ECE 6341

Spring 2016

HW 2

Assigned problems: 1-6, 9-11, 13-15.

1) Assume that a TEN models a layered structure, where the x direction (the direction perpendicular to the layers) is the direction that the transmission line in the TEN runs. Normally, we would use waves that are TE_x and TM_x. Each polarization would then have a separate TEN model (no coupling of waves at the boundaries). Within each model (TE_x or TM_x), each layer would have a wave impedance that is unique for a given value of k_x.

   What would happen if you tried to use TE_z and TM_z waves in the TEN? Would you be able to uniquely define a wave impedance for the waves? For example, would the ratio E_y / H_z be the same as the ratio – E_z / H_y if you had a TE_z or a TM_z wave? Justify your answer by showing the calculation. Compare with what happens to these ratios when you use TE_x and TM_x waves.

2) Consider a hollow rectangular waveguide of dimensions a and b in the x and y directions, where the left side wall is at x = 0 and the bottom side wall is at y = 0. Assume a TE_z (m,n) mode described by

$$F_z(x,y,z) = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z^{mn}z}.$$

   Show that this mode can also be written as a sum of TM_x and TE_x modes by solving for A_x and F_x that will produce the same field inside the waveguide as does the original F_z. Hint: In writing down expressions for A_x and F_x, consider what boundary conditions they will have to satisfy at the walls.

   Next, consider the TE_{10} mode as a special case. Solve for A_x and F_x for this mode.

3) A grounded dielectric slab of Teflon has a relative permittivity of 2.2 and a thickness of 60 mils (1.524 mm). Use a numerical search to find the normalized phase constant β_z / k_0 of the TM_0 surface-wave mode at the following frequencies: 1 GHz, 10 GHz, 100 GHz.

4) Assume that we have the same grounded slab as in the previous problem, but now we are interested in the TM_1 mode. First, find the cutoff frequency of this mode. Then, by numerically searching for the improper surface-wave solutions, find the splitting point frequency f_s. Then, for a frequency that is 10% lower than the splitting point frequency (i.e., $f = 0.9\, f_s$) do a numerical search to find k_z / k_0 for the complex TM_1 leaky-mode solution (the one that has a positive attenuation constant α_z). You may use whatever numerical search routine you want. Note that the secant method works in the complex plane, and is usually a good choice. This method is allows you to find the complex roots (zeros) of a complex function $f(z)$. The method is represented by the following iterative formula (which requires two initial guesses z_0 and z_1):

$$z_{n+1} = z_n - \left(z_n - z_{n-1}\right)\left(\frac{f\left(z_n\right)}{f\left(z_n\right) - f\left(z_{n-1}\right)}\right).$$

Note: If you have trouble finding the root 10% below the splitting point frequency, you can try lowering the frequency gradually below the splitting-point frequency, and tracking the root continuously as a function of frequency.

5) Draw a ray picture for a leaky mode that has a negative value of β_z (the phase velocity is in the negative z direction) but the wave is attenuating in the positive z direction. That leaky wave still radiates outward from the structure. That "rays" thus point in the backward direction (to the left of the positive x axis). Such waves, called "backward waves," are very important on some types of guiding structures, such as periodic structures. Explain using the ray picture why a leaky wave that is backward is proper. Next, give a mathematical proof of why this wave must be proper, similar to the proof of why a leaky wave on a simple guiding structure such as a grounded slab (which is a "forward" leaky wave) must be improper.

6) A TE$_x$ leaky mode has a field on the interface (x = 0) due to a line source at z = 0 that is represented as

$$E_y(0,z) = Ae^{-jk_z^{LW}|z|},$$

where $k_z^{LW} = \beta_z - j\alpha_z$ and A is an amplitude constant. The exact electric field above the interface is

$$E_y(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}_y(0,k_z)e^{-jk_x x}e^{-jk_z z}\,dk_z,$$

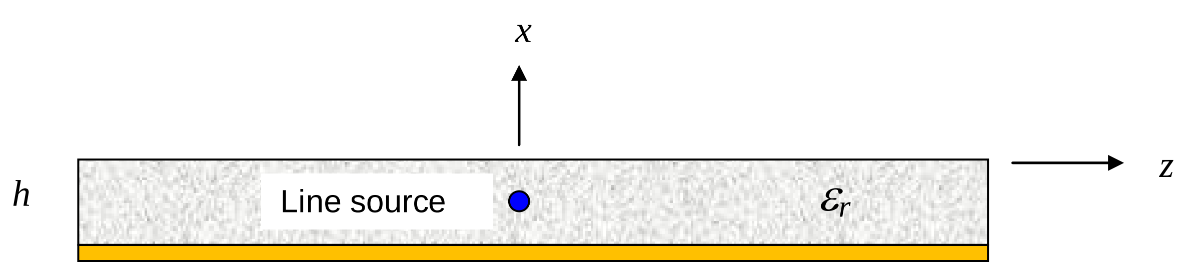
where $k_x = \left(k_0^2 - k_z^2\right)^{1/2}$. The branch of the square root is chosen so that that $\operatorname{Re}k_x > 0$ when k_x is real, and $\operatorname{Im}k_x < 0$ when k_x is imaginary.

Perform the integration above in order to show that

$$\tilde{E}_y(0,k_z) = -2jA\left(\frac{k_z^{LW}}{\left(k_z^{LW}\right)^2 - k_z^2}\right),$$

and thus derive an exact expression for the field above the interface due to the leaky mode. (The expression will be in the form of an integral, as shown above). Explain how you are handling the limit at ±∞ in your integral evaluation.

7) Assume that we have a TE$_x$ leaky mode as in Prob. 5, with $\beta_z = 0.5k_0$ and $\alpha_z = 0.005k_0$. Plot the magnitude of the field $E_y(x,z)$ versus x. Plot over the range $0 < x < 20\lambda_0$ for the following fixed values of z: $z = 1\lambda_0,\ 5\lambda_0, 10\lambda_0,\ 50\lambda_0, 100\lambda_0$. Comment on the variation that you observe vertically. The field should be calculated numerically by using the result of Prob. 5. Assume that A = 1 (the amplitude of the leaky-wave field on the interface is unity).



8) According to the method of stationary phase (or the method of steepest-descent) which will be discussed later in the semester, we can asymptotically evaluate the field radiated by the leaky mode, by using the asymptotic result for $k_0\rho \to \infty$ that says

$$\int_{-\infty}^{\infty} f(k_z) e^{-jk_x x} e^{-jk_z z}\, dk_z \sim f(k_{z0}) k_0 \cos\theta \sqrt{\frac{2\pi}{(k_0\rho)}}\, e^{-jk_0\rho}\, e^{j\frac{\pi}{4}},$$

where $f(k_z)$ is an arbitrary function of k_z, $k_x = \left(k_0^2 - k_z^2\right)^{1/2}$, $k_{z0} = k_0 \sin\theta$, and in cylindrical coordinates $x = \rho\cos\theta$, $z = \rho\sin\theta$. Assuming this relation, derive the far-field pattern $E_y^{FF}(\rho,\theta)$ of the TE$_x$ leaky mode in Problem 5.

9) The far-field pattern of a TE$_x$ bi-directional leaky mode on a grounded slab is given by

$$F(\theta) = \int_{-\infty}^{\infty} E_y(0,z) e^{+j(k_0\sin\theta)z}\, dz = \int_{-\infty}^{\infty} e^{-jk_z^{LW}|z|}\, e^{+j(k_0\sin\theta)z}\, dz.$$

Evaluate this integral and show that the result is

$$AF(\theta) = 2j\left(\frac{k_z^{LW}}{k_0^2\sin^2\theta - \left(k_z^{LW}\right)^2}\right).$$

10) Plot the far-field pattern of a TE$_x$ bi-directional leaky wave having the following properties:

a) $k_z / k_0 = \dfrac{\sqrt{3}}{2} - j(0.02)$

b) $k_z / k_0 = \dfrac{\sqrt{3}}{2} - j(0.002)$.

Plot the magnitude of the far-field pattern (i.e., the array factor given in the problem above) vs. angle θ in degrees (a rectangular plot, not a polar plot), with the angle θ in the range -90^o < θ < 90^o. (Note: The pattern should be symmetric about θ = 0.) Normalize your patterns so that the magnitude of the far-field pattern is unity at the peaks of the beam.

11) Repeat the previous problem, assuming now that $\beta_z = (1.5)k_0$ and $\alpha_z = 0.002k_0$. This corresponds to a leaky mode that is in the non-physical (slow-wave) region.

12) Assume that we have a leaky-wave field described by

$$\psi(x,z) = e^{-j(k_x x + k_z z)}$$

where

$$k_x = \left(k_0^2 - k_z^2\right)^{1/2}.$$

Introduce the following change of variables:

$$k_x = k_0 \cos\zeta,\ k_z = k_0 \sin\zeta\ ,$$

where $\zeta = \zeta_r + j\zeta_i$ is a complex number with $0 < \zeta_r < \pi/2$ and $\zeta_i < 0$ (which corresponds to $\beta_z > 0$, $\alpha_z > 0$, $\beta_x > 0$, $\alpha_x < 0$). Also, introduce cylindrical coordinates as

$$x = \rho\cos\theta,\ z = \rho\sin\theta,$$

where θ is the angle in cylindrical coordinates measured from the x axis, and ρ is the radial distance in cylindrical coordinates from the y axis (see the figure in Prob. 6). Show that the magnitude of the leaky-wave field may be written as

$$|\psi(x,z)| = e^{-(k_0\rho)\sin(\theta-\zeta_r)|\sinh\zeta_i|}.$$

Next, show that

$$\sin(\theta-\zeta_r) = \sin(\theta-\theta_0)\operatorname{sech}\zeta_i,$$

where θ_0 is the angle from the x axis to the power flow vector ($\underline{\beta}$ vector), so that

$$\beta_x = k_0\cos\theta_0,\ \beta_z = k_0\sin\theta_0.$$

(Hint: Use expressions for β_x and β_x in terms of ζ.)

Finally, conclude that

$$\left|\psi\left(x,z\right)\right| = e^{-(k_0\rho)\sin(\theta-\theta_0)\operatorname{sech}\zeta_i\left|\sinh\zeta_i\right|}.$$

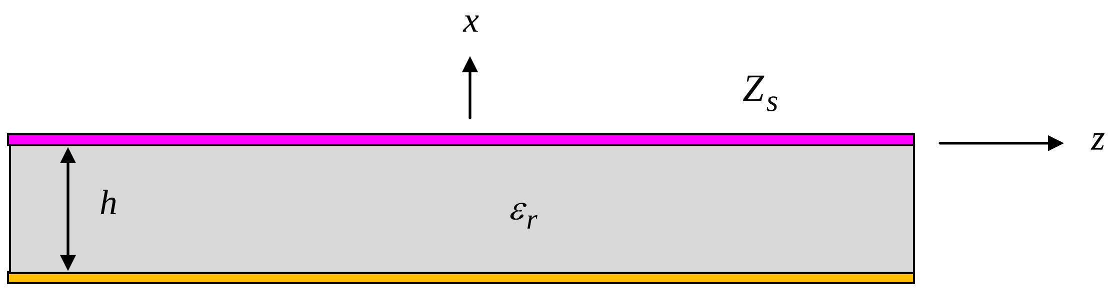
Use this result to explain why the magnitude of the leaky-wave field always decreases with distance ρ in cylindrical coordinates, as long as we are in the "leakage region" so that $\theta > \theta_0$.

13) A sheet impedance Z_s is a hypothetical sheet layer that has a continuous tangential electric field across it, but it supports a tangential (i.e., in the plane of the sheet, which we can assume is the *yz* plane) surface current $\underline{J}_s$ in the direction of the tangential electric field $\underline{E}_t$. In general, we would have, from the definite of sheet impedance, that $\underline{E}_t = Z_s \underline{J}_s$. Assume that a particular component of tangential electric field (e.g., E_z) is being modeled as voltage on a TEN model of a waveguiding structure that contains a sheet impedance, and that the corresponding perpendicular component of the tangential magnetic field (e.g., $-H_y$) is being modeled as current on the TEN. (The TEN runs in the x direction.) Show that the sheet impedance appears in the TEN as a lumped parallel impedance element, with a value of Z_s Ohms.

14) Consider a "partially reflecting surface" (or "PRS") type of leaky-wave antenna structure that consists of a lossless sheet impedance $Z_s = jX_s$ (the PRS) above a grounded dielectric slab region as shown below. There is air above the structure. Assume a TM_x leaky mode with a complex wavenumber k_z is propagating on the structure in the z direction (there is no y variation). Draw a TEN for this structure and use the TRE method to derive a transcendental equation for the unknown complex wavenumber k_z of the leaky wave. Choose a reference plane that is just below the sheet impedance.

In the transcendental equation, how do you choose the square root in order to calculate k_{x0} (the vertical wavenumber in the air region) from k_z?

Practical note: The sheet impedance could be modeling a partially-reflecting surface that is a periodic frequency selective surface (FSS) such as an array of metal patches, etc.



15) A rectangular waveguide is loaded with dielectric slabs on either side, as shown below. Derive transcendental equations for the wavenumber k_z and the cutoff frequencies for all four possible modes that can exist inside the waveguide. The four mode types are: TM_x^e, TM_x^o, TE_x^e, and TE_x^o, where the (e,o) superscript denotes even or odd mode.