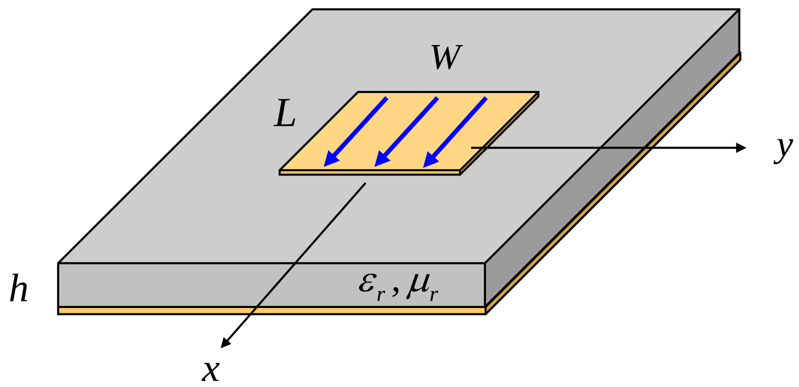


Notes 41

Patch Antenna

In this set of notes we do the following:

- Find the field E_x produced by the patch current on the interface
- Find the field E_z inside the substrate
- Find the voltage between the patch and the ground plane
- Find the input impedance of the patch (when fed by a probe)

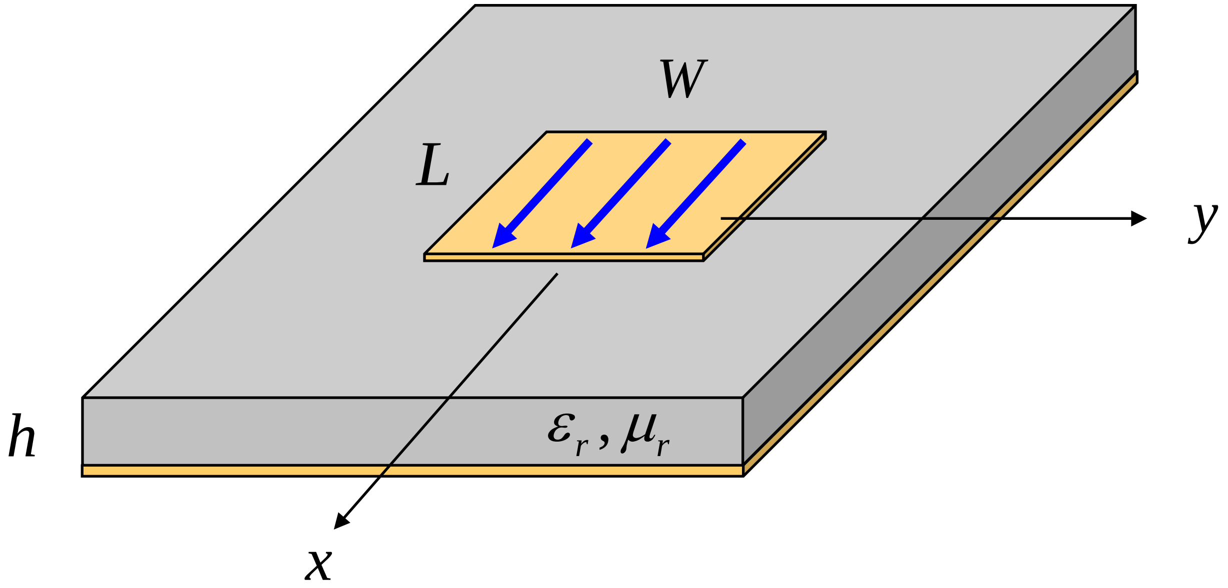


Assume that the patch current has the following form:

$$J_{sx}(x,y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Calculate the Field E_x

Find $E_x(x, y, 0)$



Dominant (1,0) mode:

$$J_{sx}(x, y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Field E_x (cont.)

Recall that

$$\tilde{E}_x = \tilde{G}_{xx} \, \tilde{J}_{sx}$$

$$\tilde{G}_{xx}\left(k_x, k_y; z, z'\right) = -\frac{1}{k_t^2}\left[k_x^2 \, V_i^{TM}\left(z, z'\right) + k_y^2 \, V_i^{TE}\left(z, z'\right)\right]$$

In this problem

$$z' = 0$$

$$z = 0$$

Field E_x (cont.)

$$J_{sx}(x,y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

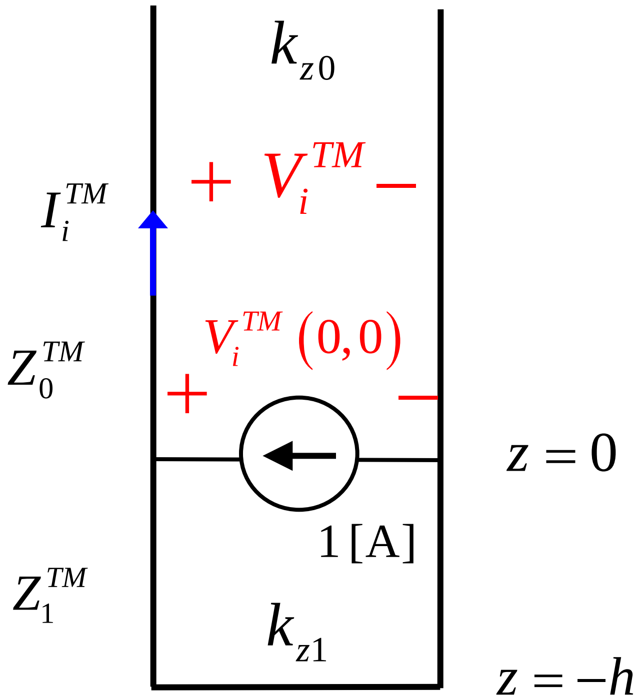
$$\tilde{J}_{sx}\left(k_x, k_y\right) = \frac{1}{W}\int_{L/2}^{L/2}\cos\left(\frac{\pi y}{L}\right)e^{jk_x x}\,dx\int_{-W/2}^{W/2}e^{jk_y y}\,dy$$

$$\tilde{J}_{sx}\left(k_x, k_y\right) = \frac{\pi L}{2}\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y\frac{W}{2}\right)$$

Field E_x (cont.)

TM_z :

$V_i^{TM}(0,0)$



$$k_{z0} = \left(k_0^2 - k_x^2 - k_y^2\right)^{1/2}$$

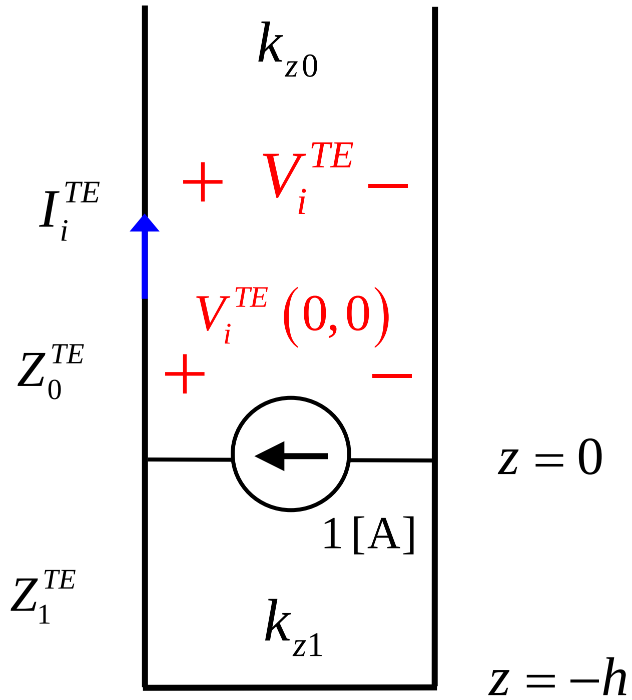
$$k_{z1} = \left(k_1^2 - k_x^2 - k_y^2\right)^{1/2}$$

$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} = \eta_0\left(\frac{k_{z0}}{k_0}\right) \qquad Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} = \frac{\eta_0}{\varepsilon_r}\left(\frac{k_{z1}}{k_0}\right)$$

Field E_x (cont.)

TE$_z$:

$V_i^{TE}(0,0)$



$$k_{z0} = \left(k_0^2 - k_x^2 - k_y^2\right)^{1/2}$$

$$k_{z1} = \left(k_1^2 - k_x^2 - k_y^2\right)^{1/2}$$

$$Z_0^{TE} = \frac{\omega\mu_0}{k_{z0}} = \frac{\eta_0}{\left(k_{z0} / k_0\right)}$$

$$Z_1^{TE} = \frac{\omega\mu_1}{k_{z1}} = \frac{\eta_0 \mu_r}{\left(k_{z1} / k_0\right)}$$

Field E_x (cont.)

At $z = 0$:

$$V_i(0,0) = Z_{in}$$
$$= Y_{in}^{-1} = \left(Y_{in}^{+} + Y_{in}^{-}\right)^{-1}$$
$$= \left[Y_0 - jY_1 \cot\left(k_{z1}h\right)\right]^{-1}$$

Hence

$$V_i^{TM}(0,0) = \frac{1}{D_m(k_t)}$$
$$V_i^{TE}(0,0) = \frac{1}{D_e(k_t)}$$

$$D_m(k_t) \equiv Y_0^{TM} - jY_1^{TM} \cot\left(k_{z1}h\right)$$
$$D_e(k_t) \equiv Y_0^{TE} - jY_1^{TE} \cot\left(k_{z1}h\right)$$

Field E_x (cont.)

$$\tilde{G}_{xx}\left(k_x,k_y;z,z'\right)=-\frac{1}{k_t^2}\left[k_x^2\,V_i^{TM}\left(z,z'\right)+k_y^2\,V_i^{TE}\left(z,z'\right)\right]$$

$$V_i^{TM}\left(0,0\right)=\frac{1}{D_m\left(k_t\right)}$$

$$V_i^{TE}\left(0,0\right)=\frac{1}{D_e\left(k_t\right)}$$

$$D_m\left(k_t\right)=Y_0^{TM}-jY_1^{TM}\cot\left(k_{z1}h\right)$$

$$D_e\left(k_t\right)=Y_0^{TE}-jY_1^{TE}\cot\left(k_{z1}h\right)$$

Hence, we have:

$$\tilde{E}_x\left(k_x,k_y,0\right)=\tilde{J}_{sx}\left(k_x,k_y\right)\left[\left(-\frac{1}{k_t^2}\right)\left[\frac{k_x^2}{D_m\left(k_t\right)}+\frac{k_y^2}{D_e\left(k_t\right)}\right]\right]$$

Field E_x (cont.)

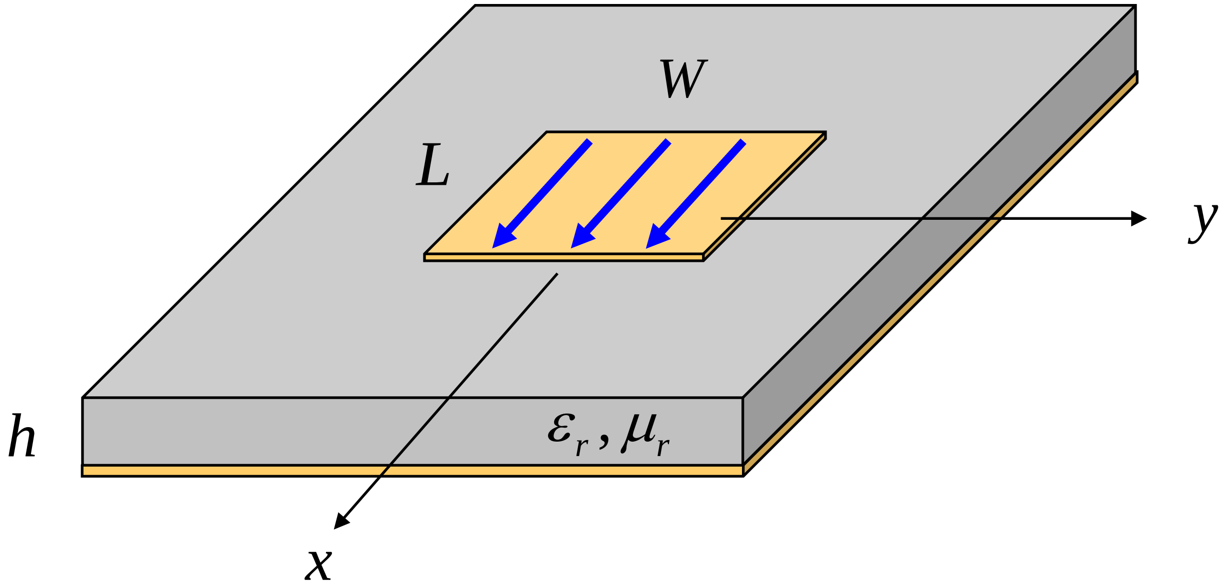
Taking the inverse Fourier transform, we have

$$E_x(x,y,0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -\frac{1}{k_t^2} \tilde{J}_{sx}(k_x,k_y) \cdot \left[\frac{k_x^2}{D_m(k_t)} + \frac{k_y^2}{D_e(k_t)} \right] e^{-j(k_x x + k_y y)} \, dk_x dk_y$$

$$\tilde{J}_{sx}(k_x,k_y) = \frac{\pi L}{2} \left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2} \right] \operatorname{sinc}\left(k_y \frac{W}{2}\right)$$

Field E_z

Find $E_z\,(x,y,z)$ inside the substrate $(-h < z < 0)$



Dominant (1,0) mode:

$$J_{sx}\left(x,y\right)=\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Field E_z (cont.)

From Notes 40 we have:

$$\tilde{E}_z\left(k_x, k_y, z\right) = \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right) I^{TM}\left(z\right)$$

$$= \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right) I_i^{TM}\left(z\right)\left(-\tilde{J}_s \cdot \underline{\hat{u}}\right)$$

$$= \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right) I_i^{TM}\left(z\right)\left(-\tilde{J}_{sx}\right)\cos\overline{\phi}$$

$$\frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_t\right) I_i^{TM}\left(z\right)\left(-\tilde{J}_{sx}\right)\left(\frac{k_x}{k_t}\right)$$

$$= \frac{-1}{\omega\varepsilon_0\varepsilon_r}\left(k_x\right) I_i^{TM}\left(z\right)\left(-\tilde{J}_{sx}\right)$$

Field E_z (cont.)

Hence, in the space domain we have

$$E_z(x,y,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\omega\varepsilon_0\varepsilon_r}(k_x)\, \tilde{J}_{sx}\, I_i^{TM}(z)\, e^{-j(k_x x + k_y y)} dk_x dk_y$$

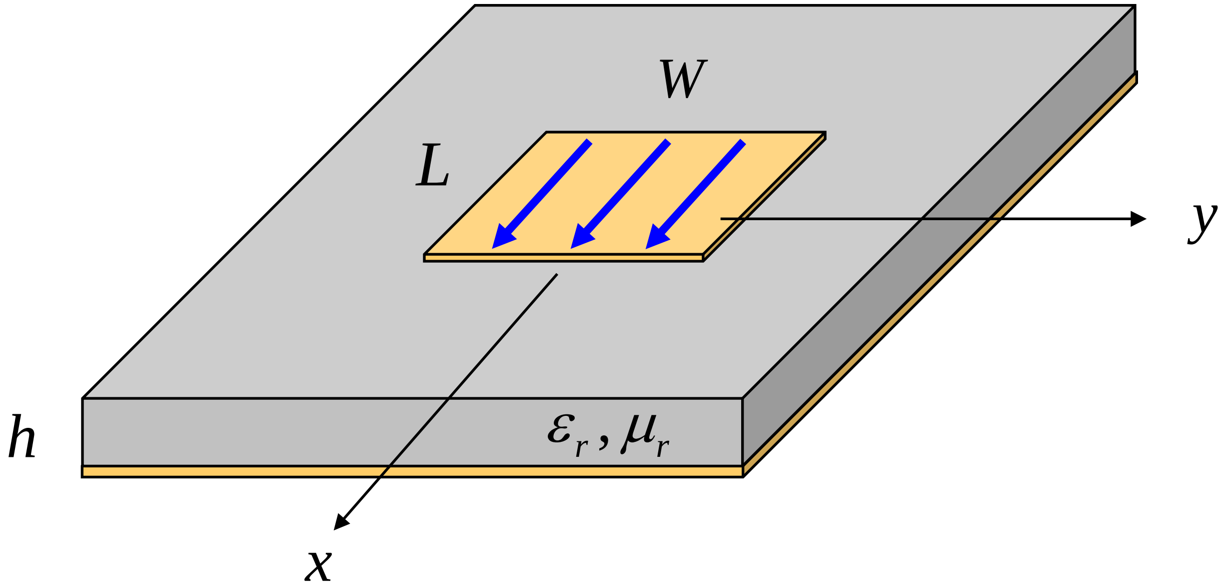
Note: Only TM_z waves contribute to the vertical electric field.

From Notes 40:

$$I_i^{TM}(z) = -\frac{1}{D_m(k_t)} \left(\frac{1}{j Z_1^{TM}} \right) \left(\frac{\cos(k_{z1}(z+h))}{\sin(k_{z1}h)} \right)$$

Voltage

Find $V(x,y)$ between the patch and the ground plane.



Dominant (1,0) mode:

$$J_{sx}(x,y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Voltage (cont.)

$$V(x,y)=\int_{0}^{-h}E_z(x,y,z)\,dz=-\int_{-h}^{0}E_z(x,y,z)\,dz$$

Using the result from the previous calculation for E_z, we have:

$$V(x,y)=-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega\varepsilon_0\varepsilon_r}(k_x)\,\tilde{J}_{sx}\,F(k_t)\,e^{-j(k_x x+k_y y)}dk_x dk_y$$

where

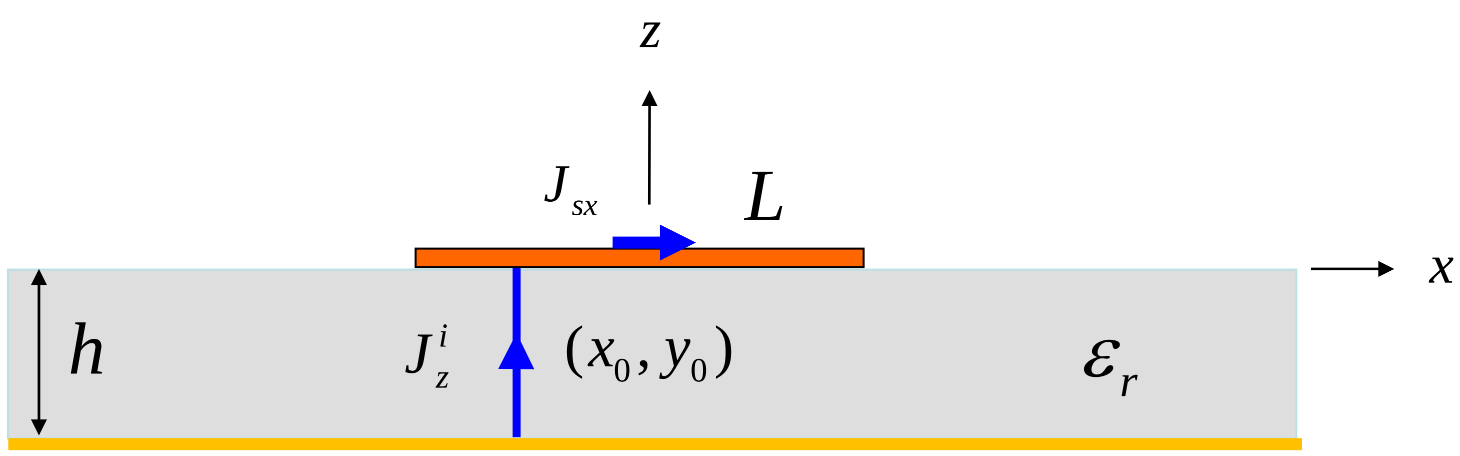
$$F(k_t)\equiv\int_{-h}^{0}I_i^{TM}(z)\,dz$$

From Notes 40:

$$F(k_t)=-\frac{1}{D_m(k_t)}\left(\frac{1}{j\,Z_1^{TM}}\right)\left(\frac{1}{k_{z1}}\right)$$

Input Impedance

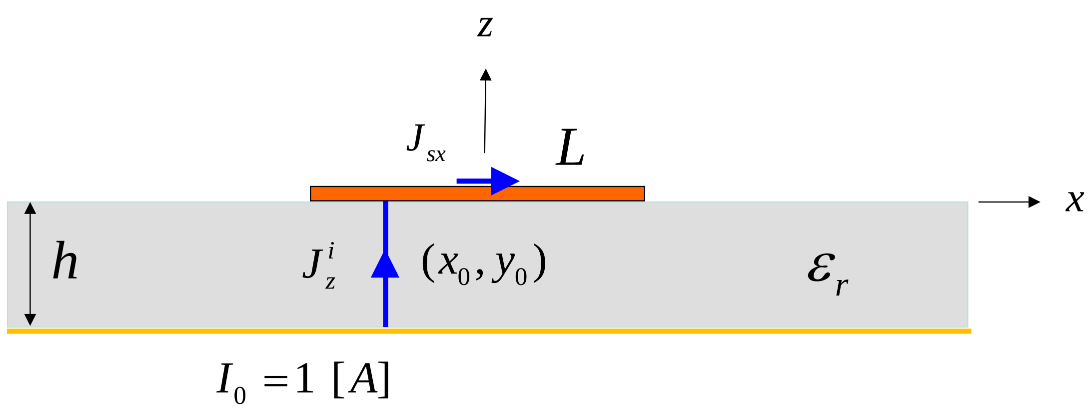
Find the input impedance Z_{in} (x_0, y_0) of the probe-fed patch antenna



$I_0 = 1\ [A]$

The probe is viewed as an impressed current.

Input Impedance (cont.)



Set $E_x = 0 \quad (x, y) \in S,\quad S$ is the patch surface

$$E_x\left[J_{sx}\right] + E_x\left[J_z^i\right] = 0, \quad (x, y) \in S$$

This is the "Electric Field Integral Equation (EFIE)"

Input Impedance (cont.)

Assume:

$$J_{sx}(x,y) = A_x B_{sx}(x,y)$$

A_x is an unknown amplitude.

$$B_{sx}(x,y) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

The EFIE is then $A_x E_x\left[B_{sx}\right] + E_x\left[J_z^i\right] = 0, \quad (x,y) \in S$

Pick a "testing" function $T(x,y)$:

$$\int_S T(x,y)\left\{A_x E_x\left[B_{sx}\right] + E_x\left[J_z^i\right]\right\} dS = 0$$

or

$$A_x \int_S T(x,y) E_x\left[B_{sx}\right] dS + \int_S T(x,y) E_x\left[J_z^i\right] dS = 0$$

Input Impedance (cont.)

Galerkin's Method: $T(x,y) = B_{sx}(x,y)$

(The testing function is the same as the basis function.)

Hence, we have:

$$A_x \int_S B_{sx}(x,y) E_x[B_{sx}]\, dS + \int_S B_{sx}(x,y) E_x[J_z^i]\, dS = 0$$

The solution for the unknown amplitude coefficient A_x is then

$$A_x = -\frac{\displaystyle\int_S B_{sx}(x,y) E_x[J_z^i]\, dS}{\displaystyle\int_S B_{sx}(x,y) E_x[B_{sx}]\, dS} = -\frac{\langle J_z^i, B_{sx} \rangle}{\langle B_x, B_{sx} \rangle}$$

$$\langle J_z^i, B_{sx} \rangle \equiv \int_S E_x[J_z^i] B_{sx}(x,y)\, dS \qquad \langle B_{sx}, B_{sx} \rangle \equiv \int_S E_x[B_{sx}] B_{sx}(x,y)\, dS$$

Input Impedance (cont.)

The input impedance is calculated as:

$$Z_{in} = \frac{V(x_0, y_0)}{I_0} = V(x_0, y_0)$$

From linearity, we have

$$Z_{in} = A_x V_N(x_0, y_0)$$

where

$$V_N(x_0, y_0) \equiv V(x_0, y_0)\Big|_{A_x = 1}$$

From the last example:

$$V_N(x_0, y_0) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\omega \varepsilon_0 \varepsilon_r} (k_x)\, \tilde{B}_{sx}(x_0, y_0) F(k_t)\, e^{-j(k_x x_0 + k_y y_0)} dk_x dk_y$$

Input Impedance (cont.)

Next, we return to the calculation of A_x:

$$A_x = -\frac{\int_S B_{sx}(x,y)\, E_x\left[J_z^i\right] dS}{\int_S B_{sx}(x,y)\, E_x\left[B_{sx}\right] dS} = -\frac{\left\langle J_z^i, B_{sx}\right\rangle}{\left\langle B_{sx}, B_{sx}\right\rangle}$$

From reciprocity:

$$\left\langle J_z^i, B_{sx}\right\rangle = \left\langle B_{sx}, J_z^i\right\rangle = \int_{-h}^{0} E_z^N\left(x_0, y_0, z\right) I_0\, dz = -V_N\left(x_0, y_0\right)$$

From the formula for the field E_x:

Note: $\tilde{B}_{sx}\left(-k_x, -k_y\right) = \tilde{B}_{sx}\left(k_x, k_y\right)$

$$Z_{xx} \equiv -\left\langle B_{sx}, B_{sx}\right\rangle = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{G}_{xx}\left(k_x, k_y\right) \tilde{B}_{sx}^2\left(k_x, k_y\right) dk_x dk_y$$

Input Impedance (cont.)

Summary

$$Z_{in} = A_x V_N(x_0, y_0)$$

$$V_N(x_0, y_0) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\omega \varepsilon_0 \varepsilon_r} (k_x) \tilde{B}_{sx}(k_x, k_y) F(k_t)\, e^{-j(k_x x_0 + k_y y_0)} dk_x dk_y$$

$$A_x = -\frac{V_N(x_0, y_0)}{Z_{xx}}$$

$$Z_{xx} \equiv -\langle B_{sx}, B_{sx} \rangle = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}_{xx}(k_x, k_y)\, \tilde{B}_{sx}^2(k_x, k_y) dk_x dk_y$$

$$\tilde{B}_{sx}(k_x, k_y) = \frac{\pi L}{2} \left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2} \right] \operatorname{sinc}\left(k_y \frac{W}{2}\right)$$

$$F(k_t) = -\frac{1}{D_m(k_t)} \left(\frac{1}{j Z_1^{TM}} \right) \left(\frac{1}{k_{z1}} \right)$$

Input Impedance (cont.)

Converting to polar coordinates, we have:

$$Z_{xx} = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\infty} \tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{B}_{sx}^2\left(k_t, \overline{\phi}\right) k_t \, dk_t d\overline{\phi}$$

$$V_N\left(x_0, y_0\right) = -\frac{1}{\left(2\pi\right)^2}\left(2\right)\left(-2j\right) \int_0^{\pi/2} \int_0^{\infty} \frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_x\right) \tilde{B}_{sx}\left(k_x, k_y\right) F\left(k_t\right)$$
$$\cdot \sin\left(k_x x_0\right) \cos\left(k_x y_0\right) k_t \, dk_t d\overline{\phi}$$

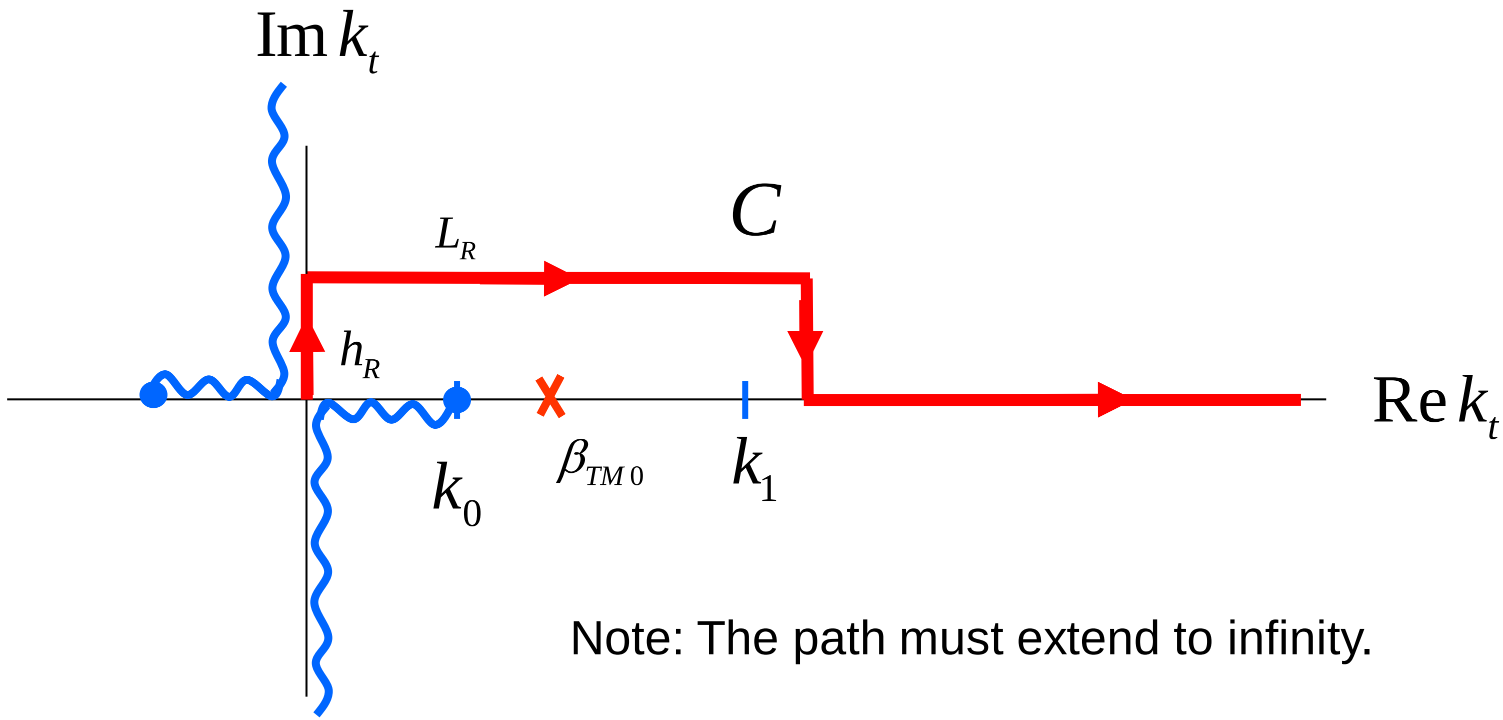
Note:
$$e^{-jk_x x_0} e^{-jk_y y_0} - e^{+jk_x x_0} e^{-jk_y y_0} - e^{+jk_x x_0} e^{+jk_y y_0} + e^{-jk_x x_0} e^{+jk_y y_0}$$
$$= e^{+jk_y y_0}\left(e^{-jk_x x_0} - e^{+jk_x x_0}\right) + e^{-jk_y y_0}\left(e^{+jk_x x_0} - e^{-jk_x x_0}\right)$$
$$= e^{+jk_y y_0}\left(-2j \sin\left(k_x x_0\right)\right) + e^{-jk_y y_0}\left(-2j \sin\left(k_x x_0\right)\right)$$
$$= 2\cos\left(k_y y_0\right)\left(-2j \sin\left(k_x x_0\right)\right)$$

Input Impedance (cont.)

The path of integration is shown below.

$$Z_{xx} = -\frac{1}{\pi^2} \int_0^{\pi/2} \int_C \tilde{G}_{xx}\left(k_t, \overline{\phi}\right) \tilde{B}_{sx}^2\left(k_t, \overline{\phi}\right) k_t \, dk_t d\overline{\phi}$$

$$V_N\left(x_0, y_0\right) = \frac{j}{\pi^2} \int_0^{\pi/2} \int_C \frac{1}{\omega \varepsilon_0 \varepsilon_r} \left(k_x\right) \tilde{B}_{sx}\left(k_x, k_y\right) F\left(k_t\right) \sin\left(k_x x_0\right) \cos\left(k_x y_0\right) k_t \, dk_t d\overline{\phi}$$



Note: The path must extend to infinity.

Input Impedance (cont.)

Improvement:

Add probe reactance to account for the stored magnetic energy near the metal probe.

$$Z_{probe} \approx jX_p$$

$$X_p = \eta_0 \mu_r \left(\frac{h}{\lambda_0} \right) \left[\ln\left(\frac{1}{a / \lambda_0} \right) - \gamma - \ln \pi - \ln \sqrt{\mu_r \varepsilon_r} \right]$$

$\gamma \doteq 0.57722$ (Euler's constant)

Input Impedance (cont.)

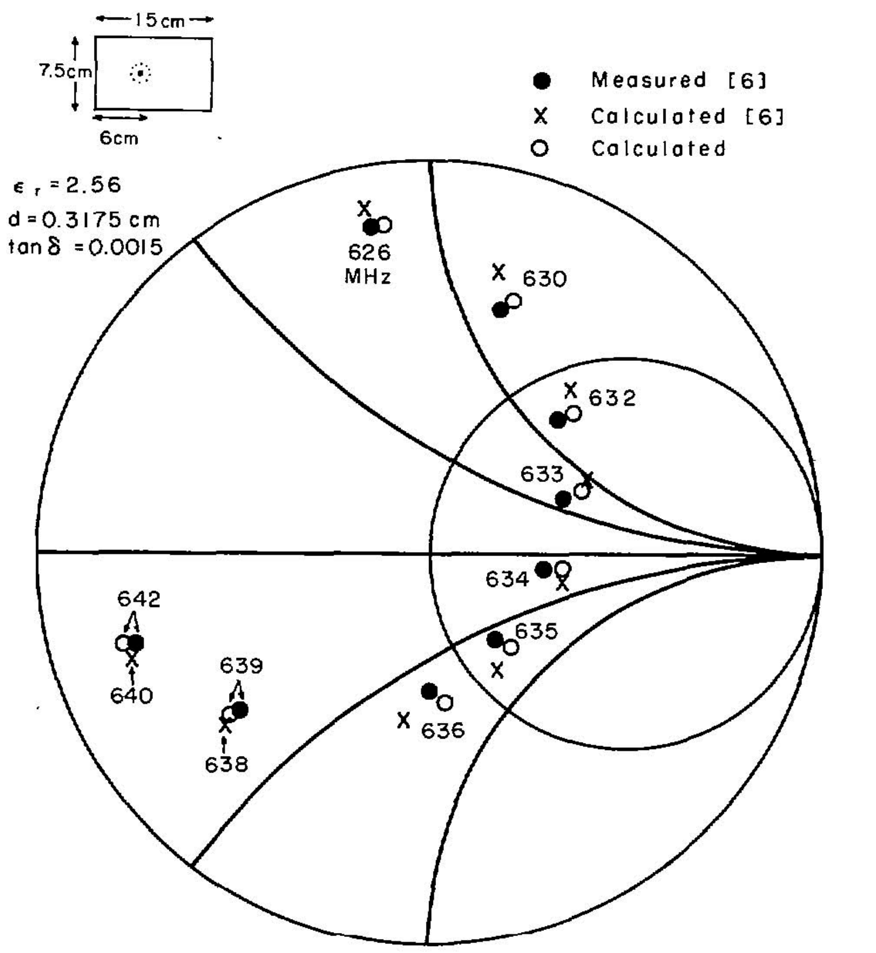


Fig. 5. Measured and calculated input impedance of a coax-fed microstrip antenna compared with calculations from [6].

D. M. Pozar, "Input impedance and mutual coupling of rectangular microstrip antennas," *IEEE Trans. Antennas Propagat.*, vol. AP-30. pp. 1191-1196, Nov. 1982.

[6] E. H. Newman and P. Tulyathan, "Analysis of microstrip antennas using moment methods," *IEEE Trans. Antennas Propagat.*, vol. AP-29. pp. 47-53, Jan. 1981.