**ECE 6341**

**Spring 2016**

**HW 1**

1) Sometimes people use the "Hertz potentials" instead of the magnetic and electric vector potentials when solving problems. The electric and magnetic Hertz potentials are related to the magnetic and electric vector potentials (in the time domain) as

$$\underline{\mathcal{A}} = \mu\varepsilon \frac{\partial \underline{\Pi}_e}{\partial t}$$

$$\underline{\mathcal{F}} = \mu\varepsilon \frac{\partial \underline{\Pi}_m}{\partial t},$$

where

$\underline{\Pi}_e$ = electric Hertz potential

$\underline{\Pi}_m$ = magnetic Hertz potential.

Write down equations for how to represent, in the most general form, the electric and magnetic fields in the phasor domain in terms of the Hertz potentials. Your result should be very similar to what we derived in class for the vector potentials, which is

$$\underline{E} = \frac{1}{j\omega\mu\varepsilon}\nabla\times(\nabla\times\underline{A}) - \frac{1}{\varepsilon}\nabla\times\underline{F}$$

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\times(\nabla\times\underline{F}).$$

Note that in the phasor domain the electric and magnetic Hertz potentials are simply proportional to the magnetic and electric vector potentials, respectively.

2) Derive the formulas in the TE$_z$/TM$_z$ tables, which are given on slides 14 and 15 in Notes 1, for the fields due to $A_z$ and $F_z$. (These results are the same as those given in Eqs. (3-86) and (3-89) in the Harrington book, except for differences with a factor of $\mu$ and $\varepsilon$, respectively.

3) Using the results from the rectangular aperture problem solved in class in Notes 2, give a final solution for the electric and magnetic vector potentials $A_z$ and $F_z$ for an aperture field that is described by a delta function,

$$E_{y0}(x, y) = \delta(x)\delta(y).$$

(Your answer will be in the form of a double integral in $k_x$ and $k_y$.)

4) Use the equivalence principle and image theory to show that the fields in the region $z > 0$ (above the aperture) in the previous problem are exactly twice those produced by a unit amplitude ($Kl = 1$) x-directed magnetic dipole in free space at the origin, as shown below. Write down the final answer for $A_z$ and $F_z$ for this unit-amplitude magnetic dipole
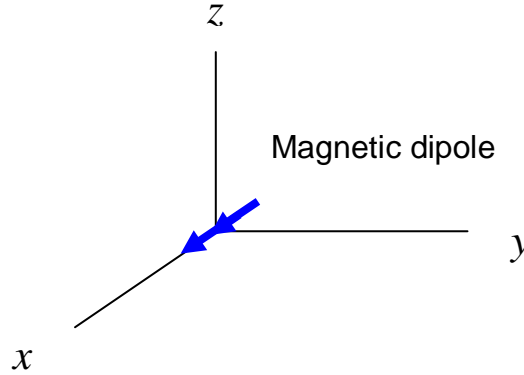


5) Consider a unit-strength ($Il = 1$) electric dipole in free space, oriented in the $x$ direction as shown below.
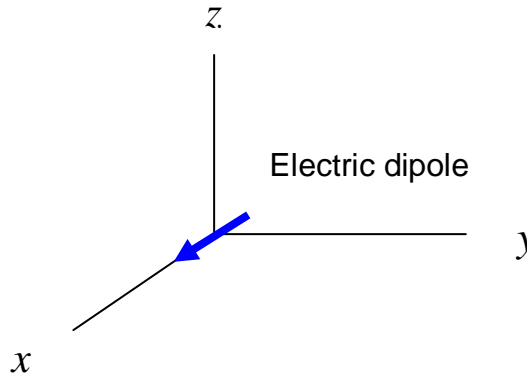


The dipole may be regarded as a surface electric current flowing in the $z = 0$ plane, having the form

$$\underline{J}_s = \hat{\underline{x}}\,\delta(x)\delta(y).$$

Show that this problem may be solved by assuming a choice of vector potentials $A_z$ and $F_z$. To do this, solve for these potential components by starting with the following representation for the Fourier transform of the potential components:

$$\tilde{A}_z^+\left(k_x, k_y, z\right) = f^+\left(k_x, k_y\right)e^{-jk_z z}$$

$$\tilde{F}_z^+\left(k_x, k_y, z\right) = g^+\left(k_x, k_y\right)e^{-jk_z z}$$

$$\tilde{F}_z^-\left(k_x,k_y,z\right)=g^-\left(k_x,k_y\right)e^{+jk_zz}$$

$$\tilde{A}_z^-\left(k_x,k_y,z\right)=f^-\left(k_x,k_y\right)e^{+jk_zz}.$$

In this notation the "+" denotes the solution in the region $z>0$, while the "-" denotes the region $z<0$.

Next, match boundary conditions at $z=0$. The $E_x$ and $E_y$ fields should be continuous, as should the $H_x$ field. For the $H_y$ field, the necessary boundary condition can be obtained from

$$\underline{J}_s=\hat{\underline{z}}\times\left(\underline{H}^+-\underline{H}^-\right).$$

As part of your derivation, you should show that

$$f^-\left(k_x,k_y\right)=-f^+\left(k_x,k_y\right)$$

$$g^-\left(k_x,k_y\right)=g^+\left(k_x,k_y\right).$$

The required task is then to solve for the two unknown functions $f^+\left(k_x,k_y\right)$ and $g^+\left(k_x,k_y\right)$.

Note that by solving this problem, you have demonstrated that the field radiated by a horizontal electric dipole can be expressed in terms of $A_z$ and $F_z$, and this is a key assumption in the proof of the TM$_z$/TE$_z$ theorem. (A similar derivation could be given for a horizontal magnetic dipole.)

6) Consider a unit-strength electric dipole in free space, oriented in the $x$ direction as in the previous problem. Solve the problem this time by using $A_x$ and $F_x$. Start by representing the Fourier transform of the potentials as

$$\tilde{A}_x^+\left(k_x,k_y,z\right)=f^+\left(k_x,k_y\right)e^{-jk_zz}$$

$$\tilde{F}_x^+\left(k_x,k_y,z\right)=g^+\left(k_x,k_y\right)e^{-jk_zz}$$

$$\tilde{F}_x^-\left(k_x,k_y,z\right)=g^-\left(k_x,k_y\right)e^{+jk_zz}$$

$$\tilde{A}_x^-\left(k_x,k_y,z\right)=f^-\left(k_x,k_y\right)e^{+jk_zz}.$$

As part of your derivation, you should show that

$$f^-\left(k_x,k_y\right)=f^+\left(k_x,k_y\right)$$

$$g^-\left(k_x,k_y\right)=-g^+\left(k_x,k_y\right).$$

You should end up with the conclusion that $F_x = 0$, so the solution only requires $A_x$. You should then solve for the unknown function $f^+\left(k_x,k_y\right)$.

7) Consider the same *x*-directed horizontal electric dipole as in the previous problem. Compare your solution for $A_x$ obtained by the Fourier transform method with the known solution (from the solution in ECE 6340) in terms of a spherical wave. From this, conclude that

$$\frac{e^{-jkr}}{r}=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}e^{-jk_z|z|}\,dk_x dk_y,$$

where the function $F$ is a function that you should obtain after doing the comparison. Note that this result shows how a spherical wave can be expanded in terms of plane waves.

8) Based on the previous problem, and what you learned in ECE 6340, find a plane-wave (spectral) representation for the magnetic vector potential $A_z$ due to a unit-amplitude infinitesimal vertical electric dipole at the origin in free space. The result should be in the form of a double integral in $k_x$ and $k_y$.