

ECE 6341
Spring 2016

HW 4

Assigned problems: 1-3, 5-9.

- 1) Consider an infinite line source of current having the form

$$I(z) = \exp(-jk_{z0}z),$$

which is flowing on the z axis in an infinite medium with wavenumber k . Assume that the wavenumber k_{z0} is real, so that the integral converges. By matching the magnetic vector potential between cylindrical coordinates and rectangular coordinates (the latter solution was done in ECE 6340), derive the following mathematical identity:

$$\int_{-\infty}^{\infty} \frac{e^{-jkR}}{R} e^{-jk_{z0}z'} dz' = -j\pi H_0^{(2)}(k_{\rho 0}\rho) e^{-jk_{z0}z},$$

where $R = \sqrt{\rho^2 + (z - z')^2}$ and $k_{\rho 0} = (k^2 - k_{z0}^2)^{1/2}$.

The radial wavenumber $k_{\rho 0}$ is chosen to be either a positive real number or a negative imaginary number.

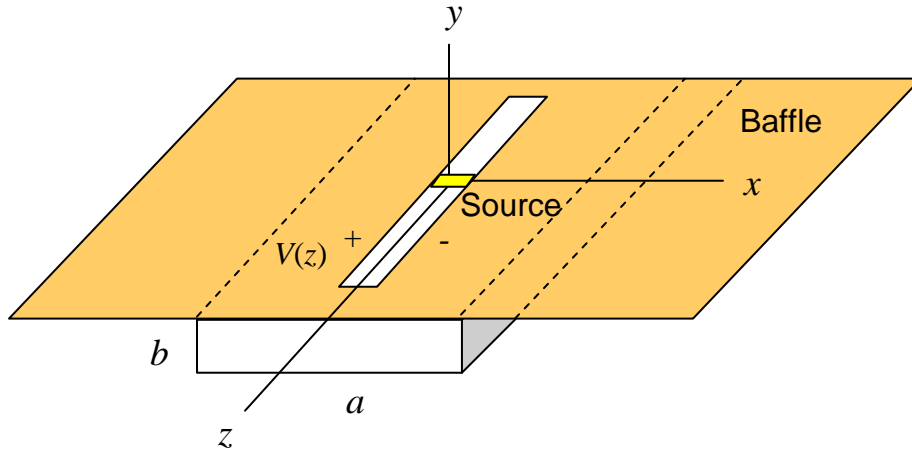
Assume next that k_{z0} becomes complex, so the current represents a leaky-wave line source. Would the integral on the left-hand side of the identity still exist? Would the right-hand side of the identity still exist? Note that in this case the right-hand side of the above identity represents the *analytic continuation* of the integral on the left-hand side into the complex wavenumber plane.

- 2) Consider a slotted rectangular waveguide leaky-wave antenna, as shown below, which has an infinite narrow slot. Assume that the structure is fed with a source at $z = 0$ and a bi-directional leaky wave propagates in both directions from this point. Therefore, the slot has a voltage given by

$$V(z) = A_0 e^{-jk_z^{LW}|z|}, \text{ where } k_z^{LW} = \beta - j\alpha.$$

Use the equivalence principle and image theory to show that the field above the infinite conducting baffle (ground plane) is that of a magnetic current flowing on the z axis in free space,

$$K(z) = 2A_0 e^{-jk_z^{LW}|z|}.$$



- 3) As a continuation of the previous problem, consider a leaky-wave magnetic current flowing on the z axis in free space that has the form

$$K(z) = 2A_0 e^{-jk_z^{LW}|z|}, \quad \text{where } k_z^{LW} = \beta - j\alpha.$$

Using duality along with Fourier transform theory, as was done in class for an arbitrary line current $I(z)$, derive an expression for the electric vector potential $F_z(\rho, \phi)$ at any point in space (your answer will be in the form of a single integral in k_z).

- 4) Derive expressions for all of the field components for the previous leaky-wave antenna. This includes $E_z, H_z, E_\rho, H_\rho, E_\phi, H_\phi$. (Note: Some of these components may be zero.)
- 5) Find the far-field components $E_\theta(r, \theta, \phi)$ and $E_\phi(r, \theta, \phi)$ of the bi-directional leaky-wave magnetic line-source current in the previous problem. Do this by first finding the electric vector potential F_z in the far field by using the “far-field identity” that was discussed in class, namely

$$\int_{-\infty}^{+\infty} f(k_z) H_n^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z \sim 2j^{n+1} \left(\frac{e^{-jkr}}{r} \right) f(k \cos \theta).$$

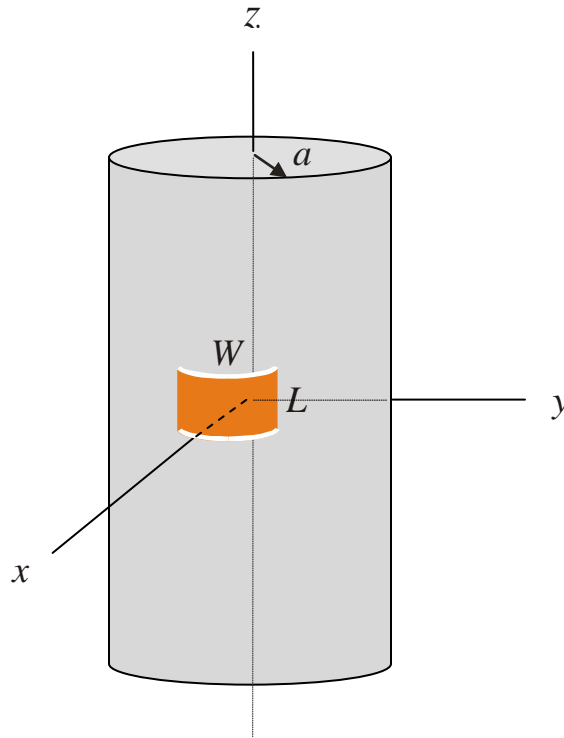
This is valid as $r \rightarrow \infty$ in spherical coordinates, with $\theta \neq 0, 180^\circ$.

- 6) A microstrip antenna is placed on the surface of a rocket (see the figure below). The center of the patch is located on the x axis. The two radiating edges of the microstrip antenna are modeled as two slots, each having a one-volt drop ($V = 1$) across the edge, measured from bottom to top. Therefore, for each slot the electric field is given by

$$\underline{E} = \hat{z} \left(\frac{1}{\Delta z} \right),$$

where Δz is the width of each slot in the z direction. The slots may be assumed to be of infinitesimal width, so that $\Delta z \rightarrow 0$. The electric field of the dominant microstrip antenna mode is assumed to be uniform across the width of the patch, in the ϕ direction, and therefore the electric field inside the slots is uniform in the ϕ direction (that is why there is no ϕ variation in the above equation for the slot field). The length of the microstrip antenna is L in the z direction. The two radiating edges, being modeled by the slots, are located at $z = L/2$ and $z = -L/2$. The width of the patch is W in the ϕ direction (the angle subtended by the two nonradiating edges is $\alpha = W/a$, with the slot fields existing in the region $-\alpha/2 < \phi < \alpha/2$).

Determine the far field pattern $E_\theta(r, \theta, \phi)$.



- 7) A circular loop of uniform current I_0 is in free space as shown below. The loop has a radius b and it lies in the xy plane. Because the current is uniform, the field produced will be purely TE_z . The field inside and outside the loop may be written as spectral integrals (integrals over a wavenumber) as

$$F_{z1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k_z) J_0(k_\rho \rho) e^{-jk_z z} dk_z, \quad \rho < b$$

$$F_{z2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} B(k_z) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z, \quad \rho > b,$$

where

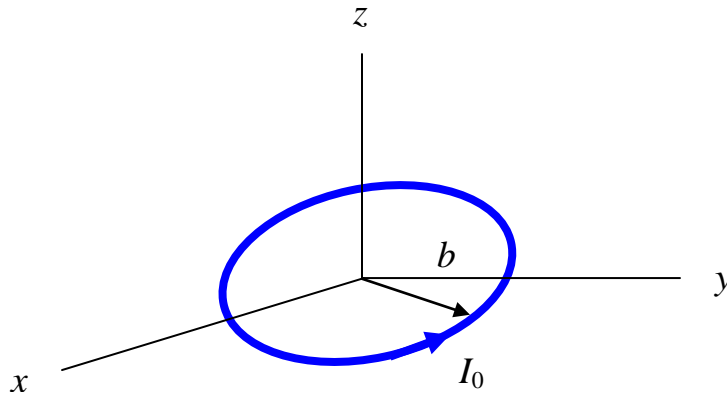
$$k_\rho = (k_0^2 - k_z^2)^{1/2}.$$

Solve for the unknown coefficient functions $A(k_z)$ and $B(k_z)$ by applying boundary conditions at $\rho = b$. These include the condition that the field E_ϕ should be continuous at this boundary, and the condition that the field H_z should be discontinuous. Note that the ring of current can be thought of as a surface current at $\rho = b$, having the form

$$\underline{J}_s = \hat{\phi} I_0 \delta(z).$$

As a helpful hint, recall that the delta-function has the following representation:

$$\delta(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jk_z z} dk_z.$$



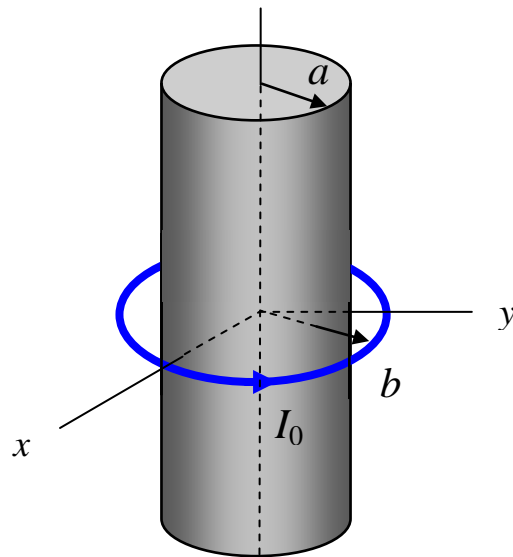
- 8) Now consider the same circular loop surrounding a perfectly conducting pipe of radius a , as shown below. (This is an important problem in some well-logging applications, where the pipe could be modeling a well-logging tool.) The total electric vector potential may be written as the sum of an incident potential and a scattered potential, as

$$F_z = F_z^i + F_z^s,$$

where the incident potential is the potential of the current loop in free space, which is what you solved for in the previous problem. The scattered potential is that due to the currents induced on the pipe. The scattered potential may be written in the form

$$F_z^s = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(k_z) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} dk_z, \quad \rho > a.$$

Solve for the unknown coefficient function $C(k_z)$ by applying the boundary condition that the total tangential electric field E_ϕ should be zero on the surface on the conducting pipe.



- 9) Now consider that there are two loops surrounding the same pipe in the previous problem. The first loop of radius b is at $z = 0$, as in the previous problem. The second loop (not shown) is identical to the first, but it is located at a height $z = h$. The first loop is thought of as a transmitter coil and the second loop is thought of as a receiver coil. This configuration is of practical interest in the well-logging area. (In many well-logging sensors, there are transmitter and receiver coils surrounding the conducting tool body.)

The mutual inductance M between the two coils may be found from the open-circuit voltage induced on the receiver coil, which is

$$V_2 = (j\omega M) I_0 = (2\pi b) E_\phi^{(1)}(h, b),$$

where V_2 is the open-circuit voltage induced at the terminals of coil 2 (assuming that we introduce a terminal pair at some point on loop 2), I_0 is the current on the transmitter coil 1, and $E_\phi^{(1)}(h, b)$ is the electric field produced by coil 1 (radiating in the presence of the pipe) at $z = h$ and $\rho = b$.

Derive a formula for the mutual inductance M between the two coils, using your solution from the previous problem to determine the field $E_\phi^{(1)}(h, b)$. Your formula will be in terms of a spectral integral over k_z .