ECE 6341

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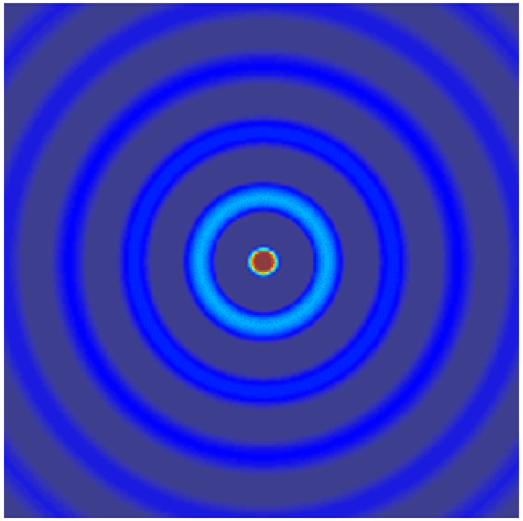
Prof. David R. Jackson
ECE Dept.



Notes 45

Periodic SDI

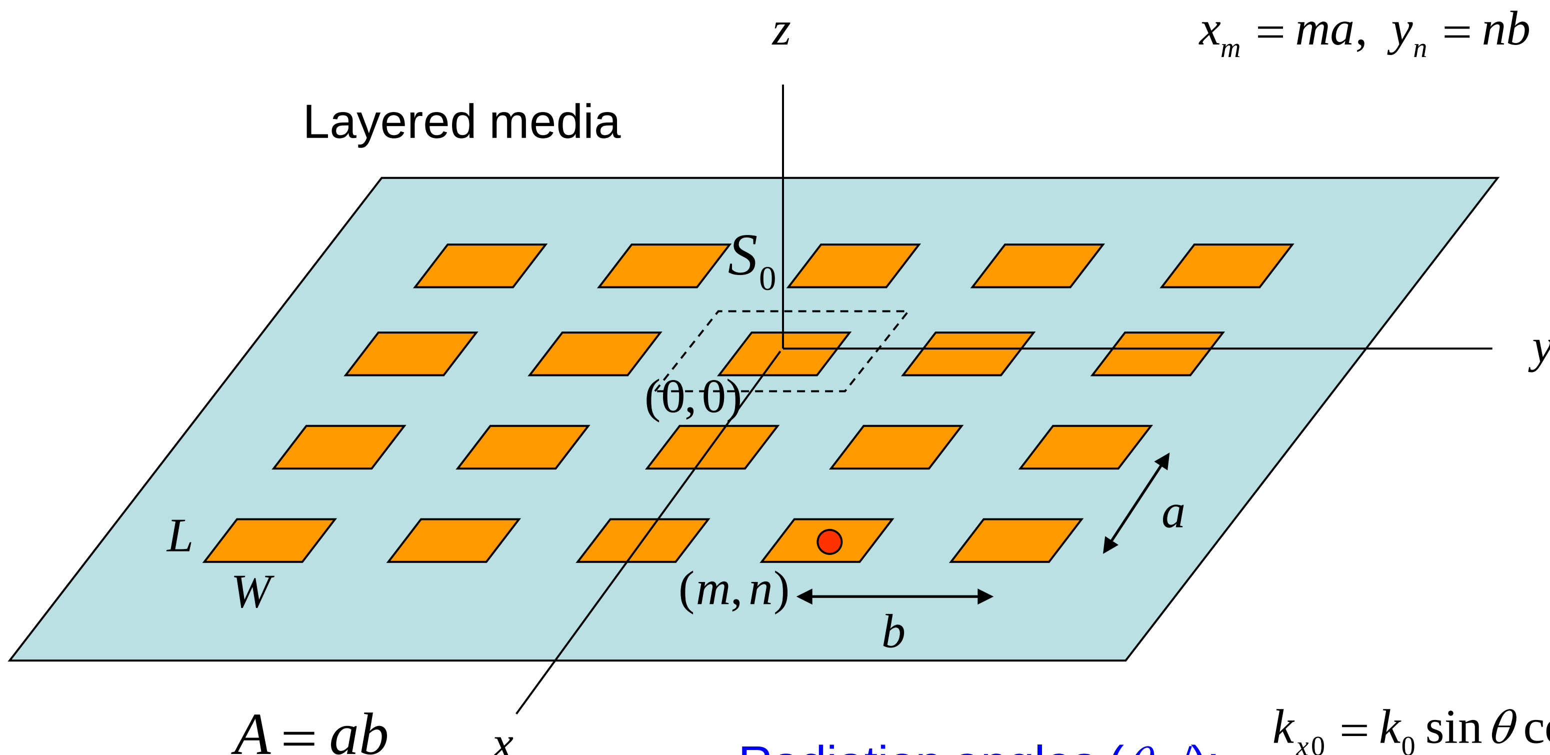
Calculate the fields from a
2D periodic array of currents in a layered media.

$$\underline{J}_s^{mn}(x,y) = \underline{J}_s^{00}(x - x_m, y - y_n)\, e^{-j\left(k_{x0}x_m + k_{y0}y_m\right)}$$

Phased array of patches

(m,n) patch centered at (x_m, y_n)

$$x_m = ma,\;\; y_n = nb$$



$$A = ab$$

Radiation angles (θ, ϕ):

$$k_{x0} = k_0 \sin\theta \cos\phi$$
$$k_{y0} = k_0 \sin\theta \sin\phi$$

Periodic SDI (cont.)

The current on the periodic structure is represented in terms of Floquet waves:

$$\underline{J}_s(x,y) = e^{-j\left(k_{x0}x + k_{y0}y\right)} \underline{P}(x,y)$$

where

$$\underline{P}(x,y) = \text{2D periodic function}$$

From Fourier series theory, we can write:

$$\underline{P}(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\left(\left(\frac{2\pi p}{a}\right)x + \left(\frac{2\pi q}{b}\right)y\right)}$$

Periodic SDI (cont.)

Hence, we have:

$$\underline{J}_s(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\left(k_{xp}x + k_{yq}y\right)}$$

"Floquet expansion"

where

$$k_{xp} = k_{x0} + \frac{2\pi p}{a}$$

$$k_{yq} = k_{y0} + \frac{2\pi q}{b}$$

To solve for the unknown coefficients, multiply both sides by $e^{+j\left(k_{xp'}x + k_{yq'}y\right)}$ and integrate over the $(0,0)$ unit cell.

Periodic SDI (cont.)

$$\underline{J}_s(x,y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\left(k_{xp}x + k_{yq}y\right)}$$

$$\Rightarrow \quad \int_{S_0} \underline{J}_s(x,y)\, e^{+j\left(k_{xp'}x + k_{yq'}y\right)}\, dS = \int_{S_0} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \underline{a}_{pq}\, e^{-j\left(k_{xp}x + k_{yq}y\right)}\, e^{+j\left(k_{xp'}x + k_{yq'}y\right)}\, dS$$

Use orthogonality for the RHS, this reduces to:

$$\int_{S_0} \underline{J}_s(x,y)\, e^{+j\left(k_{xp'}x + k_{yq'}y\right)}\, dS = \underline{a}_{p'q'}\, A$$

$$\Rightarrow \quad \underline{a}_{pq} = \frac{1}{A} \int_{S_0} \underline{J}_s(x,y)\, e^{+j\left(k_{xp}x + k_{yq}y\right)}\, dS$$

Periodic SDI (cont.)

$$\underline{a}_{pq} = \frac{1}{A}\int_{S_0} \underline{J}_s\left(x,y\right) e^{+j\left(k_{xp}x + k_{yq}y\right)}\, dS$$

$$= \frac{1}{A}\int_{S_0} \underline{J}_{s0}\left(x,y\right)\; e^{+j\left(k_{xp}x + k_{yq}y\right)}\, dS$$

$$= \frac{1}{A}\underline{\tilde{J}}_{s0}\left(k_{xp}, k_{yq}\right)$$

The current $\underline{J}_{s0}$ denotes the current on the (0,0) patch.

$$\underline{J}_{s0}\left(x,y\right) \equiv \underline{J}_s^{00}\left(x,y\right)$$

For example:

$$\underline{J}_{s0}\left(x,y\right) = \underline{\hat{x}}\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

(TM_{10} patch mode)

$$\underline{\tilde{J}}_{s0}\left(k_x, k_y\right) = \underline{\hat{x}}\frac{\pi L}{2}\left[\frac{\cos\left(k_x \frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_x L}{2}\right)^2}\right]\operatorname{sinc}\left(k_y \frac{W}{2}\right)$$

Hence

$$\underline{a}_{pq} = \frac{1}{A}\underline{\tilde{J}}_{s0}\left(k_{xp}, k_{yq}\right)$$

Periodic SDI (cont.)

Hence, the current on the 2D periodic structure can be expressed as

$$\underline{J}_s(x,y)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_{s0}\left(k_{xp},k_{yq}\right)\,e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

We now calculate the Fourier transform of the 2D periodic current (this is what we need in the SDI method):

Use

$$F\left[e^{-j\left(k_{xp}x+k_{yq}y\right)}\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-jk_{xp}x}e^{-jk_{yq}y}\,e^{+j\left(k_xx+k_yy\right)}\,dxdy$$

$$=\int_{-\infty}^{\infty}e^{-jk_{xp}x}\,e^{+jk_xx}\,dx\int_{-\infty}^{\infty}e^{-jk_{yq}y}\,e^{+jk_yy}\,dy$$

$$=2\pi\delta\left(k_x-k_{xp}\right)\,2\pi\delta\left(k_y-k_{yq}\right)$$

Periodic SDI (cont.)

Hence

$$\underline{\tilde{J}}_s\left(k_x,k_y\right)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_{s0}\left(k_{xp},k_{yq}\right)2\pi\delta\left(k_x-k_{xp}\right)2\pi\delta\left(k_y-k_{yq}\right)$$

Next, calculate the field:

$$\underline{\tilde{E}}\left(k_x,k_y,z\right)=\underline{\underline{\tilde{G}}}\left(k_x,k_y;z\right)\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)$$

$$\underline{E}\left(x,y,z\right)=\frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}\left(k_x,k_y;z\right)\cdot\underline{\tilde{J}}_s\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}dk_x dk_y$$

Periodic SDI (cont.)

Hence, we have:

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_x,k_y;z)$$

$$\cdot\left[\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\tilde{J}}_{s0}(k_{xp},k_{yq})2\pi\delta(k_x-k_{xp})\,2\pi\delta(k_y-k_{yq})\right]$$

$$e^{-j(k_x x+k_y y)}\,dk_x dk_y$$

Periodic SDI (cont.)

Therefore, we have

$$\underline{E}(x,y,z)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\underline{\tilde{G}}}\left(k_{xp},k_{yq};z\right)\cdot\underline{\tilde{J}}_{s0}\left(k_{xp},k_{yq}\right)e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

The field is in the form of an infinite series of <u>Floquet waves</u> (space harmonics).

Each Floquet wave is a plane wave that has a wavenumber given by:

$$\underline{k}_{pq}=\underline{k}_{tpq}+\underline{\hat{z}}k_{zpq}$$

$$\underline{k}_{tpq}=\left(\underline{\hat{x}}\,k_{x0}+\underline{\hat{y}}\,k_{y0}\right)+\underline{\hat{x}}\left(\frac{2\pi p}{a}\right)+\underline{\hat{y}}\left(\frac{2\pi q}{b}\right)$$

$$k_{zpq}=\left(k^2-k_{tpq}^2\right)^{1/2} \qquad k_{tpq}^2=k_{xp}^2+k_{yq}^2$$

Periodic SDI (cont.)

Compare:

Single element (non-periodic):

$$\underline{E}(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_x,k_y;z)\cdot\,\underline{\tilde{J}}_s^{single}(k_x,k_y)\,e^{-j(k_x x+k_y y)}dk_x dk_y$$

Periodic array of elements:

$$\underline{E}(x,y,z)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\underline{\underline{\tilde{G}}}(k_{xp},k_{yq};z)\cdot\underline{\tilde{J}}_{s0}(k_{xp},k_{yq})e^{-j(k_{xp}x+k_{yq}y)}$$

Note: $\underline{J}_s^{single}(x,y)=\underline{J}_{s0}(x,y)$

Periodic SDI (cont.)

Conclusion:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F\left(k_x,k_y\right)\,dk_x dk_y \rightarrow \frac{(2\pi)^2}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F\left(k_{xp},k_{yq}\right)$$

where

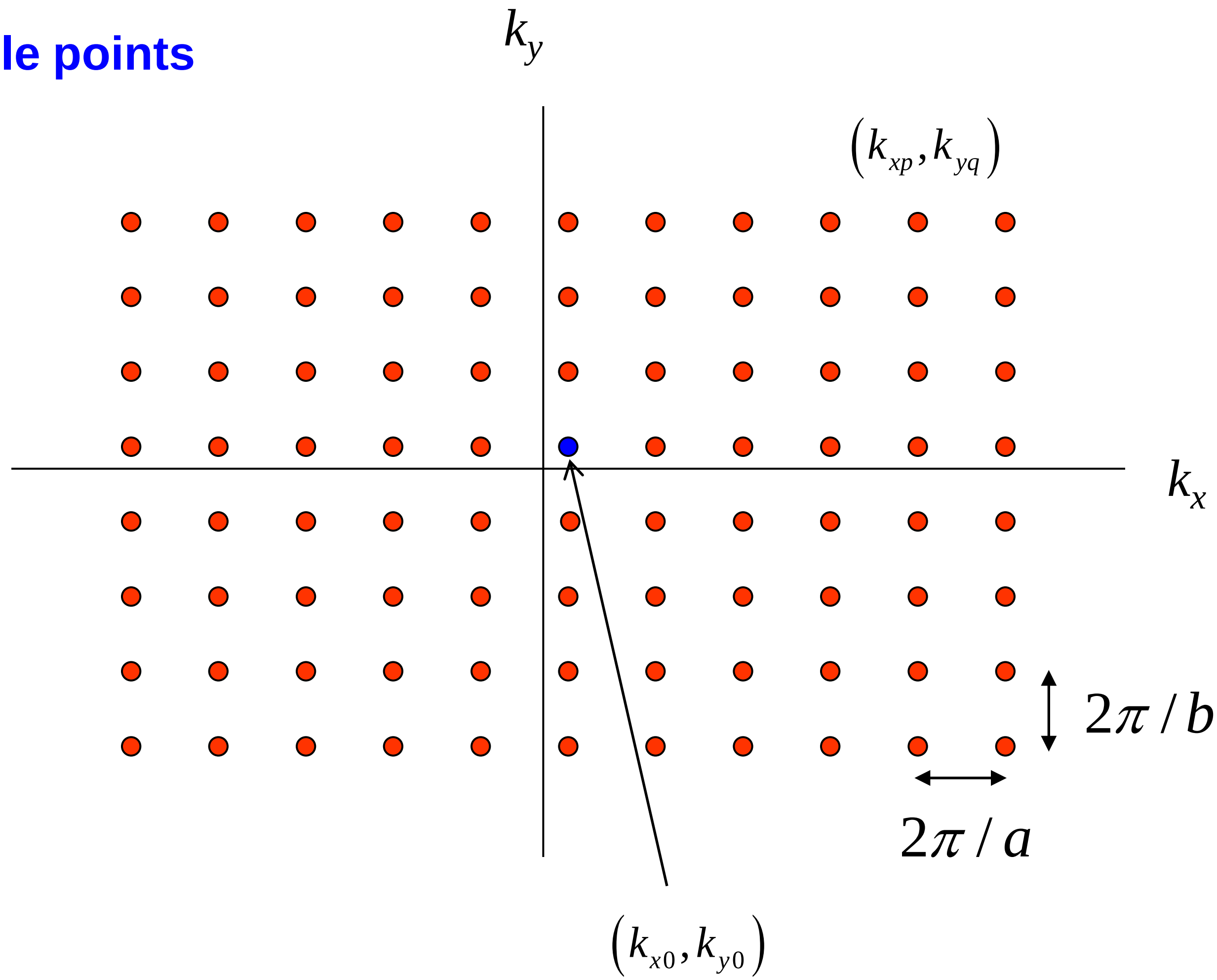
$$k_{xp} = k_{x0} + \frac{2\pi p}{a}$$

$$k_{yq} = k_{y0} + \frac{2\pi q}{b}$$

The double integral is replaced by a double sum, and a factor is introduced.

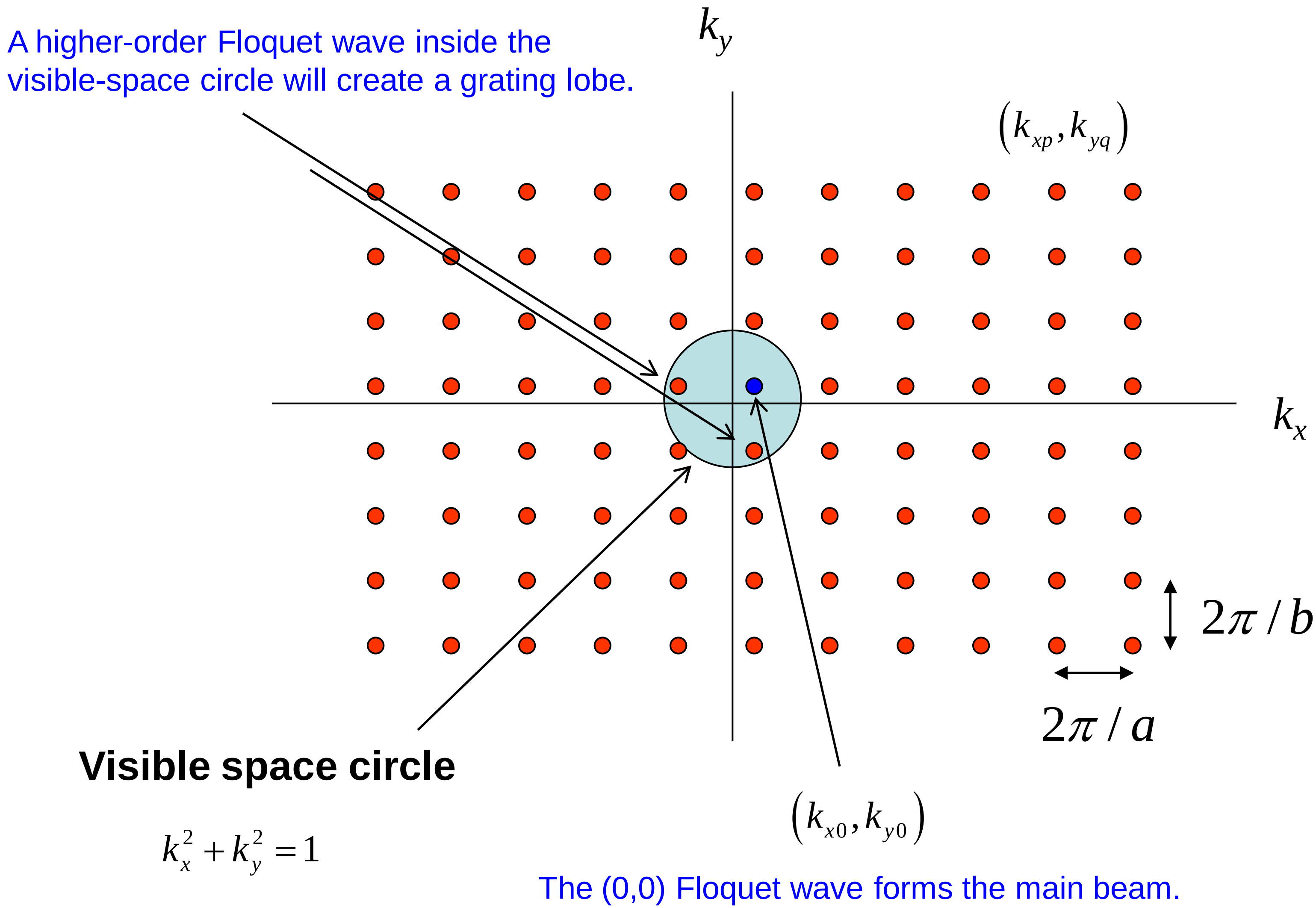
Periodic SDI (cont.)

Sample points



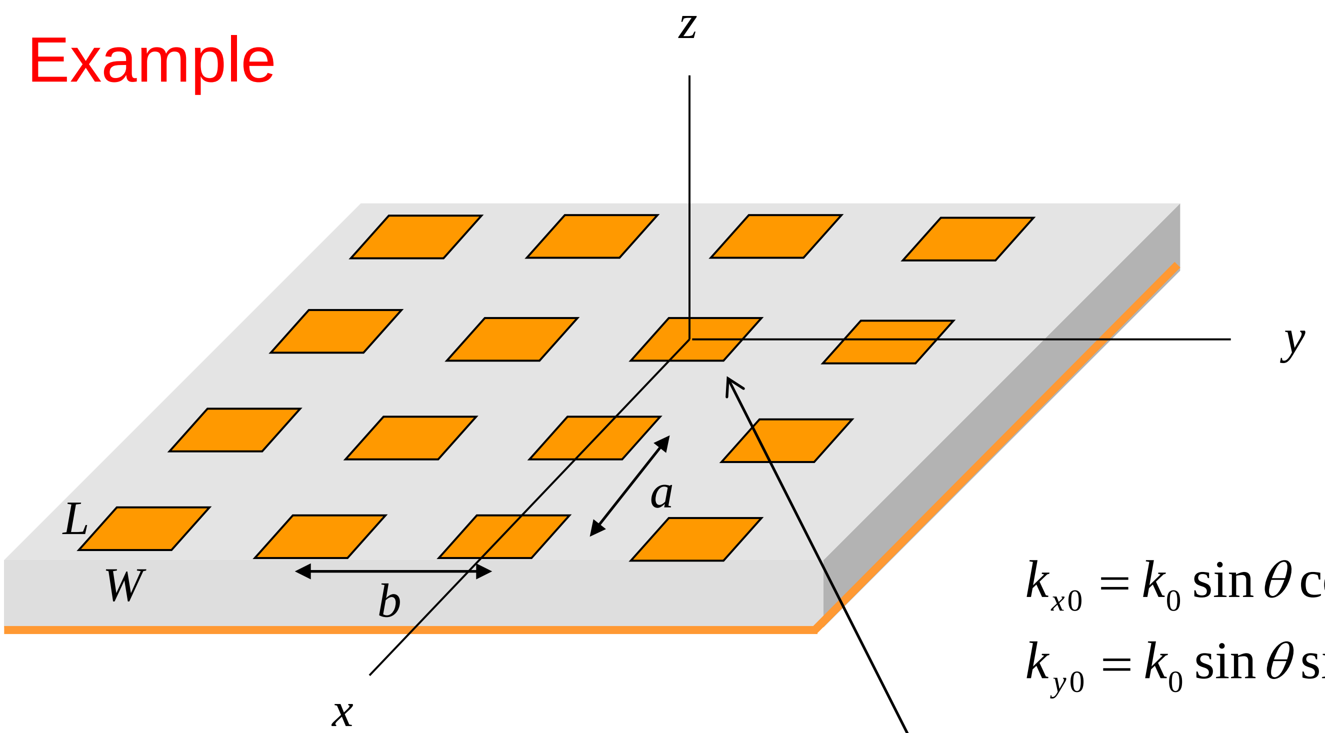
Periodic SDI (cont.)

For a finite size array, the Floquet waves will form beams or lobes in the pattern.



Microstrip Patch Phased Array

Example



Microstrip Patch Phased Array

$$k_{x0} = k_0 \sin\theta \cos\phi$$
$$k_{y0} = k_0 \sin\theta \sin\phi$$

Find $E_x(x,y,0)$

$$\underline{J}_{s0}(x,y) = \underline{\hat{x}} \frac{1}{W} \cos\left(\frac{\pi x}{L}\right)$$

Phased Array (cont.)

Single Patch:

$$E_x(x,y,0)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{J}_{sx}^{single}(k_x,k_y)$$

$$\cdot\frac{-1}{k_t^2}\left[\frac{k_x^2}{D_m(k_t)}+\frac{k_y^2}{D_e(k_t)}\right]e^{-j(k_x x+k_y y)}\;dk_x dk_y$$

$\tilde{G}_{xx}$

$$J_{sx}^{single}(x,y)=\frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$

Phased Array (cont.)

2D Phased Array:

$$E_x(x,y,0)=\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}-\frac{1}{k_{tpq}^2}\tilde{J}_{sx0}\left(k_{xp},k_{yq}\right)\left[\frac{k_{xp}^2}{D_m\left(k_{tpq}\right)}+\frac{k_{yq}^2}{D_e\left(k_{tpq}\right)}\right]e^{-j\left(k_{xp}x+k_{yq}y\right)}$$

where

$$\tilde{J}_{sx0}\left(k_x,k_y\right)=\tilde{J}_{sx}^{single}\left(k_x,k_y\right)=\frac{\pi L}{2}\left[\frac{\cos\left(k_x\frac{L}{2}\right)}{\left(\frac{\pi}{2}\right)^2-\left(\frac{k_xL}{2}\right)^2}\right]\operatorname{sinc}\left(k_y\frac{W}{2}\right)$$

$$k_{xp}=k_{x0}+\frac{2\pi p}{a}$$

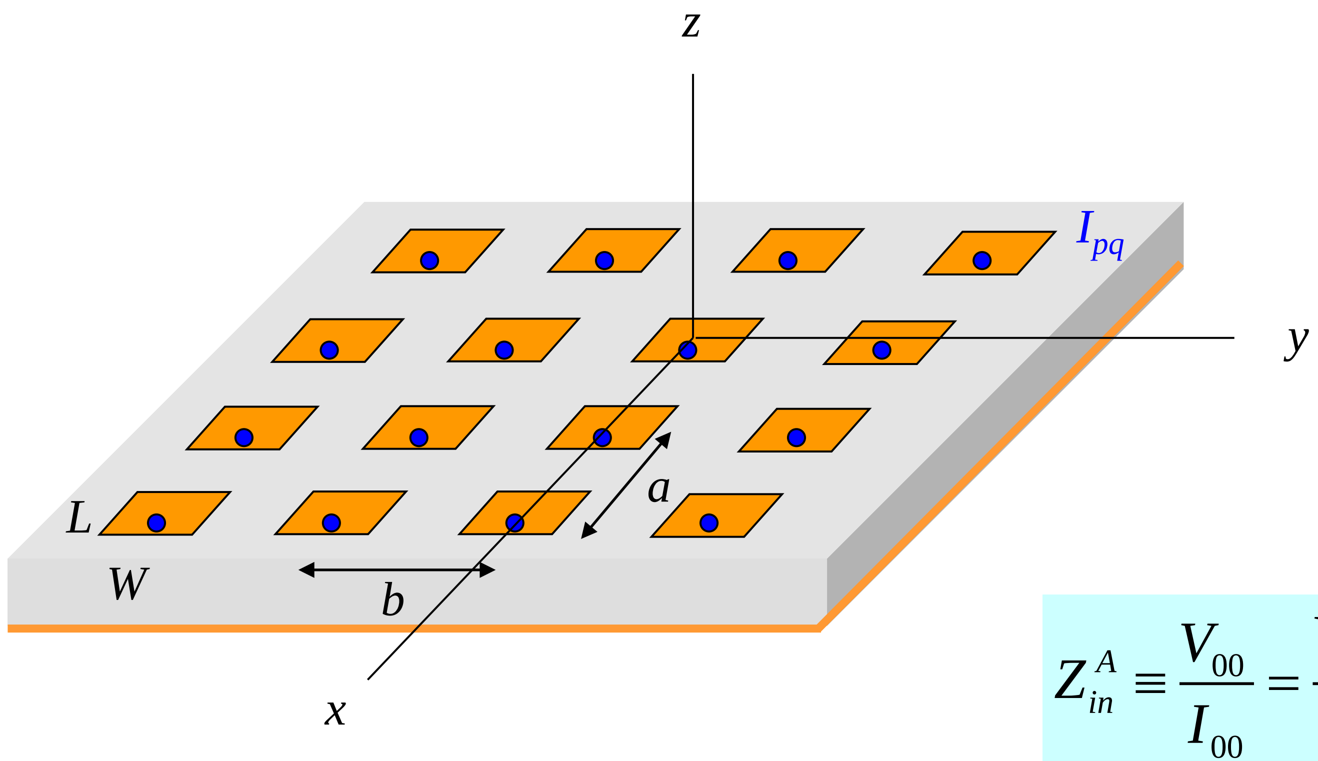
$$k_{xq}=k_{y0}+\frac{2\pi q}{b}$$

$$k_{tpq}^2=k_{xp}^2+k_{yq}^2$$

Phased Array (cont.)

Active impedance (scan impedance) of 2D phased array

Each patch is fed with an identical coaxial feed.



$$Z_{in}^{A} \equiv \frac{V_{00}}{I_{00}} = \frac{V_{pq}}{I_{pq}}$$

Phased Array (cont.)

Active impedance (scan impedance) of 2D phased array

Using the results from a single patch, we have:

$$Z_{in}^{A} = -\frac{V_N^2}{Z_{xx}}$$

$$V_N\left(x_0, y_0\right) = -\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\frac{1}{\omega\varepsilon_0\varepsilon_r}\left(k_{xp}\right)\tilde{B}_{sx}\left(k_{xp}, k_{yq}\right)F\left(k_{tpq}\right)e^{-j\left(k_{xp}x_0 + k_{yq}y_0\right)}$$

$$Z_{xx} = -\frac{1}{A}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\tilde{G}_{xx}\left(k_{xp}, k_{yq}\right)\tilde{B}_{sx}^{2}\left(k_{xp}, k_{yq}\right)$$

$$F\left(k_t\right) = -\frac{1}{D_m\left(k_t\right)}\left(\frac{1}{j Z_1^{TM}}\right)\left(\frac{1}{k_{z1}}\right)$$

$$B_{sx}\left(x, y\right) = \frac{1}{W}\cos\left(\frac{\pi x}{L}\right)$$