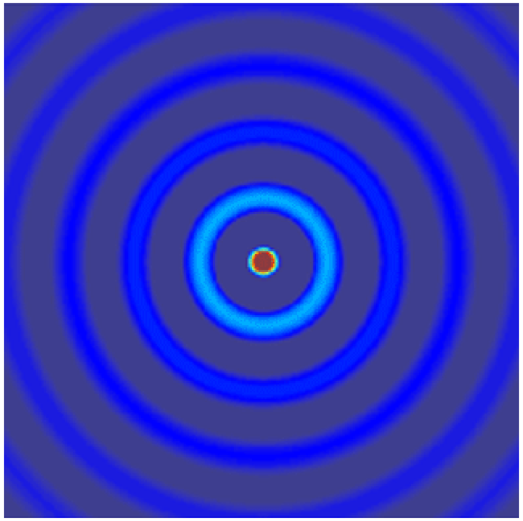
ECE 6341

Spring 2016

Prof. David R. Jackson
ECE Dept.

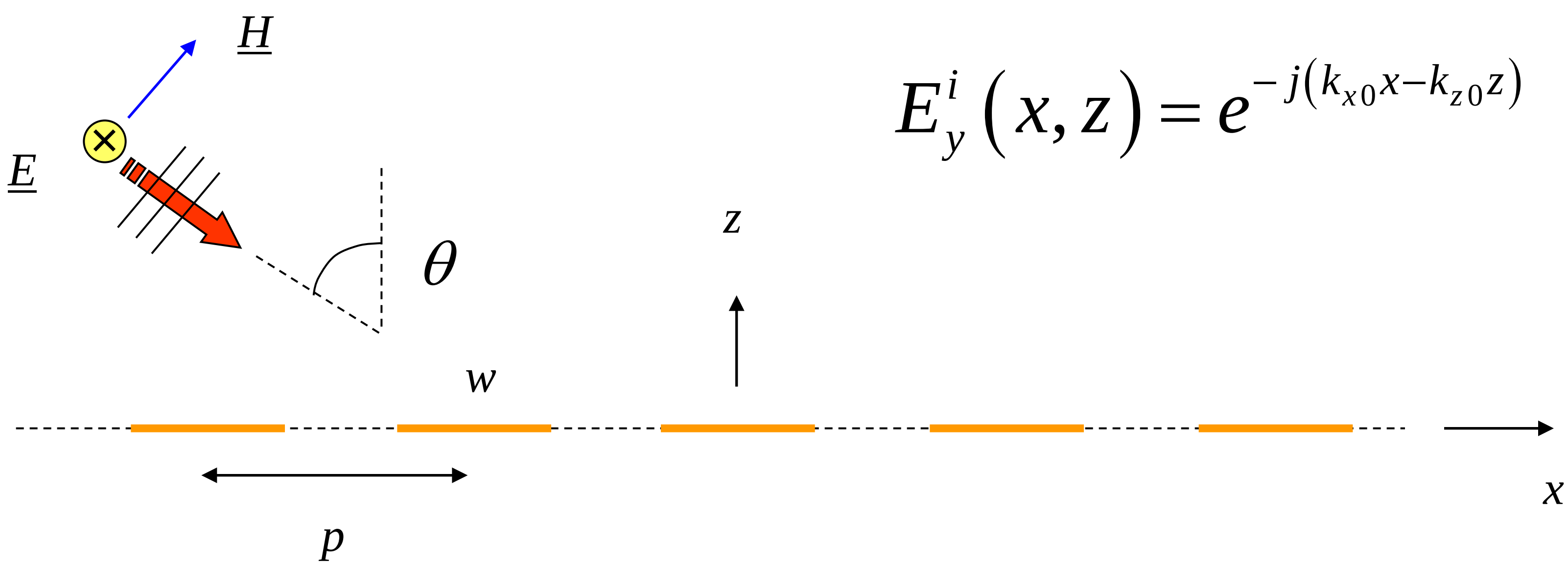


Notes 47

1D Frequency Selective Surface (FSS)

Infinite Periodic Metal Strip Grating

Scattering from a 1-D array of metal strips (metal-strip grating)



$$E_y^i(x,z) = e^{-j(k_{x0}x - k_{z0}z)}$$

$$k_{x0} = k_0 \sin\theta$$

$$k_{z0} = k_0 \cos\theta$$

1D FSS (cont.)

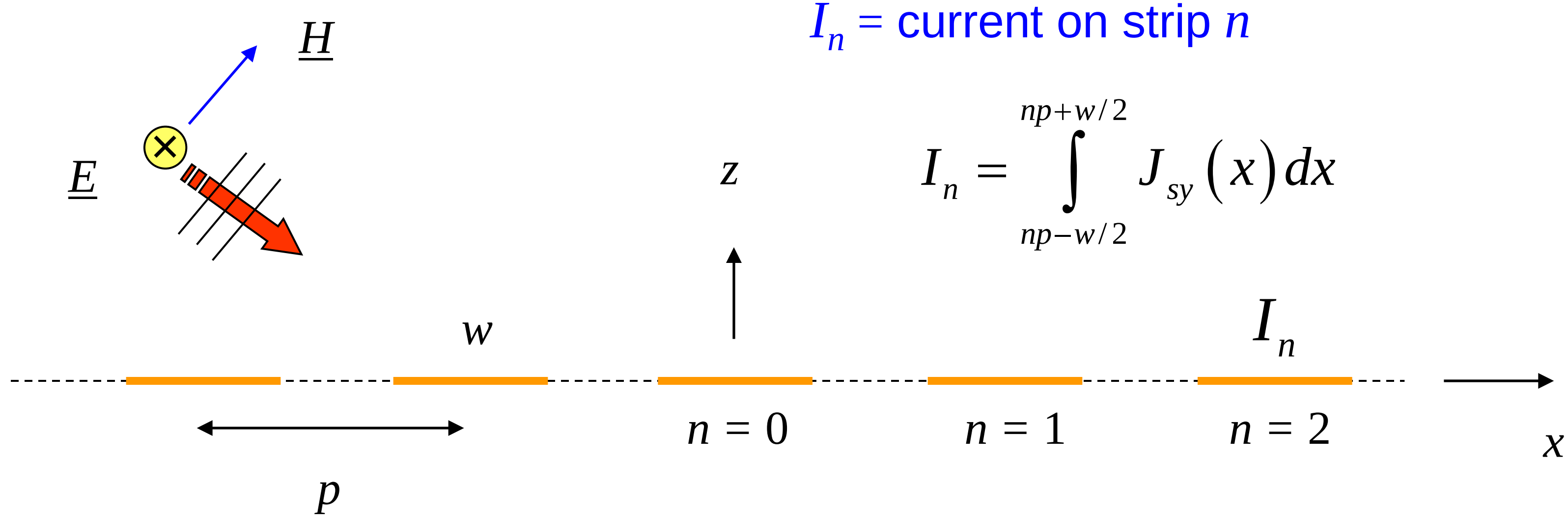
Incident field at interface: $E_y^i(x,0) = e^{-j(k_{x0}x)}$

$$\Rightarrow \quad E_y^i(x+p,0) = e^{-j(k_{x0}p)}\, E_y^i(x)$$

From symmetry, $$I_n = I_0\, e^{-jk_{x0}(np)}$$

I_n = current on strip n

$$I_n = \int_{np-w/2}^{np+w/2} J_{sy}(x)\,dx$$

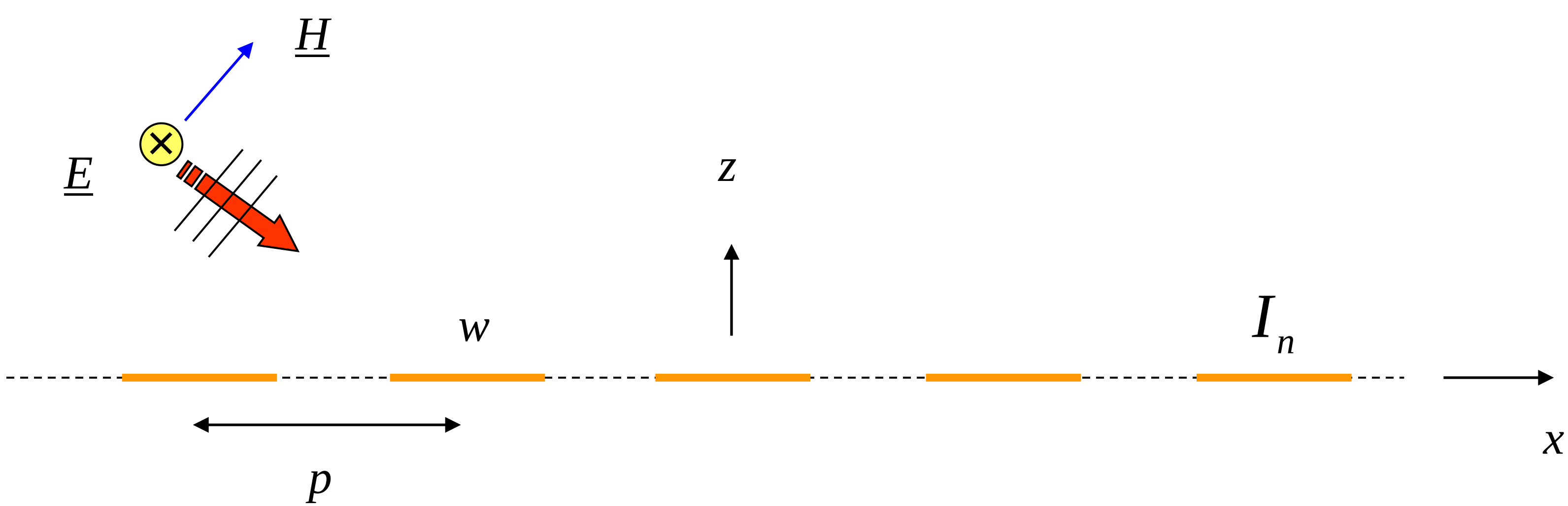


1D FSS (cont.)

The strip currents are periodic, except for a uniform progressive phase shift.

⇒ The scattered field should have this same property.

$$E_y^s\left(x+p,0\right)=e^{-j\left(k_{x0}p\right)}E_y^s\left(x,0\right)$$



1D FSS (cont.)

Denote $$E_y^s(x,0) = e^{-j(k_{x0}x)}\,P(x)$$

Then $$E_y^s(x+p,0) = e^{-j(k_{x0}p)}\,E_y^s(x,0)$$

$$\Downarrow$$

$$e^{-j(k_{x0}(x+p))}\,P(x+p) = e^{-j(k_{x0}p)}\left[\,e^{-j(k_{x0}x)}\,P(x)\right]$$

Hence $$P(x+p) = P(x)$$ (periodic function)

1D FSS (cont.)

Assume a complex Fourier series representation:

$$P(x) = \sum_{n=-\infty}^{\infty} A_n e^{-j\left(\frac{2\pi n}{p}\right)x}$$

We then have

$$E_y^s(x,0) = e^{-j(k_{x0}x)} P(x)$$

$$= e^{-j(k_{x0}x)} \sum_{n=-\infty}^{\infty} A_n e^{-j\left(\frac{2\pi n}{p}\right)x}$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{-j\left(k_{x0}+\left(\frac{2\pi n}{p}\right)\right)x}$$

1D FSS (cont.)

Denote:
$$k_{xn} = k_{x0} + \left(\frac{2\pi n}{p}\right)$$

We then have

$$E_y^s(x,0) = \sum_{n=-\infty}^{\infty} A_n \, e^{-jk_{xn}x}$$

For $z > 0$
$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \, e^{-jk_{xn}x} \, e^{-jk_{zn}z}$$

For $z < 0$
$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \, e^{-jk_{xn}x} \, e^{+jk_{zn}z}$$

$$k_{zn} = \left(k_0^2 - k_{xn}^2\right)^{1/2}$$

1D FSS (cont.)

We can write this as

For $z > 0$
$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \psi_n^+(x,z)$$

For $z < 0$
$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \psi_n^-(x,z)$$

$$\psi_n^+(x,z) = e^{-jk_{xn}x}\, e^{-jk_{zn}z}$$

"Floquet waves"

$$\psi_n^-(x,z) = e^{-jk_{xn}x}\, e^{+jk_{zn}z}$$

$$k_{zn} = \left(k_0^2 - k_{xn}^2\right)^{1/2}$$

1D FSS (cont.)

Magnetic field:
$$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla \times \underline{E}$$

$$\underline{E} = \underline{\hat{y}}\,E_y\left(x,z\right)$$

We then have

$$H_x = \frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial z}$$

$$H_y = 0$$ (The field is TM_y)

$$H_z = -\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial x}$$ (We don't need this equation.)

1D FSS (cont.)

For $z > 0$

$$H_x^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \left(\frac{-k_{zn}}{\omega \mu_0} \right) \psi_n^+(x,z)$$

For $z < 0$

$$H_x^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \left(\frac{+k_{zn}}{\omega \mu_0} \right) \psi_n^-(x,z)$$

Boundary condition at $z = 0$:

$$H_x^s(x,z^+) - H_x^s(x,z^-) = J_{sy}(x) \qquad -w/2 < x < w/2$$

Note: We only need to satisfy this over one strip, since the BC is then automatically satisfied over the other strips.

1D FSS (cont.)

Hence $$\sum_{n=-\infty}^{\infty} 2A_n \left(\frac{-k_{zn}}{\omega\mu_0} \right) e^{-jk_{xn}x} = J_{sy}(x)$$

$$-w/2 < x < w/2$$

Multiply both sides by $e^{+jk_{xm}x}$ and then integrate over the period.

$$\sum_{n=-\infty}^{\infty} 2A_n \left(\frac{-k_{zn}}{\omega\mu_0} \right) \int_{-p/2}^{p/2} e^{-j(k_{xn}-k_{xm})x}\, dx = \int_{-p/2}^{p/2} e^{jk_{xm}x} J_{sy}(x)\, dx$$

1D FSS (cont.)

Examine the integral:

$$I = \int_{-p/2}^{p/2} e^{-j\left(k_{xn}-k_{xm}\right)x}\,dx$$

$$= \int_{-p/2}^{p/2} e^{-j\left(\frac{2\pi}{p}(n-m)\right)x}\,dx$$

$$= \begin{cases} 0, & m \neq n \\ p, & m = n \end{cases}$$

$$k_{xn} = k_{x0} + \left(\frac{2\pi n}{p}\right)$$

$$k_{xm} = k_{x0} + \left(\frac{2\pi m}{p}\right)$$

1D FSS (cont.)

Hence we have:

$$2A_m\left(\frac{-k_{zm}}{\omega\mu_0}\right)p=\int_{-p/2}^{p/2}e^{jk_{xm}x}J_{sy}\left(x\right)dx$$

or

$$2A_m\left(\frac{-k_{zm}}{\omega\mu_0}\right)p=\int_{-p/2}^{p/2}e^{jk_{xm}x}J_{sy0}\left(x\right)dx=\int_{-w/2}^{w/2}e^{jk_{xm}x}J_{sy0}\left(x\right)dx$$

where $J_{sy0}\left(x\right)=$ current on $n=0$ strip

1D FSS (cont.)

Hence

$$2A_m\left(\frac{-k_{zm}}{\omega\mu_0}\right)p=\tilde{J}_{sy}\left(k_{xm}\right)$$

or

$$A_m=\left(\frac{-\omega\mu_0}{2\,pk_{zm}}\right)\tilde{J}_{sy}\left(k_{xm}\right)$$

Next, we enforce the EFIE on the $n = 0$ strip.

$$E_y\left(x,0\right)=0 \qquad -w/2<x<w/2$$

The EFIE is then automatically satisfied on the other strips.

1D FSS (cont.)

The total electric field on the interface is

$$E_y(x,0) = E_y^i(x,0) + E_y^s(x,0)$$

$$E_y^i(x,0) + \sum_{n=-\infty}^{\infty} A_n \psi_n^+(x,0)$$

$$= e^{-jk_{x0}x} + \sum_{n=-\infty}^{\infty} A_n e^{-jk_{xn}x}$$

EFIE: $$e^{-jk_{x0}x} + \sum_{n=-\infty}^{\infty} A_n e^{-jk_{xn}x} = 0 \qquad -w/2 < x < w/2$$

1D FSS (cont.)

Hence

$$e^{-jk_{x0}x} + \sum_{n=-\infty}^{\infty} \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \tilde{J}_{sy}\left(k_{xn}\right) e^{-jk_{xn}x} = 0 \qquad -w/2 < x < w/2$$

Introduce basis functions:

$$J_{sy}(x) = \sum_{m=1}^{M} c_m B_m(x)$$

so

$$\tilde{J}_{sy}\left(k_x\right) = \sum_{m=1}^{M} c_m \tilde{B}_m\left(k_x\right)$$

1D FSS (cont.)

We then have

$$e^{-jk_{x0}x} + \sum_{n=-\infty}^{\infty} \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \left[\sum_{m=1}^{M} c_m \tilde{B}_m\left(k_{xn}\right) \right] e^{-jk_{xn}x} = 0$$

$$-w/2 < x < w/2$$

Introduce testing function:

$$\int_{-w/2}^{w/2} T_l\left(x\right) E_y\left(x,0\right) dx = 0$$

$$l = 1, 2 \ldots M$$

1D FSS (cont.)

We then have

$$\int_{-w/2}^{w/2} T_l(x) e^{-jk_{x0}x}\, dx + \int_{-w/2}^{w/2} T_l(x) \left(\sum_{n=-\infty}^{\infty} \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \left[\sum_{m=1}^{M} c_m \tilde{B}_m(k_{xn}) \right] e^{-jk_{xn}x} \right) dx = 0$$

or

$$\int_{-w/2}^{w/2} T_l(x) e^{-jk_{x0}x}\, dx + \sum_{n=-\infty}^{\infty} \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \left[\sum_{m=1}^{M} c_m \tilde{B}_m(k_{xn}) \right] \left(\int_{-w/2}^{w/2} T_l(x) e^{-jk_{xn}x} dx \right) = 0$$

or

$$\tilde{T}_l(-k_{x0}) + \sum_{n=-\infty}^{\infty} \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \left[\sum_{m=1}^{M} c_m \tilde{B}_m(k_{xn}) \tilde{T}_l(-k_{xn}) \right] = 0$$

1D FSS (cont.)

Hence

$$\tilde{T}_l\left(-k_{x0}\right)+\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)\left[\sum_{m=1}^{M}c_m\tilde{B}_m\left(k_{xn}\right)\tilde{T}_l\left(-k_{xn}\right)\right]=0$$

or

$$\tilde{T}_l\left(-k_{x0}\right)+\sum_{m=1}^{M}c_m\left(\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)\left[\tilde{B}_m\left(k_{xn}\right)\tilde{T}_l\left(-k_{xn}\right)\right]\right)=0$$

Define

$$Z_{lm}=\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)\left[\tilde{B}_m\left(k_{xn}\right)\tilde{T}_l\left(-k_{xn}\right)\right]$$

$$R_l=-\tilde{T}_l\left(-k_{x0}\right)$$

1D FSS (cont.)

We can then write

$$\sum_{m=1}^{M} Z_{lm}\, c_m = R_l$$

or

$$\left[Z_{lm}\right]\left[c_m\right] = \left[R_l\right]$$

This is an $M \times M$ matrix equation for the unknown coefficients c_m.

1D FSS (cont.)

Approximate solution: $M = 1$

Choose:

$$B_1(x) = T_1(x) = \frac{1}{\pi} \frac{1}{\sqrt{\left(\frac{w}{2}\right)^2 - x^2}}$$

(accurate for narrow strips)

(Galerkin's method with a single Maxwell basis function.)

$$\tilde{B}_1(k_x) = \tilde{T}_1(k_x) = J_0\left(\frac{k_x w}{2}\right)$$

1D FSS (cont.)

$$\left[Z_{lm}\right]\left[c_m\right]=\left[R_l\right]$$

$$\Downarrow$$

$$Z_{11}\,c_1 = R_1$$

or

$$c_1 = \frac{R_1}{Z_{11}}$$

1D FSS (cont.)

Hence we have

$$c_1 = \frac{-\tilde{T}_1(-k_{x0})}{\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)\left[\tilde{B}_1(k_{xn})\tilde{T}_1(-k_{xn})\right]}$$

which becomes

$$c_1 = \frac{-J_0\left(-\frac{k_{x0}w}{2}\right)}{\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)J_0\left(\frac{k_{xn}w}{2}\right)J_0\left(\frac{-k_{xn}w}{2}\right)}$$

1D FSS (cont.)

Since the Bessel function is an even function, we have

$$c_1 = \frac{-J_0\left(\dfrac{k_{x0}w}{2}\right)}{\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{-\omega\mu_0}{2pk_{zn}}\right)J_0^2\left(\frac{k_{xn}w}{2}\right)}$$

or

$$c_1 = \frac{J_0\left(\dfrac{k_{x0}w}{2}\right)}{\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{\omega\mu_0}{2pk_{zn}}\right)J_0^2\left(\frac{k_{xn}w}{2}\right)}$$

1D FSS (cont.)

Recall that

$$A_n = \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \tilde{J}_{sy}\left(k_{xn}\right)$$

$$= \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) c_1 J_0\left(\frac{k_{xn}w}{2} \right)$$

Hence

$$A_n = \left(\frac{-\omega\mu_0}{2pk_{zn}} \right) \left[\frac{J_0\left(\frac{k_{x0}w}{2} \right)}{\sum_{q=-\infty}^{\infty} \left(\frac{\omega\mu_0}{2pk_{zq}} \right) J_0^2\left(\frac{k_{xq}w}{2} \right)} \right] J_0\left(\frac{k_{xn}w}{2} \right)$$

Note: The summation index has been changed to q to avoid confusion with n.

1D FSS (cont.)

Hence

$$A_n = \left(\frac{-1}{k_{zn}}\right)\left[\frac{J_0\left(\frac{k_{x0}w}{2}\right)J_0\left(\frac{k_{xn}w}{2}\right)}{\sum_{q=-\infty}^{\infty}\left(\frac{1}{k_{zq}}\right)J_0^2\left(\frac{k_{xq}w}{2}\right)}\right]$$

For $z > 0$

$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n\, e^{-jk_{xn}x}\, e^{-jk_{zn}z}$$

For $z > 0$

$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n\, e^{-jk_{xn}x}\, e^{+jk_{zn}z}$$

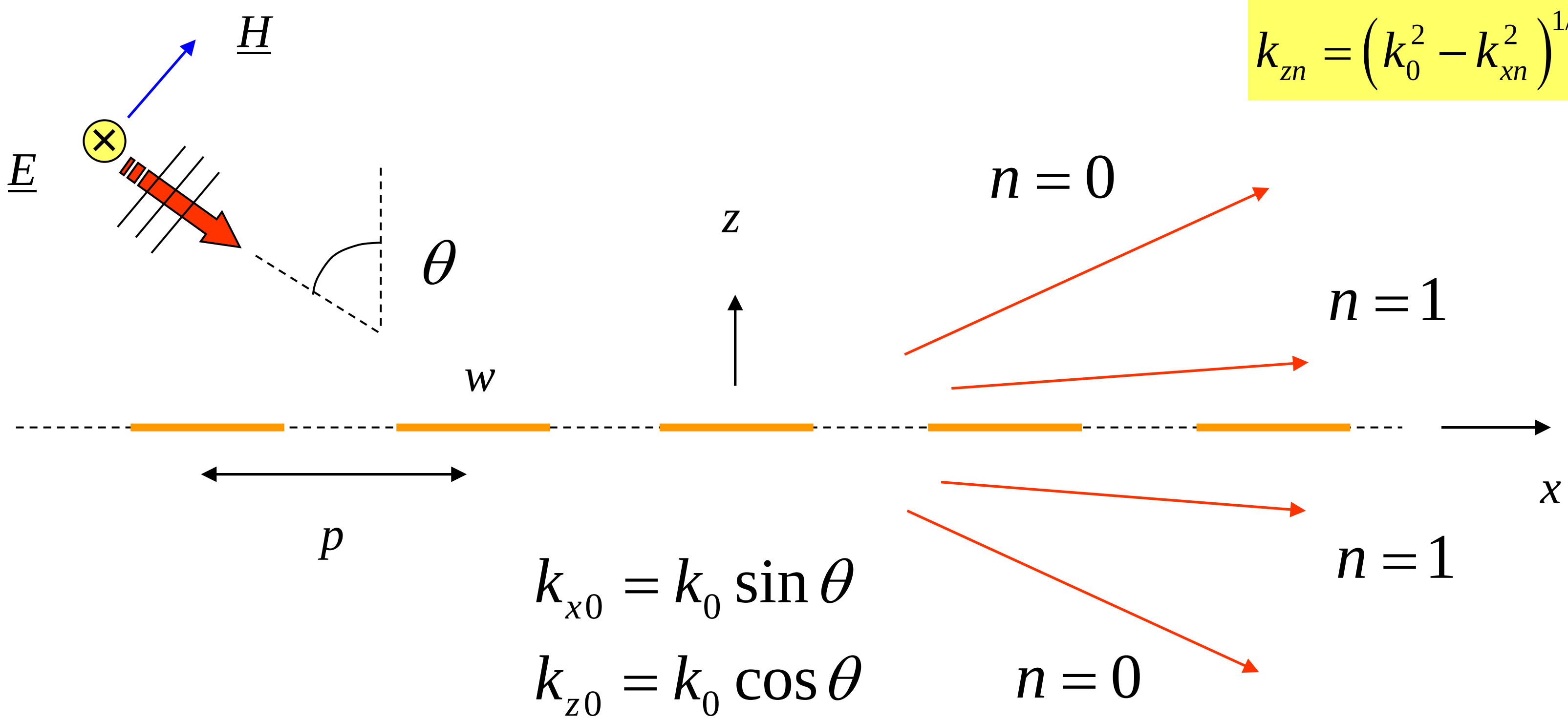
1D FSS (cont.)

Grating Waves

$$E_y^s(x,z) = \sum_{n=-\infty}^{\infty} A_n \, e^{-jk_{xn}x} \, e^{-jk_{zn}|z|}$$

$$k_{xn} = k_{x0} + \left(\frac{2\pi n}{p}\right)$$

$$k_{zn} = \left(k_0^2 - k_{xn}^2\right)^{1/2}$$



$$k_{x0} = k_0 \sin\theta$$

$$k_{z0} = k_0 \cos\theta$$

1D FSS (cont.)

To avoid grating waves (waves that propagate):

$$\left|k_{xn}\right| > k_0, \qquad n \neq 0$$

or
$$\left|k_{x0} + \frac{2\pi n}{p}\right| > k_0, \qquad n \neq 0$$

Set $n = 1$:
$$\left|k_{x0} + \frac{2\pi}{p}\right| > k_0$$

The Floquet waves with $n > 1$ must then also be cutoff.

This will always be satisfied if
$$-k_0 + \frac{2\pi}{p} > k_0$$

since
$$-k_0 < k_{x0} < k_0$$

1D FSS (cont.)

$$-k_0 + \frac{2\pi}{p} > k_0$$

$$\Rightarrow \quad \frac{2\pi}{p} > 2k_0$$

$$\Rightarrow \quad \frac{\pi}{p} > k_0$$

$$\Rightarrow \quad k_0 p < \pi$$

so

$$\frac{p}{\lambda_0} < \frac{1}{2}$$

Note: the same conclusion results from using $n = -1$.