ECE 6341
Spring 2016

Homework 6

Please do Probs. 1-3, 5-6, and 8-11.

Please also note the extra-credit problem at the end, which is optional.

1) Use integration by parts to asymptotically evaluate the following integral:

$$I(\Omega) = \int_0^1 e^{-x} \cos(\Omega x)\, dx \,.$$

2) Use the stationary-phase method to asymptotically evaluate the Bessel function $J_n(x)$ for large x, starting with

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta)\, d\theta \,.$$

Compare your result with what is available in math handbooks (e.g., Eq. 24.103 of the *Schaum's Outline Mathematical Handbook*).

3) An infinite uniform line source carries I Amps along the z axis, at a radian frequency ω. By using the Fourier transform method, the exact vector potential $A_z(x,y)$ for $y > 0$ is found to be

$$A_z(x,y) = \frac{\mu_0 I}{4\pi j} \int_{-\infty}^{\infty} \frac{1}{k_y} e^{-jk_y y}\, e^{-jk_x x}\, dk_x \,,$$

where $k_y = \left(k_0^2 - k_x^2\right)^{1/2}$.

Apply the stationary-phase method to find $A_z(\rho)$ in the far field ($k\rho >> 1$). First convert to polar coordinates for the observation point, but do <u>not</u> do a change of variables for k_x (i.e., leave the integration variable as k_x). Check your result by starting with the known exact expression

$$A_z = \frac{\mu_0 I}{4j} H_0^{(2)}(k\rho)$$

and then approximating this expression in the far field.

4) The following integral $I(r,\theta)$ is a typical "Sommerfeld" type integral (in Hankel form) that often appears in the analysis of dipoles in layered-media:

$$I(r,\theta) = \int_{-\infty}^{\infty} f(k_\rho) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} \, dk_\rho .$$

In this equation $k_z = \left(k_0^2 - k_\rho^2\right)^{1/2}$. (The square root has the usual interpretation of being either a positive real number or a negative imaginary number.)

Evaluate this integral for $kr >> 1$ using the stationary-phase method. The variables r and θ denote the usual spherical coordinates here, with ρ and z denoting the usual cylindrical coordinates. Assume that $z > 0$. Do <u>not</u> do a change of variables for k_ρ (i.e., leave the integration variable as k_ρ).

(Hint: Approximate the Hankel function with its asymptotic approximation first. Also, convert to spherical coordinates.)

5) The modified Bessel function of the first kind has an integral definition that is

$$I_0(\Omega) = \frac{1}{2\pi} \int_0^{2\pi} e^{\Omega \sin\theta} \, d\theta .$$

Use Laplace's method to asymptotically evaluate the function $I_0(\Omega)$ for large Ω. Compare your result with what is available in math handbooks (e.g., Eq. 24.107 of the *Schaum's Outline Mathematical Handbook*).

6) Determine the first two leading terms of the asymptotic approximation to the following integral, as Ω becomes large, using Watson's Lemma.

$$I(\Omega) = \int_{-1}^{1} \cos(s) \, e^{-\Omega s^2} \, ds .$$

7) Use Watson's lemma to derive the first two leading terms of the asymptotic expansion of the modified Bessel function $I_0(x)$, defined by the integral in Problem 5.

Hint: Establish that $s^2 = 1 - \sin\theta$. Then use this to establish that $h(s)$ is given by

$$h(s) = \frac{2}{\sqrt{2 - s^2}} ,$$

where the principle square root is chosen. Discuss the reason for the choice of sign in the above equation (i.e., $h(s)$ is positive at $s = 0$). Please evaluate all necessary constants in your final answer (do not leave special functions in your final answer).

8) Determine the first two leading terms (i.e., the first two non-zero terms) of the asymptotic approximation to the following integral, as Ω becomes large, using the "alternative form" of Watson's Lemma.

$$I(\Omega) = \int_0^1 \sin(s) e^{-\Omega s} \, ds \, .$$

9) Determine the first two leading terms (i.e., the first two non-zero terms) of the asymptotic expansion to the following integral, as Ω becomes large, using the "alternative form" of Watson's Lemma. Please evaluate all necessary constants in your final answer (do not leave special functions in your final answer).

$$I(\Omega) = \int_0^1 \sqrt{\sin(s)} \, e^{-\Omega s} \, ds \, .$$

10) Evaluate the integral

$$I(r,\theta) = \int_{-\infty}^{\infty} f(k_\rho) H_0^{(2)}(k_\rho \rho) e^{-jk_z z} \, dk_\rho \, ,$$

for $kr >> 1$ with θ fixed, using the steepest-descent method. In this integral

$$k_z = \left(k_0^2 - k_\rho^2\right)^{1/2},$$

and the variables r and θ denote the usual spherical coordinates here, with ρ and z denoting the usual cylindrical coordinates. To cast the integrand into a form that is suitable for the method of steepest descent, use the asymptotic expansion of the Hankel function. Use the steepest-descent transformation as part of your derivation (i.e., perform the asymptotic analysis in the steepest-descent ζ plane).

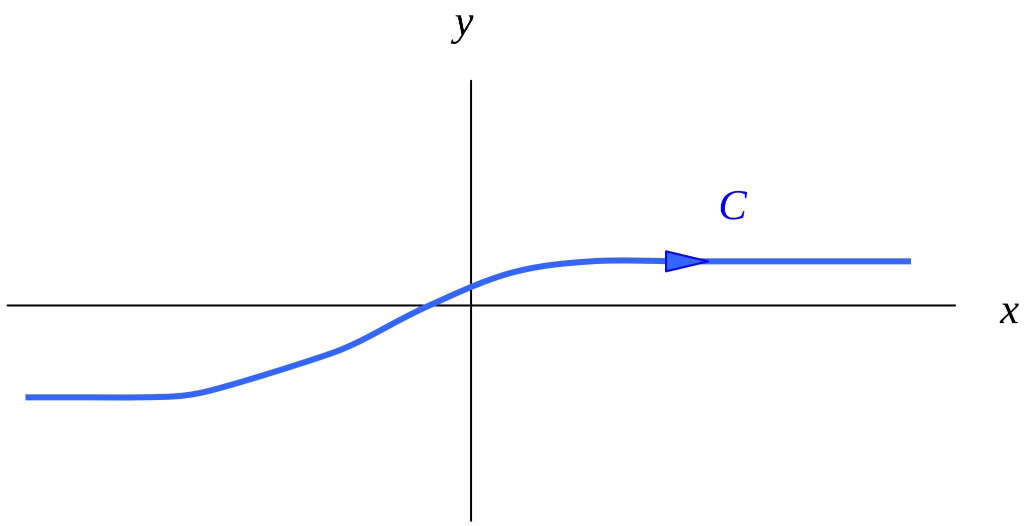
Carefully discuss the SDP and how you are choosing the value of the departure angle θ_{SDP}.

Note: This is the same integral evaluated previously in problem 4, using the stationary-phase method.

11) Evaluate the integral

$$I(\Omega) = \int_C \frac{1}{z} e^{j\Omega\left(\frac{z^2}{2} - z\right)} dz$$

For $\Omega >> 1$, where C is the contour extending from $(-\infty - j2)$ to $(-\infty + j1)$, going above the origin as shown below. Carefully discuss how you are treating any singularities that are present when you deform the integration path to the SDP.



EXTRA CREDIT

Consider an infinitively long perfectly conducting cylinder of radius $a = 10\ \lambda_0$ running along the z axis, with the z axis at the center of the cylinder. The cylinder is illuminated by a unit-amplitude plane wave, described by

$$\underline{E}^{inc} = \underline{\hat{z}}\, e^{-jk_0 x}\ .$$

Plot the magnitude of the geometrical optics scattered far field (from Notes 29) vs. angle ϕ, and then plot the magnitude of the exact scattered field vs. angle ϕ, and compare the two (where $0 < \phi < 2\pi$ in your plot). The exact scattered field may be found from Notes 13. Note that in the exact solution from Notes 13, you can approximate asymptotically the Hankel function to obtain the scattered field in the far field. (The final result in Notes 29 assumes that we are in the far field.) Normalize the plots so that they are both unity at $\phi = \pi$.

If your program allows it, try the case $a = 100\ \lambda_0$ as well. (The normalized geometrical optics result will not change, only the exact solution will.) Note that the solution given in Notes 13 will converge more slowly as the radius of the cylinder becomes larger compared with a wavelength, so you may encounter numerical trouble.