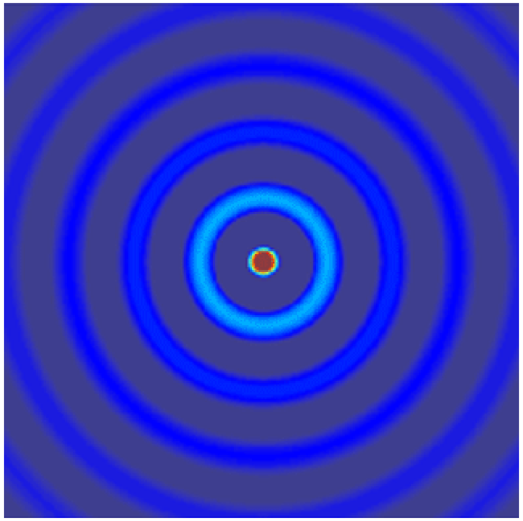
ECE 6341

Spring 2016

Prof. David R. Jackson
ECE Dept.

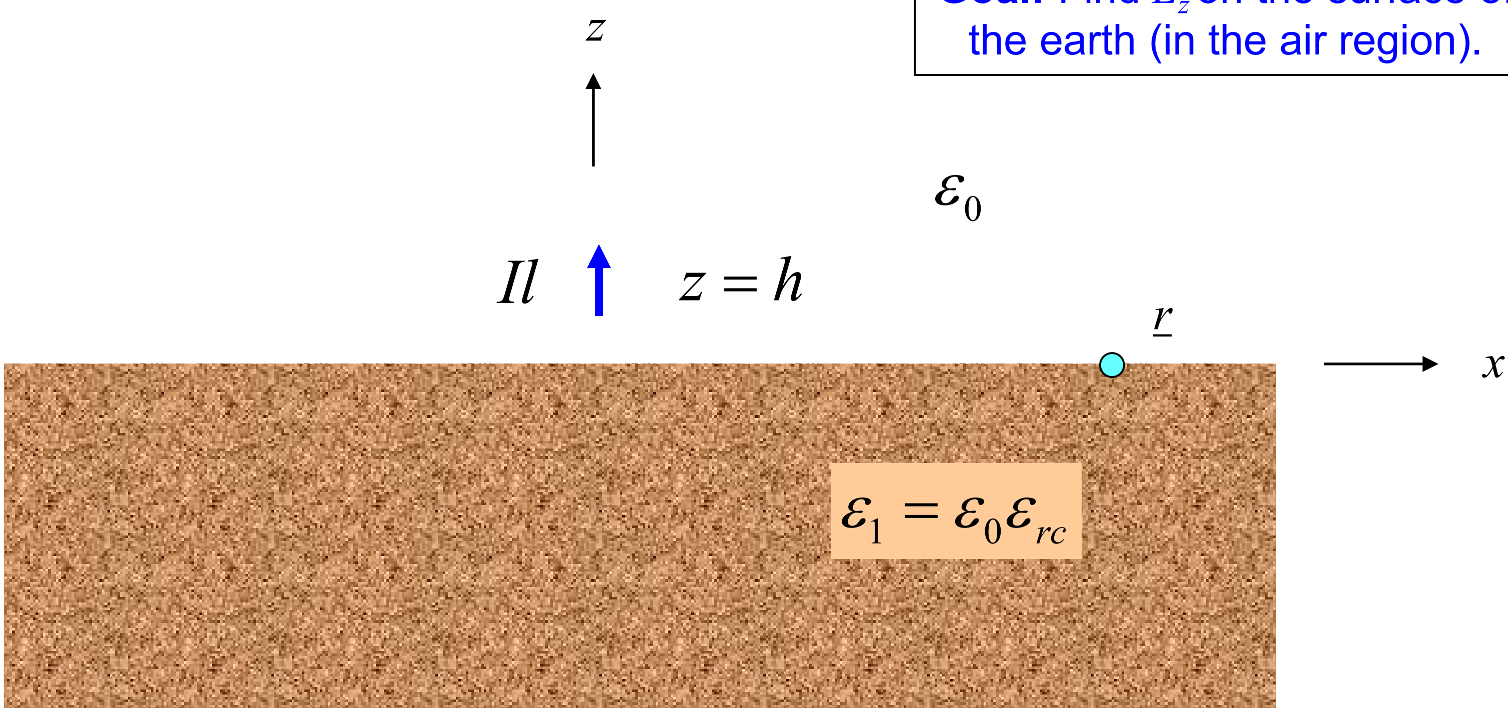


Notes 43

Sommerfeld Problem

In this set of notes we use SDI theory to solve the classical "Sommerfeld problem" of a vertical dipole over an semi-infinite earth.

Goal: Find E_z on the surface of the earth (in the air region).



Sommerfeld Problem (cont.)

Planar vertical electric current (from Notes 39):

$$V_s^{TM} = \left(\frac{k_t}{\omega \varepsilon_0}\right) \tilde{J}_{sz} \quad \left(\text{where } J_z\left(x,y,z\right) = J_{sz}\left(x,y\right)\delta(z-h)\right)$$

(the impressed source current)

For a vertical electric dipole of amplitude Il, we have

$$J_z(x,y) = Il\,\delta\left(x\right)\delta\left(y\right)\delta\left(z-h\right)$$

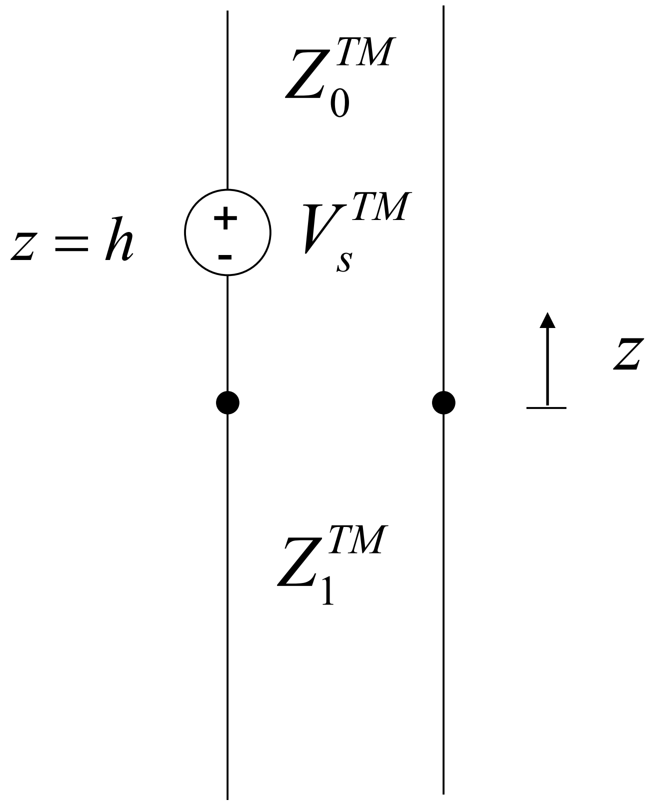
Hence $J_{sz}(x,y) = Il\,\delta\left(x\right)\delta\left(y\right)$

Therefore, we have

$$V_s^{TM} = \left(\frac{k_t}{\omega \varepsilon_0}\right)\left(Il\right)$$

Sommerfeld Problem (cont.)

TEN:



$$V_s^{TM} = Il\left(\frac{k_t}{\omega\varepsilon_0}\right)$$

The vertical electric dipole excites TM_z waves only.

Vertical Field

Find $E_z\,(x,y,z)$ inside the air region ($z > 0$).

$$\nabla \times \underline{H} = j\omega\varepsilon_0\,\underline{E}$$

$$E_z = \frac{1}{j\omega\varepsilon_0}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\tilde{E}_z = \frac{1}{j\omega\varepsilon_0}\left(-jk_x\tilde{H}_y + jk_y\tilde{H}_x\right)$$

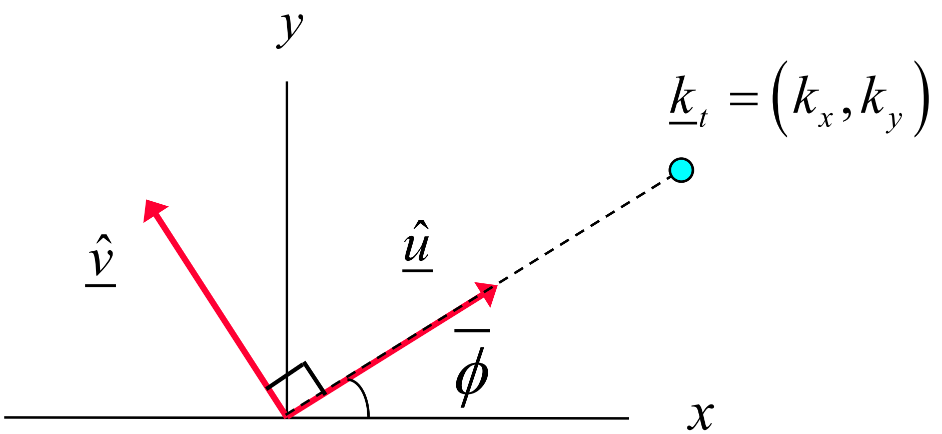
$$= \frac{1}{\omega\varepsilon_0}\left(-k_x\tilde{H}_y + k_y\tilde{H}_x\right)$$

Example (cont.)

$$\tilde{H}_x = \tilde{H}_u\left(\underline{\hat{u}}\cdot\underline{\hat{x}}\right) + \tilde{H}_v\left(\underline{\hat{v}}\cdot\underline{\hat{x}}\right)$$

$$= \tilde{H}_u\left(\cos\overline{\phi}\right) + \tilde{H}_v\left(-\sin\overline{\phi}\right)$$

$$= \tilde{H}_u\left(\frac{k_x}{k_t}\right) + \tilde{H}_v\left(-\frac{k_y}{k_t}\right)$$



$$\tilde{H}_y = \tilde{H}_u\left(\underline{\hat{u}}\cdot\underline{\hat{y}}\right) + \tilde{H}_v\left(\underline{\hat{v}}\cdot\underline{\hat{y}}\right)$$

$$\tilde{H}_u\left(\sin\overline{\phi}\right) + \tilde{H}_v\left(\cos\overline{\phi}\right)$$

$$= \tilde{H}_u\left(\frac{k_y}{k_t}\right) + \tilde{H}_v\left(\frac{k_x}{k_t}\right)$$

Example (cont.)

Hence

cancels

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0}\left(-k_x\left[\tilde{H}_u\left(\frac{k_y}{k_t}\right)+\tilde{H}_v\left(\frac{k_x}{k_t}\right)\right]+k_y\left[\tilde{H}_u\left(\frac{k_x}{k_t}\right)+\tilde{H}_v\left(-\frac{k_y}{k_t}\right)\right]\right)$$

or

$$\tilde{E}_z = \frac{1}{\omega\varepsilon_0}\left(\tilde{H}_v\left[-k_x^{\,2}-k_y^{\,2}\right]\frac{1}{k_t}\right)$$

or

$$\tilde{E}_z = \frac{-1}{\omega\varepsilon_0}\left(k_t\tilde{H}_v\right)$$

Sommerfeld Problem (cont.)

Hence

$$\tilde{E}_z\left(k_x, k_y, z\right) = \frac{-1}{\omega\varepsilon_0}\left(k_t\right) I^{TM}\left(z\right)$$

We use the Michalski normalized current function:

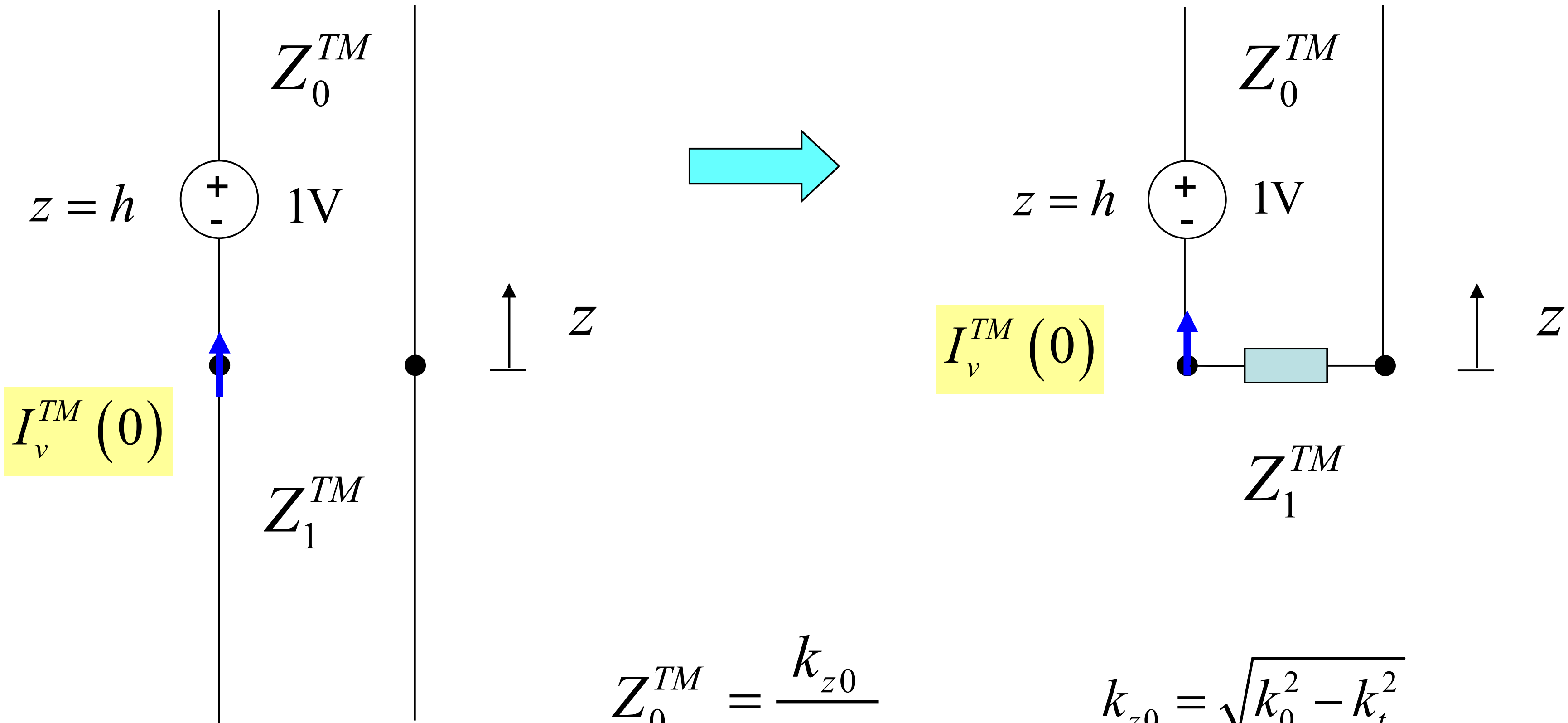
$$I^{TM}\left(z\right) = I_v^{TM}\left(z\right) V_s^{TM} = I_v^{TM}\left(z\right)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right]$$

(The v subscript indicates a 1V series source.)

We need to calculate the Michalski normalized current function at $z = 0$ (since we want the field on the surface of the earth).

Sommerfeld Problem (cont.)

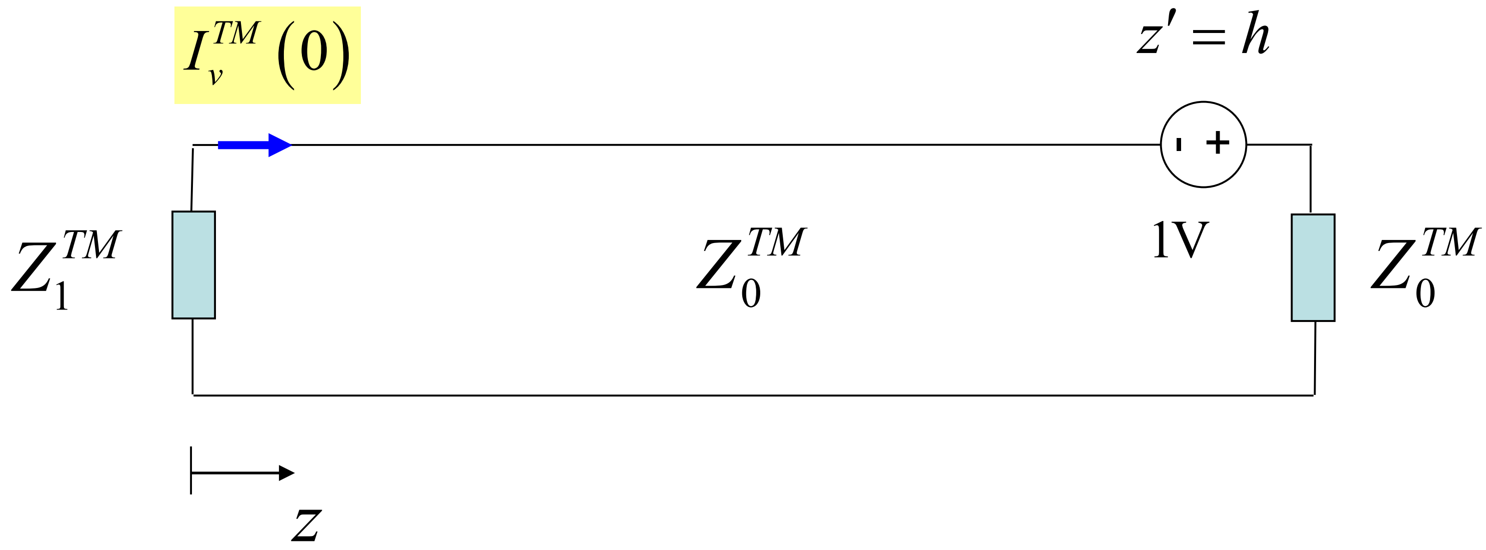
Calculation of the Michalski normalized current function



$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0} \qquad k_{z0} = \sqrt{k_0^2 - k_t^2}$$

$$Z_1^{TM} = \frac{k_{z1}}{\omega\varepsilon_1} \qquad k_{z1} = \sqrt{k_1^2 - k_t^2}$$

Sommerfeld Problem (cont.)



This figure shows how to calculate the Michalski normalized current function: it will be calculated later in these notes.

Sommerfeld Problem (cont.)

Return to the calculation of the field:

$$\tilde{E}_z\left(k_x, k_y, 0^+\right) = \frac{-1}{\omega\varepsilon_0}\left(k_t\right) I_v^{TM}(0)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right]$$

Hence we have

$$E_z\left(x, y, 0^+\right) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-1}{\omega\varepsilon_0}\left(k_t\right) I_v^{TM}(0)\left[Il\left(\frac{k_t}{\omega\varepsilon_0}\right)\right] e^{-j\left(k_x x + k_y y\right)}\, dk_x dk_y$$

or

$$E_z\left(x, y, 0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_v^{TM}(0)\, e^{-j\left(k_x x + k_y y\right)}\, k_t^2\, dk_x dk_y$$

Sommerfeld Problem (cont.)

$$E_z\left(x,y,0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_v^{TM}(0)\, e^{-j\left(k_x x + k_y y\right)}\, k_t^2\, dk_x dk_y$$

Note: $I_v^{TM}(0)$ is only a function of k_t

Change to polar coordinates:

$$x = \rho\cos\phi \qquad k_x = k_t\cos\overline{\phi}$$

$$y = \rho\sin\phi \qquad k_y = k_t\sin\overline{\phi}$$

$$dk_x\, dk_y \to k_t\, dk_t\, d\overline{\phi}$$

Sommerfeld Problem (cont.)

$$E_z\left(x,y,0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_v^{TM}(0)\, e^{-j\left(k_x x + k_y y\right)}\, k_t^2\, dk_x dk_y$$

Switch to polar coordinates

$$dk_x\, dk_y \rightarrow k_t\, dk_t\, d\overline{\phi}$$

$$E_z\left(x,y,0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^{\infty} I_v^{TM}(0)\, k_t^3 \int_0^{2\pi} e^{-j\left(k_t \cos\overline{\phi}\,\rho\cos\phi + k_t \sin\overline{\phi}\,\rho\sin\phi\right)}\, d\overline{\phi}\, dk_t$$

$$E_z\left(x,y,0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^{\infty} I_v^{TM}(0)\, k_t^3 \int_0^{2\pi} e^{-j\left(k_t\rho\right)\cos\left(\overline{\phi}-\phi\right)}\, d\overline{\phi}\, dk_t$$

Sommerfeld Problem (cont.)

$$E_z\left(x,y,0^+\right) = -\frac{Il}{(2\pi)^2}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^{\infty} I_v^{TM}(0)\, k_t^3 \int_0^{2\pi} e^{-j(k_t\rho)\cos\left(\bar{\phi}-\phi\right)}\, d\bar{\phi}$$

Use $\alpha = \bar{\phi} - \phi$

$$\int_0^{2\pi} e^{-j(k_t\rho)\cos\left(\bar{\phi}-\phi\right)}\, d\bar{\phi} = \int_{-\phi}^{2\pi-\phi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha = \int_0^{2\pi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha$$

We see from this result that the vertical field of the vertical electric dipole should not vary with angle ϕ.

Integral identity: $$I = \int_0^{2\pi} e^{-j(k_t\rho)\cos\alpha}\, d\alpha = 2\pi J_0\left(k_t\rho\right)$$

Sommerfeld Problem (cont.)

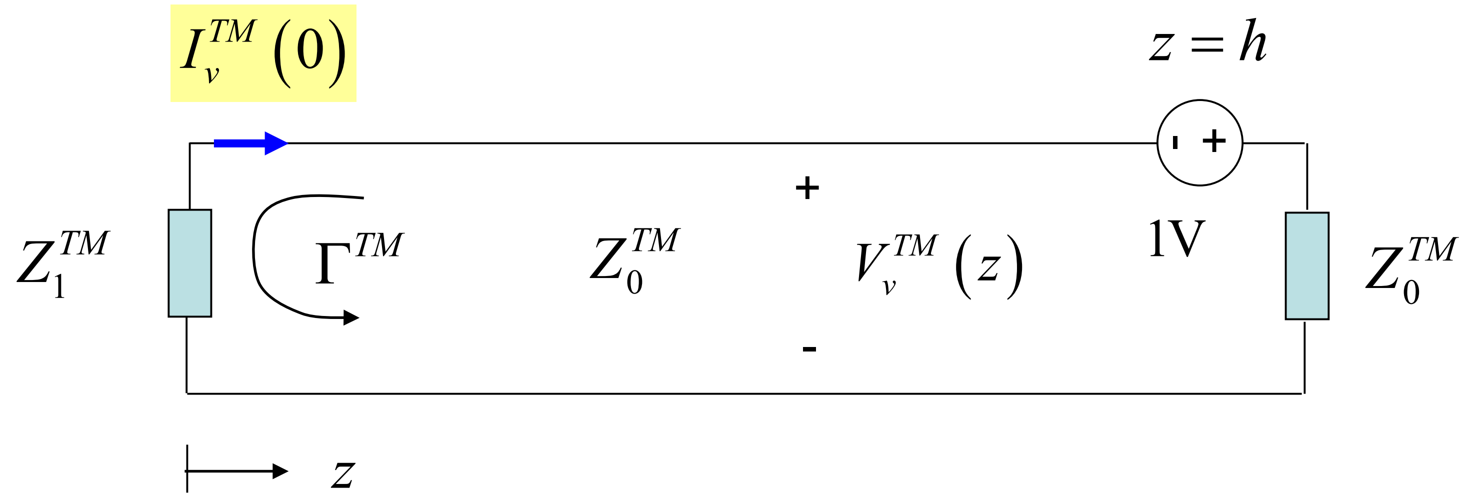
Hence we have

$$E_z\left(\rho,0^+\right)=-\frac{Il}{(2\pi)}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_0^{\infty} J_0\left(k_t\rho\right)\, I_v^{TM}\left(0\right)\, k_t^3\, dk_t$$

This is the “Sommerfeld form” of the field.

Sommerfeld Problem (cont.)

We now return to the calculation of the Michalski normalized current function.



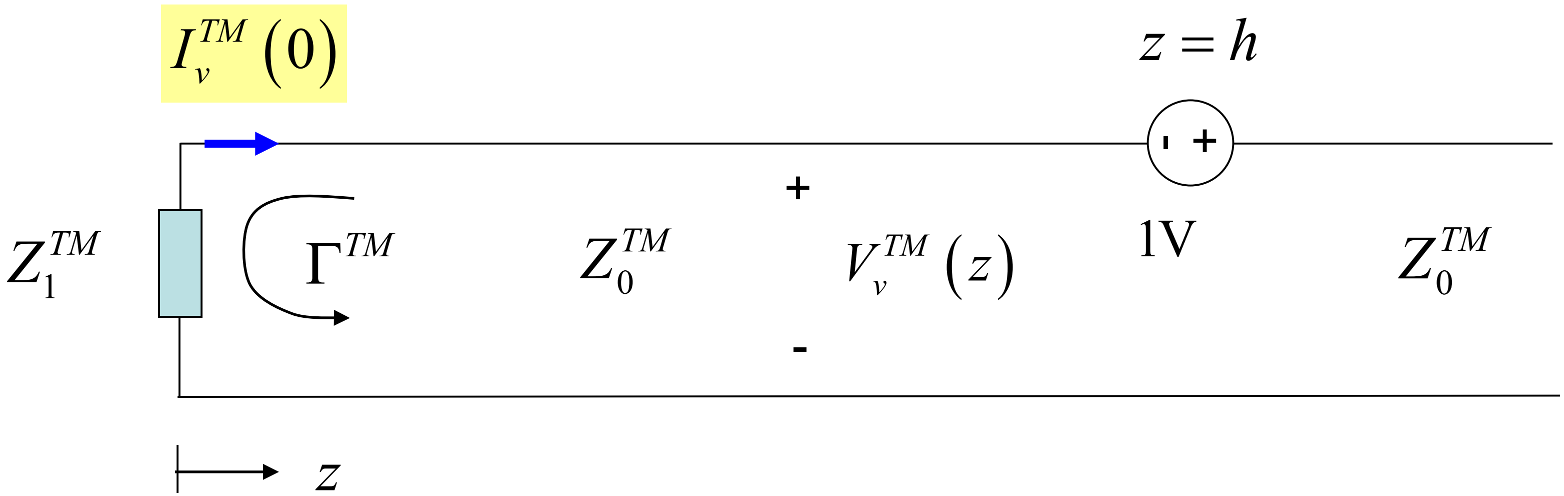
$0 < z < h$

$$V_v^{TM}(z) = Ae^{-jk_{z0}z} + Be^{+jk_{z0}z}$$
$$= B\left(e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

$$\Gamma^{TM} = \frac{A}{B} = \frac{Z_1^{TM} - Z_0^{TM}}{Z_1^{TM} + Z_0^{TM}}$$

$$I_v^{TM}(z) = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

Sommerfeld Problem (cont.)



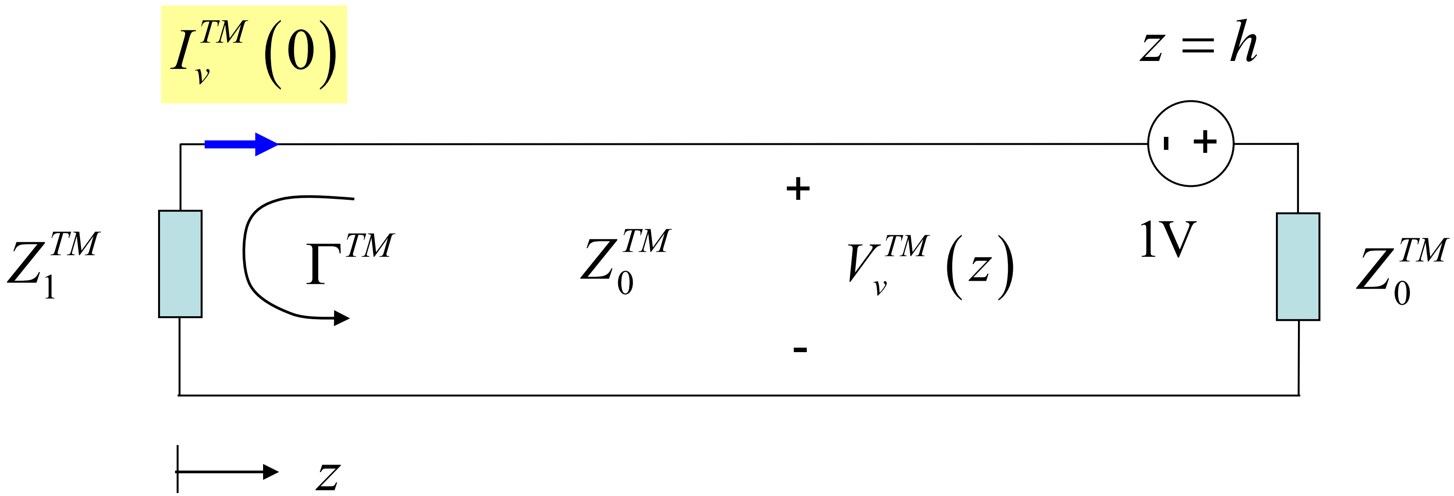
$$V_v^{TM}(z) = Ce^{-jk_{z0}z}$$

$z > h$

$$I_v^{TM}(z) = \frac{C}{Z_0^{TM}} e^{-jk_{z0}z}$$

Here we visualize the transmission line as infinite beyond the voltage source.

Sommerfeld Problem (cont.)



Boundary conditions:

$$V_v^{TM}(h^+) - V_v^{TM}(h^-) = 1$$

$$I_v^{TM}(h^+) = I_v^{TM}(h^-)$$

Sommerfeld Problem (cont.)

Hence we have

$$V_v^{TM}\left(h^+\right)-V_v^{TM}\left(h^-\right)=1$$

$$\Rightarrow \quad Ce^{-jk_{z0}h}-B\left(e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)=1$$

$$I_v^{TM}\left(h^+\right)=I_v^{TM}\left(h^-\right)$$

$$\Rightarrow \quad \frac{C}{Z_0^{TM}}e^{-jk_{z0}h}=\frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)$$

Sommerfeld Problem (cont.)

Substitute the first of these into the second one:

$$Ce^{-jk_{z0}h} = 1 + B\left(e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

$$\frac{C}{Z_0^{TM}} e^{-jk_{z0}h} = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

This gives us

$$\frac{1}{Z_0^{TM}}\left[1 + B\left(e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)\right] = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h} + \Gamma^{TM} e^{-jk_{z0}h}\right)$$

Sommerfeld Problem (cont.)

$$\frac{1}{Z_0^{TM}}\left[1+B\left(e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)\right]=\frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}h}+\Gamma^{TM}e^{-jk_{z0}h}\right)$$

$$\Downarrow$$

$$\left[1+B\left(e^{+jk_{z0}h}+\cancel{\Gamma^{TM}e^{-jk_{z0}h}}\right)\right]=B\left(-e^{+jk_{z0}h}+\cancel{\Gamma^{TM}e^{-jk_{z0}h}}\right)$$

cancels

$$\Downarrow$$

$$B\left(2e^{+jk_{z0}h}\right)=-1$$

Hence we have

$$B=-\frac{1}{2}e^{-jk_{z0}h}$$

Sommerfeld Problem (cont.)

For the current we then have

$$I_v^{TM}(z) = \frac{B}{Z_0^{TM}}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

with $B = -\frac{1}{2}e^{-jk_{z0}h}$

Hence

$$I_v^{TM}(z) = -\frac{1}{2Z_0^{TM}} e^{-jk_{z0}h}\left(-e^{+jk_{z0}z} + \Gamma^{TM} e^{-jk_{z0}z}\right)$$

For $z = 0$:

$$I_v^{TM}(0) = \frac{1}{2Z_0^{TM}}\left(1 - \Gamma^{TM}\right)e^{-jk_{z0}h}$$

Sommerfeld Problem (cont.)

We thus have

$$E_z\left(\rho,0^+\right) = -\frac{Il}{(2\pi)}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_0^{\infty} J_0\left(k_t\rho\right) I_v^{TM}\left(0\right) k_t^3\, dk_t$$

with $$I_v^{TM}\left(0,h\right) = \frac{1}{2Z_0^{TM}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}$$

Hence

$$E_z\left(\rho,0^+\right) = -\frac{Il}{(2\pi)}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_0^{\infty} J_0\left(k_t\rho\right)\left[\frac{1}{2Z_0^{TM}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\, dk_t$$

or

$$E_z\left(\rho,0^+\right) = -\frac{Il}{(2\pi)}\left(\frac{1}{\left(\omega\varepsilon_0\right)^2}\right)\int_0^{\infty} J_0\left(k_t\rho\right)\left[\frac{1}{2\left(\dfrac{k_{z0}}{\omega\varepsilon_0}\right)}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\, dk_t$$

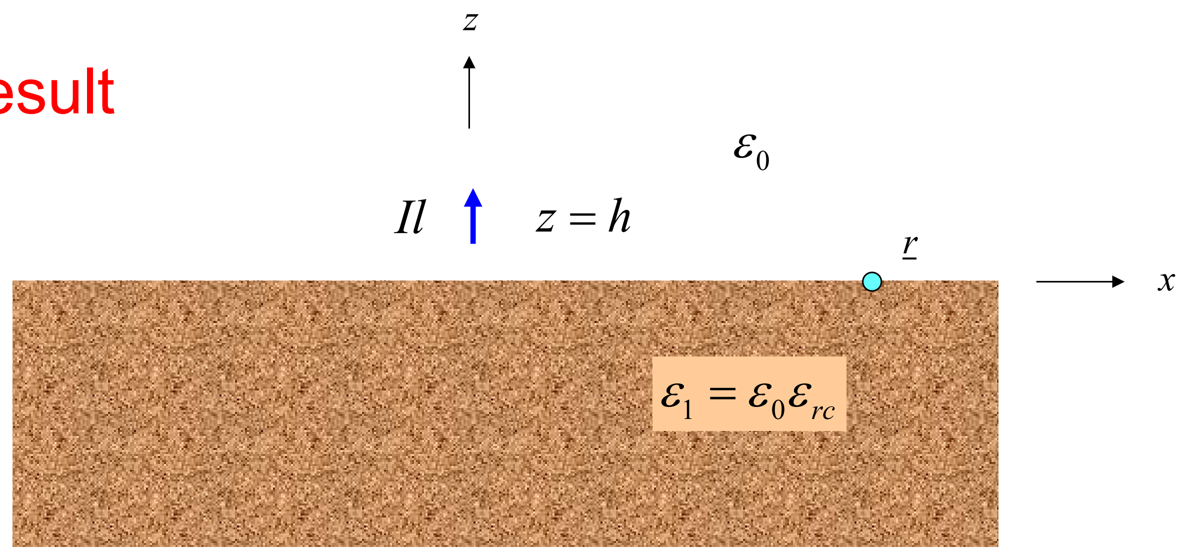
Sommerfeld Problem (cont.)

$$E_z\left(\rho,0^+\right) = -\frac{Il}{(2\pi)}\left(\frac{1}{(\omega\varepsilon_0)^2}\right)\int_0^\infty J_0\left(k_t\rho\right)\left[\frac{1}{2\left(\frac{k_{z0}}{\omega\varepsilon_0}\right)}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

$$E_z\left(\rho,0^+\right) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\int_0^\infty J_0\left(k_t\rho\right)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

Sommerfeld Problem (cont.)

Final Result



$$E_z\left(\rho,0^+\right) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\int_0^\infty J_0\left(k_t\rho\right)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

$$\Gamma^{TM} = \Gamma^{TM}\left(k_t\right) = \frac{Z_1^{TM} - Z_0^{TM}}{Z_1^{TM} + Z_0^{TM}}$$

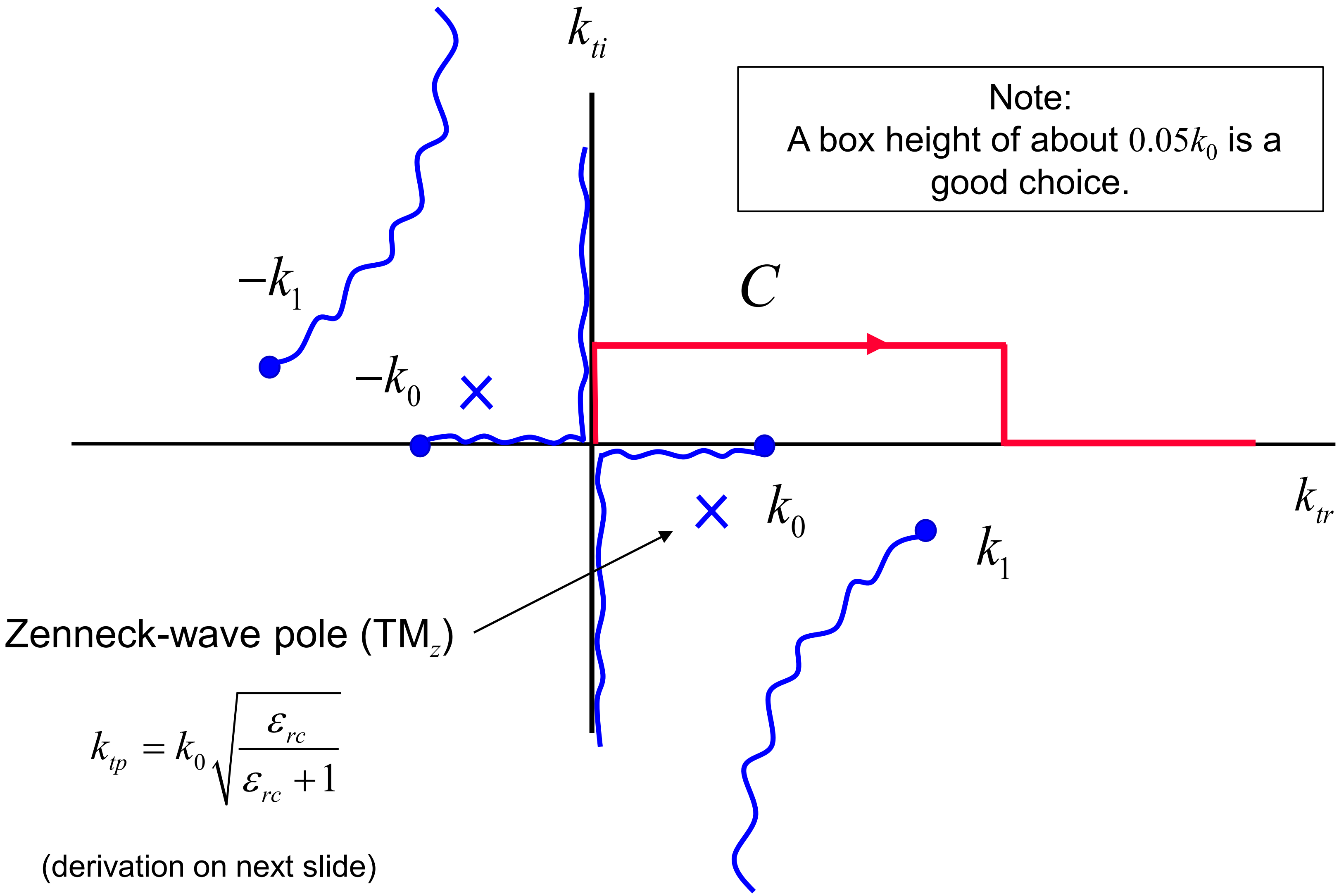
$$Z_0^{TM} = \frac{k_{z0}}{\omega\varepsilon_0}$$

$$Z_{`1}^{TM} = \frac{k_{z1}}{\omega\varepsilon_1}$$

$$k_{z0} = \sqrt{k_0^2 - k_t^2}$$

$$k_{z1} = \sqrt{k_1^2 - k_t^2}$$

Sommerfeld Problem (cont.)



Note:
A box height of about $0.05 k_0$ is a good choice.

$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc} + 1}}$$

(derivation on next slide)

ε_{rc} = complex relative permittivity of the earth (accounting for the conductivity.)

Sommerfeld Problem (cont.)

Zenneck-wave pole (TM_z)

The TRE is:

$$\overleftarrow{Z}^{TM} = -\overrightarrow{Z}^{TM}$$

$$Z_0^{TM} = -Z_1^{TM}$$

$$\frac{k_{z0}}{\omega\varepsilon_0} = -\frac{k_{z1}}{\omega\varepsilon_1}$$

$$\varepsilon_{rc} k_{z0} = -k_{z1}$$

$$\varepsilon_{rc}^2 \left(k_0^2 - k_{tp}^2\right) = \left(k_1^2 - k_{tp}^2\right)$$

$$k_{tp}^2 \left(\varepsilon_{rc}^2 - 1\right) = \varepsilon_{rc}^2 k_0^2 - k_1^2$$

$$k_{tp}^2 \left(\varepsilon_{rc}^2 - 1\right) = \varepsilon_{rc}^2 k_0^2 - k_0^2 \varepsilon_{rc}$$

$$k_{tp}^2 = \frac{\varepsilon_{rc}^2 k_0^2 - k_0^2 \varepsilon_{rc}}{\varepsilon_{rc}^2 - 1}$$

$$k_{tp}^2 = k_0^2 \varepsilon_{rc} \left(\frac{\varepsilon_{rc} - 1}{\varepsilon_{rc}^2 - 1}\right)$$

$$k_{tp}^2 = k_0^2 \varepsilon_{rc} \left(\frac{\cancel{\varepsilon_{rc} - 1}}{\cancel{(\varepsilon_{rc} - 1)}(\varepsilon_{rc} + 1)}\right)$$

$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc} + 1}}$$

Note:
Both vertical wavenumbers (k_{z0} and k_{z1}) are proper for the Zenneck wave (proof omitted).

Sommerfeld Problem (cont.)

Alternative form

The path is extended to the entire real axis.

$$E_z\left(\rho,0^+\right)=\int_0^\infty J_0\left(k_t\rho\right)\mathrm{Odd}\left(k_t\right)dk_t$$

We use

$$J_0\left(z\right)=\frac{1}{2}\left(H_0^{(1)}\left(z\right)+H_0^{(2)}\left(z\right)\right)$$ (see note 1)

$$H_0^{(1)}\left(z\right)=-H_0^{(2)}\left(-z\right)$$ (see note 2)

Transform the $H_0^{(1)}$ term:

$$\int_0^\infty \mathrm{Odd}\left(k_t\right)H_0^{(1)}\left(k_t\rho\right)dk_t=-\int_0^\infty \mathrm{Odd}\left(k_t\right)H_0^{(2)}\left(-k_t\rho\right)dk_t$$

$$=\int_0^{-\infty} \mathrm{Odd}\left(-k_t'\right)H_0^{(2)}\left(k_t'\rho\right)dk_t'$$ Use $k_t'=-k_t$

$$=\int_{-\infty}^{0} \mathrm{Odd}\left(k_t'\right)H_0^{(2)}\left(k_t'\rho\right)dk_t'$$

$$=\int_{-\infty}^{0} \mathrm{Odd}\left(k_t\right)H_0^{(2)}\left(k_t\rho\right)dk_t$$

Note 1: z is not on the negative real axis.

Note 2: $\mathrm{Im}(z)$ is positive.

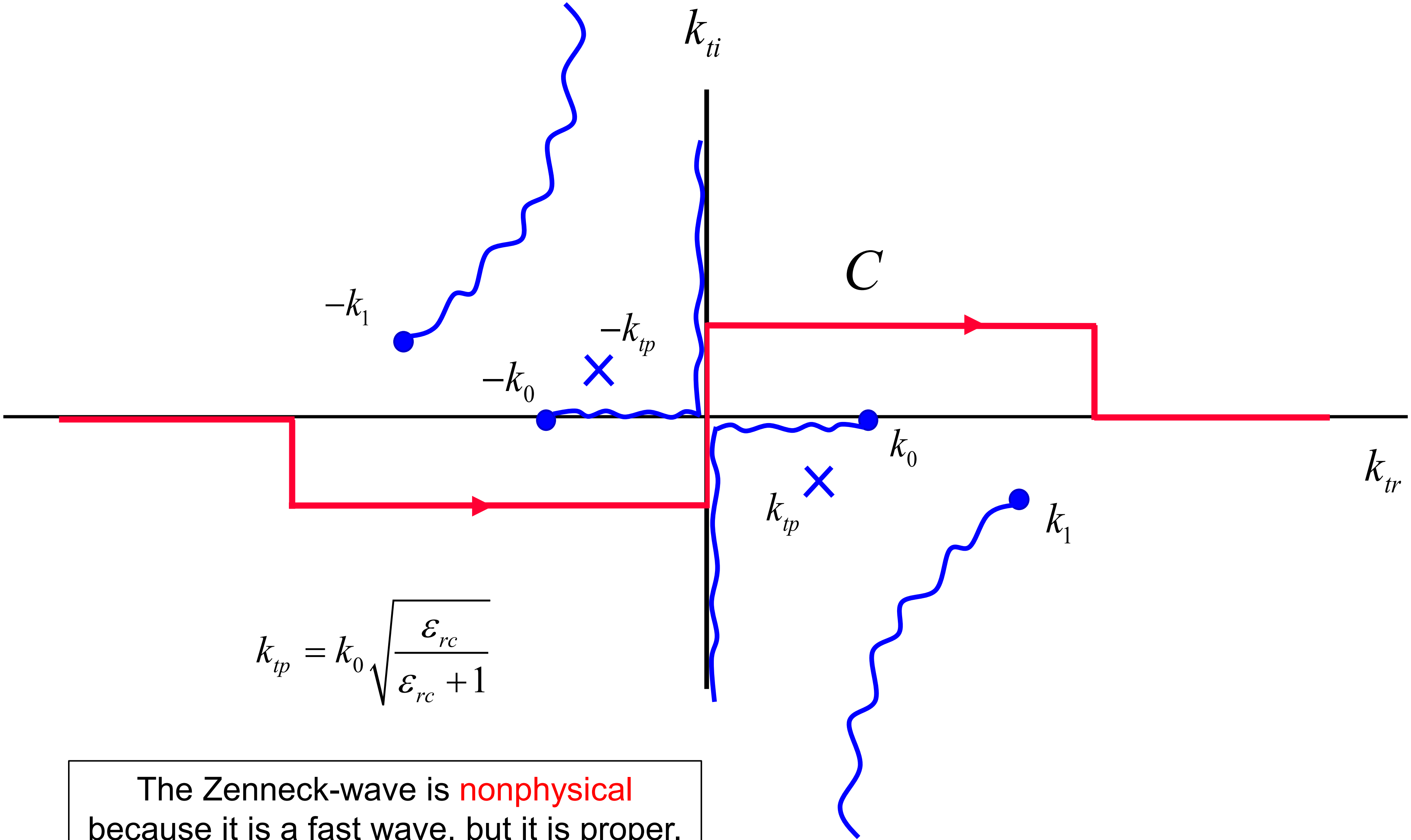
Sommerfeld Problem (cont.)

Hence we have

$$E_z\left(\rho,0^+\right) = -\frac{Il}{4\pi}\left(\frac{1}{\omega\varepsilon_0}\right)\frac{1}{2}\int_{-\infty}^{\infty} H_0^{(2)}\left(k_t\rho\right)\left[\frac{1}{k_{z0}}\left(1-\Gamma^{TM}\right)e^{-jk_{z0}h}\right]k_t^3\,dk_t$$

This is a convenient form for deforming the path.

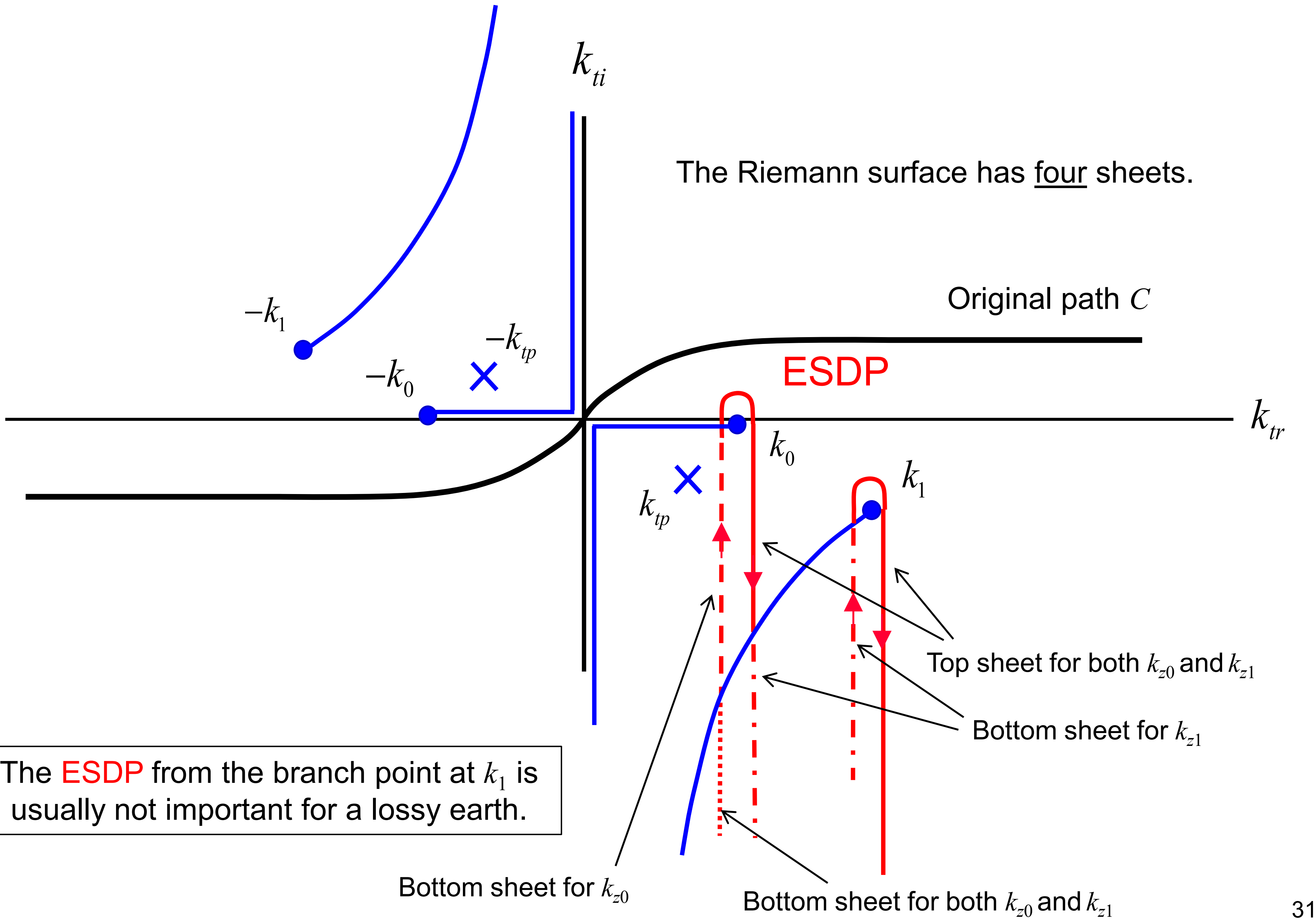
Sommerfeld Problem (cont.)



$$k_{tp} = k_0 \sqrt{\frac{\varepsilon_{rc}}{\varepsilon_{rc}+1}}$$

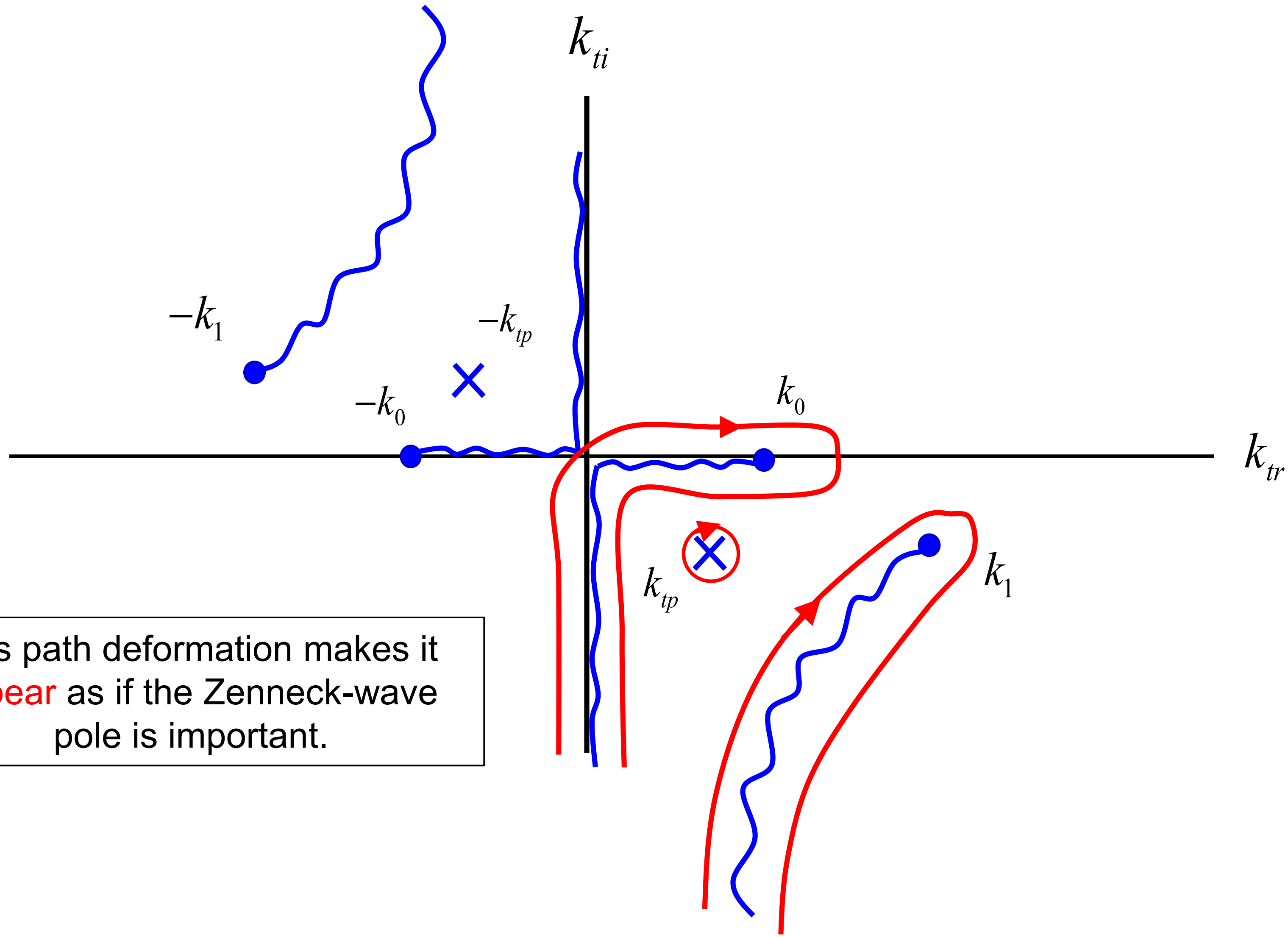
The Zenneck-wave is nonphysical because it is a fast wave, but it is proper. It is not captured when deforming to the ESDP (vertical path from k_0).

Sommerfeld Problem (cont.)



The ESDP from the branch point at k_1 is usually not important for a lossy earth.

Sommerfeld Problem (cont.)



This path deformation makes it appear as if the Zenneck-wave pole is important.

Sommerfeld Problem (cont.)

Throughout much of the 20th century, a controversy raged about the "reality of the Zenneck wave."

- Arnold Sommerfeld predicted a surface-wave like field coming from the residue of the Zenneck-wave pole (1909).
- People took measurements and could not find such a wave.
- Hermann Weyl solved the problem in a different way and did not get the Zenneck wave (1919).
- Some people (Norton, Niessen) blamed it on a sign error that Sommerfeld had made, though Sommerfeld never admitted to a sign error.
- Eventually it was realized that there was no sign error (Collin, 2004).
- The limitation in Sommerfeld's original asymptotic analysis (which shows a Zenneck-wave term) is that the pole must be well separated from the branch point – the asymptotic expansion that he used neglects the effects of the pole on the branch point (the saddle point in the steepest-descent plane).
- When the asymptotic evaluation of the branch-cut integral around k_0 includes the effects of the pole, it turns out that there is no Zenneck-wave term in the total solution (branch-cut integrals + pole-residue term).
- The easiest way to explain the fact that the Zenneck wave is not important far away is that the pole is not captured in deforming to the ESDP paths.

Sommerfeld Problem (cont.)

R. E. Collin, "Hertzian Dipole Radiating Over a Lossy Earth or Sea: Some Early and Late 20th-Century Controversies," AP-S Magazine, pp. 64-79, April 2004.

Hertzian Dipole Radiating Over a Lossy Earth or Sea: Some Early and Late 20th-Century Controversies

R. E. Collin

Department of Electrical Engineering and Computer Science, Case Western Reserve University
Cleveland, OH 44106 USA
Tel: +1 (440) 442 3712; E-mail: rec2@po.cwru.edu

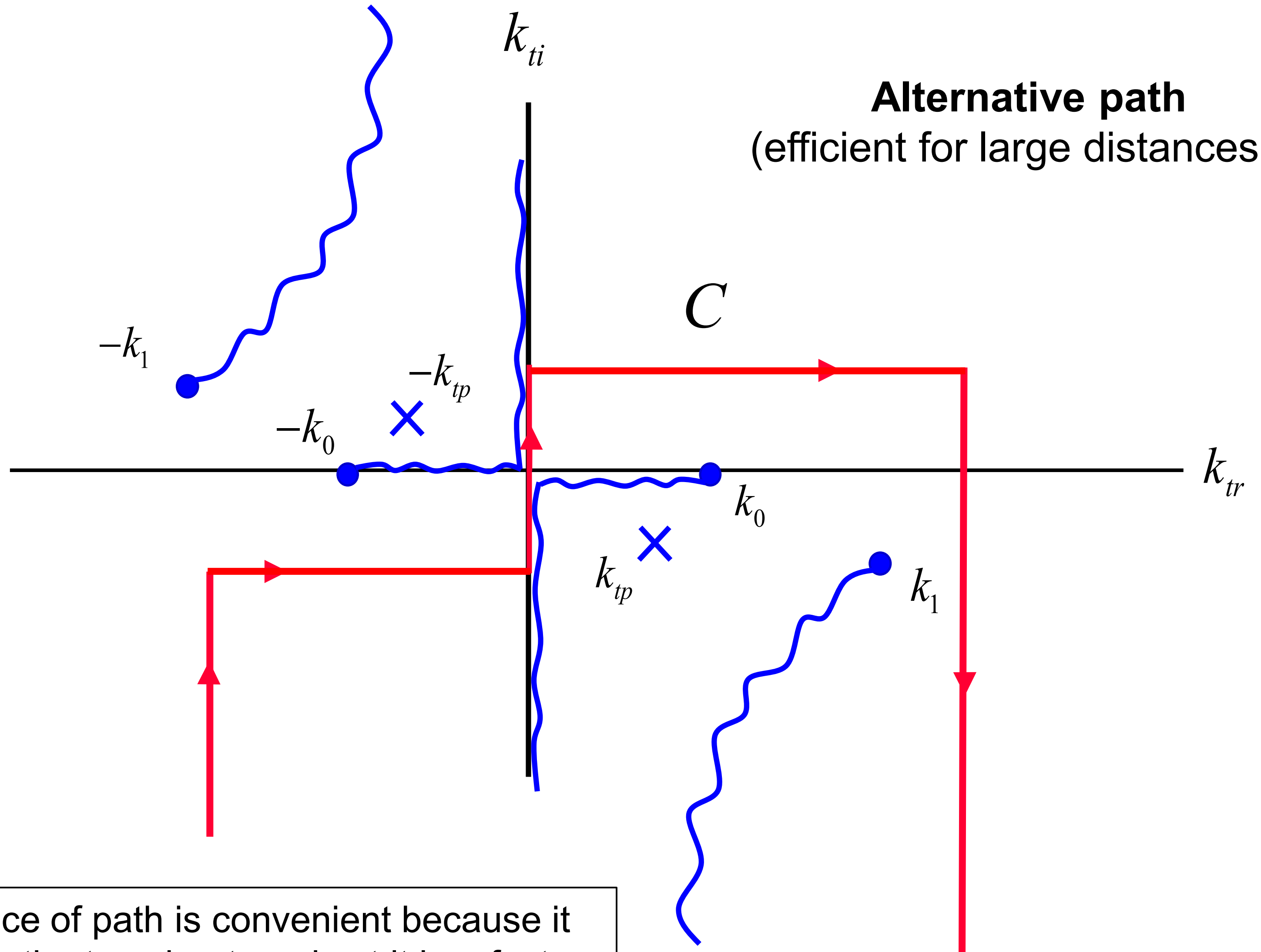
Abstract

This paper presents a contemporary solution to the problem of radiation from a vertical Hertzian dipole over a lossy Earth. Sommerfeld's 1909 solution to the problem is re-examined. It is demonstrated that a change in sign in the square root of the numerical distance is mathematically not allowed. Thus, the sign error that has been claimed in the technical literature for more than 65 years is a myth. Recent work by King and Sandler is also examined. It is found that due to an incorrect asymptotic expansion of the complementary error function for the problem of a lossy earth or sea covered with a thin dielectric layer, a trapped surface wave was missed in their solution.

Keywords: Dipole antennas; electromagnetic radiation; Zenneck surface wave; asymptotic solution; electromagnetic surface waves

Sommerfeld Problem (cont.)

Alternative path
(efficient for large distances ρ)



This choice of path is convenient because it stays on the top sheet, and yet it has fast convergence as the distance ρ increases, due to the Hankel function.